10

Asymptotics

This chapter discusses some basic consequences of the notion of *asymptotic simplicity* introduced in Chapter 7. As already mentioned, the main motivation behind this definition is to provide a characterisation of a broad class of spacetimes in which *universal structures* can be identified. Once this has been done, the idea is to use these structures to define in a rigorous manner concepts of physical interest.

The characterisation of the gravitational field through the analysis of its asymptotic behaviour has a long tradition dating back to the early works by Bondi et al. (1962), Sachs (1962b) and Newman and Penrose (1962). These studies culminated in the identification of gravitational radiation as a real physical phenomenon. The developments of this *classical theory* have been treated extensively in the literature; see, for example, Geroch (1976), Penrose and Rindler (1986), Stewart (1991) and Frauendiener (2004). The readers interested in the historic development of this idea are referred to Kennefick (2007).

Despite the important insights provided by the classical theory of asymptotics of general relativity, this approach has the weakness of being, to some extent, *formal*. More precisely, it relies on a number of assumptions about the nature of solutions to the Einstein field equations – say, for example, the regularity of the conformal boundary – which are hard to verify for a *suitably large class of spacetimes*. This point is key: the theory of asymptotics of the gravitational field comes fully into life when combined with the (conformal) field equations and methods of the theory of partial differential equations. This remark does not disown the fundamental insights into the behaviour of the gravitational field that formal asymptotic analyses have produced, but rather insists on the need to carry the subject further.

Arguably, the most important consequence of asymptotic simplicity is the set of results collectively known as ***peeling*** – that is, a detailed description of the asymptotic behaviour of the gravitational field expressed in terms of the components of the Weyl tensor. The peeling behaviour is the main subject of this

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chapter. The basic assumptions behind the peeling results are the main subject of Chapter 20. Complementary to the discussion of the peeling behaviour, this chapter contains a detailed discussion of a gauge prescription for the analysis of the structure of the gravitational field at the conformal boundary of Minkowski-like spacetimes, the so-called ***NP gauge***. The chapter concludes with a brief overview of other aspects of the theory of the asymptotics of the gravitational field which are sligthly outside the main focus of this book: the Bondi mass, the BMS group and the so-called Newman-Penrose constants.

10.1 Basic set up: general structure of the conformal boundary

In what follows let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ be an *asymptotically simple spacetime* in the sense of Definition 7.1 and let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ denote an associated conformal extension. As in Section 7.1, let $\mathscr{I}$ denote part of the conformal boundary characterised by the requirements

$$\Xi = 0, \qquad \mathbf{d}\Xi \neq 0. \tag{10.1}$$

Much of the analysis of the present chapter is based on the evaluation of the various conformal field equations at $\mathscr{I}$. In what follows, *the notation $\simeq$ will be used to indicate that a certain equality holds at $\mathscr{I}$*. In terms of this notation, the conditions in (10.1) can be rewritten as

$$\Xi \simeq 0, \qquad \mathbf{d}\Xi \not\simeq 0.$$

The basic observation concerning the set $\mathscr{I}$ is that its causal nature is determined by the sign of the cosmological constant λ. This result follows from a direct inspection of the conformal Einstein field equations; see, for example, Equations (8.26a)–(8.26e) in Section 8.2.5. One has that:

Theorem 10.1 (***causal nature of the conformal boundary***) *Suppose that the Friedrich scalar s is finite at $\mathscr{I}$ and that $T = o(\Xi^{-4})$. Then $\mathscr{I}$ is a null, spacelike or timelike hypersurface, respectively, depending on whether $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$.*

Proof The normal to the hypersurface $\mathscr{I}$ is given by $\nabla_a \Xi$. From Equation (8.24) one directly has that

$$\nabla_a \Xi \nabla^a \Xi \simeq -\frac{1}{3}\lambda, \tag{10.2}$$

as by hypothesis $\Xi^4 T \to 0$ if $\Xi \to 0$ and s is finite at $\mathscr{I}$. □

A discussion of the ***order symbols*** o and O used in the previous and other results of this chapter can be found in the Appendix to Chapter 11.

Remark. Spacetimes with $\lambda = 0$ will be said to be ***Minkowski-like***, those with $\lambda < 0$ ***de Sitter-like*** and those with $\lambda > 0$ ***anti-de Sitter-like***.

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The regularity of s at $\mathscr{I}$ can be rephrased in terms of a sufficiently rapid decay of the physical energy-momentum tensor $\tilde{T}_{ab}$. Using the conformal field Equation (8.13) it follows that a sufficient condition for $\nabla_a \nabla_b \Xi$ and s to be finite at $\mathscr{I}$ is that $T_{\{ab\}} = o(\Xi^{-3})$. In this case one concludes that

$$\nabla_a \nabla_b \Xi \simeq s g_{ab}. \tag{10.3}$$

It follows from the transformation formulae of the energy-momentum tensor, Equation (9.2), that if $T_{\{ab\}} = o(\Xi^{-3})$, then, in fact, $\tilde{T}_{\{ab\}} = O(\Xi^3)$; see also the discussion in Stewart (1991). If, in addition, one has that R is finite at $\mathscr{I}$, then expression (10.3) reduces to

$$\nabla_a \nabla_b \Xi \simeq \frac{1}{4} \nabla^c \nabla_c \Xi g_{ab}.$$

The spinorial version of the above expression is

$$\nabla_{A(A'} \nabla_{B')B} \Xi \simeq 0. \tag{10.4}$$

The latter is usually known as the ***asymptotic Einstein condition***; see, for example, Penrose and Rindler (1986).

10.1.1 Topology of the conformal boundary

As will be seen in Chapter 15, there exists considerable freedom in the specification of the topology of de Sitter-like spacetimes. By contrast, the case of a vanishing cosmological constant is much more restrictive:

Theorem 10.2 (***topology of $\mathscr{I}$ for asymptotically Minkowskian spacetimes***) *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote an asymptotically simple spacetime with $\lambda = 0$ and let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ denote a conformal extension thereof. Then $\mathscr{I}$ consists of two disjoint components $\mathscr{I}^-$ and $\mathscr{I}^+$, each one having the topology of $\mathbb{R} \times \mathbb{S}^2$.*

A discussion of the proof of the above theorem goes beyond the scope of this book. The interested reader is referred to Newman (1989) for a proof and for a discussion on pitfalls in earlier arguments in Penrose (1965) and Geroch (1971b, 1976); see also Hawking and Ellis (1973). Remarkably, this result depends on the satisfactory resolution of the so-called *Poincaré conjecture*; see, for example, Gowers (2008) for an introduction to this (now solved) classical problem in mathematics. Vacuum spacetimes with a vanishing cosmological constant and a conformal infinity with sections which are toroidal, that is, having the topology of $\mathbb{R} \times \mathbb{S} \times \mathbb{S}$ have been considered in the literature; see Schmidt (1996). Note that as a consequence of Theorem 10.2 these spacetimes must exhibit some type of pathology – and, in particular, they cannot be asymptotically simple.

The behaviour of points in the conformal extension of an asymptotically simple spacetime for which both $\Xi = 0$ and $\mathbf{d}\Xi = 0$ will be analysed from various perspectives in Chapters 16, 18 and 20.

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10.1.2 Further properties of the case $\lambda = 0$

In this section let $\lambda = 0$ throughout so that the asymptotically simple spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ has a null conformal boundary. For ease of the exposition, attention is restricted to the vacuum case.

As a consequence of Theorem 10.1 the physical spacetime manifold $\tilde{\mathcal{M}}$ must lie either towards the past or the future of $\mathscr{I}$ – intuitively, this assertion seems natural; however, a detailed argument requires the ideas of the discussion on Lorentzian causality in Chapter 14. Consistent with the discussion of conformal extensions of exact solutions in Chapter 6, $\mathscr{I}^+$ (i.e. ***future null infinity***) will denote the set on which null geodesics attain a future endpoint while $\mathscr{I}^-$ (i.e. ***past null infinity***) corresponds to the set of past endpoints of null geodesics. A null hypersurface has the property of being generated by null geodesics; that is, each $p \in \mathscr{I}^\pm$ lies on exactly one null geodesic which is everywhere tangent to $\mathscr{I}^\pm$. Accordingly, each of $\mathscr{I}^+$ and $\mathscr{I}^-$ can be regarded as the union of these ***generators*** (or ***rays***). Complementary to the latter is the notion of a ***cut of null infinity***, that is, a two-dimensional surface $\mathscr{C}$ which intersects each generator exactly once. As a result of Theorem 10.2 one has that $\mathscr{C} \approx \mathbb{S}^2$.

The subsequent discussion will, for simplicity, be restricted to $\mathscr{I}^+$ – an analogous discussion follows, *mutatis mutandis*, for $\mathscr{I}^-$. By definition, the normal to $\mathscr{I}^+$ is given by $\mathbf{d}\Xi$. As $\boldsymbol{g}^\sharp(\mathbf{d}\Xi, \mathbf{d}\Xi) \simeq 0$, it follows that the vector $\boldsymbol{N} \equiv -\boldsymbol{g}^\sharp(\mathbf{d}\Xi, \cdot)$ satisfies $\langle \mathbf{d}\Xi, \boldsymbol{N}\rangle = 0$ and, thus, is tangent to $\mathscr{I}^+$ – and, in particular, to its null generators.

As $\mathscr{I}^+$ is a hypersurface of $\mathcal{M}$, there exists an embedding $\varphi : \mathscr{I}^+ \to \mathcal{M}$. Let $\boldsymbol{q} \equiv \varphi^*\boldsymbol{g}$ denote the metric induced on $\mathscr{I}^+$ by $\boldsymbol{g}$. *The metric $\boldsymbol{q}$ is degenerate.* To see this, write $\mathbf{d}\Xi$ in coordinates adapted to $\mathscr{I}^+$; it follows that $\varphi^*(\mathbf{d}\Xi) = 0$ so that $\varphi^*(\boldsymbol{N}^\flat) = 0$. Thus, from $\boldsymbol{N}^\flat = \boldsymbol{g}(\boldsymbol{N}, \cdot)$ one concludes that $\boldsymbol{q}(\boldsymbol{N}, \cdot) = 0$ as claimed – observe that as $\boldsymbol{N}$ is tangent to $\mathscr{I}^+$, it follows that it has a well-defined pull-back.

To analyse the behaviour of the metric $\boldsymbol{q}$ along the generators of $\mathscr{I}^+$ consider the Lie derivative $\pounds_{\boldsymbol{N}}\boldsymbol{q}$. To compute it start from

$$
\begin{aligned}
\pounds_{\boldsymbol{N}} g_{ab} &= N^c \nabla_c g_{ab} + \nabla^c N_a g_{cb} + \nabla^c N_b g_{ac} \\
&= \nabla_b N_a + \nabla_a N_b = 2\nabla_a N_b,
\end{aligned}
$$

as $\nabla_a \nabla_b \Xi = -\nabla_a N_b = -\nabla_b N_a$. Hence, using Equation (10.3) it follows that

$$
\pounds_{\boldsymbol{N}}\boldsymbol{q} = -s\boldsymbol{q}. \tag{10.5}
$$

The trace-free part of $\pounds_{\boldsymbol{N}}\boldsymbol{q}$ is called the ***shear tensor*** $\boldsymbol{\varsigma}$ of the congruence of generators of $\mathscr{I}^+$ – it describes the tendency of a sphere of points in the congruence to be deformed into an ellipsoid with the same volume. As Equation (10.5) is pure trace, it follows that $\boldsymbol{\varsigma} = 0$. Thus, *the congruence of generators of $\mathscr{I}^+$ is shear free.* This result is a consequence of the conformal field equations via Equation (10.3) so that from the conformal invariance of the equations it follows

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that the shear-freeness of the congruence of generators is a property independent of the particular choice of conformal factor.

The conformal gauge freedom inherent in the construction of a conformal extension can be exploited to gain further insight into the structure of null infinity. Given a conformal extension $(\mathcal{M}, \boldsymbol{g}, \Xi)$ consider $\vartheta > 0$ and define a conformally related metric $\boldsymbol{g}'$ via $\boldsymbol{g}' = \vartheta^2 \boldsymbol{g}$. The transformation rule of the Friedrich scalar s – see Equation (8.29b) – yields that

$$s' \simeq \vartheta^{-1} s - \vartheta^{-2} N^c \nabla_c \vartheta.$$

Thus, if initially $s \neq 0$, one can always find a further conformal representation $(\mathcal{M}, \boldsymbol{g}', \Xi')$ for which $s' = 0$ if one imposes the condition

$$N^c \nabla_c \vartheta = \vartheta s. \tag{10.6}$$

Notice that the above equation can be rewritten as $\pounds_{\boldsymbol{N}} \vartheta = \vartheta s$, and, accordingly, it can be read as an ordinary differential equation along the generators of null infinity. It is important to observe that once condition (10.6) has been imposed, one is still left with the freedom of specifying a further rescaling $\boldsymbol{g}'' = \varkappa^2 \boldsymbol{g}'$ such that $\pounds_{\boldsymbol{N}'} \varkappa = 0$.

The conformal gauge implied by condition (10.6) yields, together with Equation (10.5), that

$$\pounds_{\boldsymbol{N}'} \boldsymbol{q}' = 0; \tag{10.7}$$

that is, the intrinsic metric of $\mathscr{I}^+$ is *Lie dragged* along the generators of null infinity. Each of the cuts $\mathscr{C}$ of null infinity inherits from the metric $\boldsymbol{q}$ on $\mathscr{I}^+$ a metric $\boldsymbol{k}$ which is *non-degenerate*. As a consequence of Equation (10.7), if one considers any other cut $\mathscr{C}'$, one obtains the same induced metric $\boldsymbol{k}$. Now, any metric on a two-dimensional surface which is topologically $\mathbb{S}^2$ is conformal to the standard metric of $\mathbb{S}^2$, $\boldsymbol{\sigma}$ – this fact is a consequence of the so-called ***Riemann mapping theorem***; see, for example, Krantz (2006), chapter 4. Hence, one can write $\boldsymbol{k} = \theta^2 \boldsymbol{\sigma}$ for some conformal factor $\theta > 0$ on $\mathbb{S}^2$. Under a further conformal gauge transformation $\boldsymbol{g}'' = \varkappa^2 \boldsymbol{g}'$ such that $\pounds_{\boldsymbol{N}'} \varkappa = 0$ (see the previous paragraph), one can then always assume that the gauge has been chosen so that $\boldsymbol{k} = \boldsymbol{\sigma}$. Under these circumstances the conformal gauge freedom is reduced to a function $\varkappa$ such that $\varkappa \simeq 1$.

10.2 Peeling properties

One of the most important results of the theory of asymptotics of the gravitational field is the so-called ***Peeling theorem*** – a precise prescription of the decay of the Weyl tensor of an asymptotically simple spacetime. The Peeling theorem is based on the important observation that the Weyl tensor of an asymptotically simple spacetime must vanish on $\mathscr{I}$. As will be seen in the

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following, this observation follows in a quite straight forward manner if $\lambda \neq 0$. A more subtle argument is required if $\lambda = 0$.

In what follows, let Ψ_{ABCD} denote the Weyl spinor, and recall that $\Psi_{ABCD} = \Xi\phi_{ABCD}$. The subsequent analysis is best carried out with the spinorial conformal Einstein field equations expressed with respect to a spin dyad $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$; see Section 8.3.2. *In this formulation of the field equations the fields are scalars. Hence, they can readily be evaluated at the conformal boundary without the need of pull-backs.* One has the following:

Theorem 10.3 (***vanishing of the Weyl tensor at*** $\mathscr{I}$) *Assume that* $\Psi_{\boldsymbol{ABCD}}$ *is smooth at* $\mathscr{I}$*. If* $\lambda \neq 0$ *and the physical Cotton tensor satisfies* $\tilde{Y}_{\boldsymbol{abc}} = o(\Xi^{-1})$ *at* $\mathscr{I}$*, then* $\Psi_{\boldsymbol{ABCD}} = 0$ *at* $\mathscr{I}$*. If* $\lambda = 0$*, the same conclusion follows if* $\tilde{Y}_{\boldsymbol{abc}} = o(\Xi^{-1})$ *and* $\nabla_{\boldsymbol{d}}\tilde{Y}_{\boldsymbol{abc}} = o(\Xi^{-1})$.

Proof (case $\lambda \neq 0$) The starting point of the analysis is the Bianchi equation

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\phi_{\boldsymbol{ABCQ}} + T_{\boldsymbol{BCAA}'} = 0;$$

compare the spinorial conformal Einstein Equation (8.37b). Now, recalling that $\phi_{\boldsymbol{ABCD}} = \Xi^{-1}\Psi_{\boldsymbol{ABCD}}$ and $T_{\boldsymbol{BCAA}'} = \Xi^{-1}\tilde{Y}_{\boldsymbol{BCAA}'}$ it follows that

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\Xi\Psi_{\boldsymbol{ABCQ}} - \Xi\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\Psi_{\boldsymbol{ABCQ}} = \Xi\tilde{Y}_{\boldsymbol{BCAA}'}. \tag{10.8}$$

Hence, using $\tilde{Y}_{\boldsymbol{abc}} = o(\Xi^{-1})$ one finds that $\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\Xi\ \Psi_{\boldsymbol{ABCQ}} \simeq 0$. Contracting with $\nabla_{\boldsymbol{DA}'}\Xi$ one obtains

$$\nabla_{\boldsymbol{DA}'}\Xi\nabla_{\boldsymbol{Q}}{}^{\boldsymbol{A}'}\Xi\ \Psi^{\boldsymbol{Q}}{}_{\boldsymbol{ABC}} \simeq 0. \tag{10.9}$$

Now, using Equation (10.2) one has

$$\nabla_{\boldsymbol{DA}'}\Xi\nabla_{\boldsymbol{Q}}{}^{\boldsymbol{A}'}\Xi = \frac{1}{2}\nabla_{\boldsymbol{PP}'}\Xi\nabla^{\boldsymbol{PP}'}\Xi\ \epsilon_{\boldsymbol{DQ}} \simeq -\frac{3}{2}\lambda\ \epsilon_{\boldsymbol{DQ}}.$$

Substituting the latter in (10.9) one finds that $\lambda\Psi_{\boldsymbol{ABCD}}\simeq 0$. Hence, $\Psi_{\boldsymbol{ABCD}} = 0$ on $\mathscr{I}$. □

Proof (case $\lambda = 0$) Again, one has that

$$\nabla^{\boldsymbol{AA}'}\Xi\ \Psi_{\boldsymbol{ABCD}} \simeq 0. \tag{10.10}$$

In this case, however, $\nabla^{\boldsymbol{AA}'}\Xi$ is the spinorial counterpart of a null vector. Hence, there exists a spinor ι^{A} such that

$$\nabla^{\boldsymbol{AA}'}\Xi = \iota^{\boldsymbol{A}}\bar{\iota}^{\boldsymbol{A}'}. \tag{10.11}$$

It follows from Equation (10.10) that there exists a scalar field ψ such that

$$\Psi_{\boldsymbol{ABCD}} \simeq \psi\iota_{\boldsymbol{A}}\iota_{\boldsymbol{B}}\iota_{\boldsymbol{C}}\iota_{\boldsymbol{D}}. \tag{10.12}$$

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In order to extract further information consider Equation (10.8) – which is also valid in the case $\lambda = 0$ – and apply $\nabla_{\boldsymbol{EE'}}$ to both sides. The assumptions on $\tilde{Y}_{\boldsymbol{CDBB'}}$ imply that

$$\nabla_{\boldsymbol{EE'}}\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\Xi\Psi_{\boldsymbol{ABCQ}} + \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\Xi\nabla_{\boldsymbol{EE'}}\Psi_{\boldsymbol{ABCQ}} - \nabla_{\boldsymbol{EE'}}\Xi\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\Psi_{\boldsymbol{ABCQ}} \simeq 0.$$

Symmetrising on $_{\boldsymbol{E'}}$ and $_{\boldsymbol{B'}}$, and using the asymptotic Einstein condition (10.4) one concludes that

$$\nabla^{\boldsymbol{Q}}{}_{(\boldsymbol{B'}}\Xi\nabla_{\boldsymbol{E'})\boldsymbol{E}}\Psi_{\boldsymbol{ABCQ}} - \nabla_{\boldsymbol{E}(\boldsymbol{E'}}\Xi\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'})}\Psi_{\boldsymbol{ABCQ}} \simeq 0. \tag{10.13}$$

Now, using identity (3.6) to interchange the indices $_{\boldsymbol{E}}$ and $_{\boldsymbol{Q}}$ one obtains

$$\begin{aligned}\nabla^{\boldsymbol{Q}}{}_{(\boldsymbol{B'}}\Xi\nabla_{\boldsymbol{E'})\boldsymbol{Q}}\Psi_{\boldsymbol{ABCE}} - \nabla_{\boldsymbol{Q}(\boldsymbol{E'}}\Xi\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'})}\Psi_{\boldsymbol{ABCE}}&\\ - \epsilon_{\boldsymbol{EQ}}\epsilon^{\boldsymbol{ST}}\big(\nabla^{\boldsymbol{Q}}{}_{(\boldsymbol{B'}}\Xi\nabla_{\boldsymbol{E'})\boldsymbol{S}}\Psi_{\boldsymbol{ABCT}} + \nabla_{\boldsymbol{S}(\boldsymbol{E'}}\Xi\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'})}\Psi_{\boldsymbol{ABCT}}\big) &\simeq 0,\end{aligned}$$

which in view of Equation (10.13) reduces to

$$\nabla^{\boldsymbol{Q}}{}_{(\boldsymbol{B'}}\Xi\nabla_{\boldsymbol{E'})\boldsymbol{Q}}\Psi_{\boldsymbol{ABCE}} \simeq 0.$$

Using the decomposition (10.11) in this last equation one obtains

$$\iota^{\boldsymbol{Q}}\bar{\iota}_{(\boldsymbol{B'}}\nabla_{\boldsymbol{E'})\boldsymbol{Q}}\Psi_{\boldsymbol{ABCE}} \simeq 0.$$

Contracting the latter with $\bar{\iota}^{\boldsymbol{B'}}$ and observing that $\bar{\iota}_{\boldsymbol{E'}} \neq 0$, one concludes that

$$\iota^{\boldsymbol{Q}}\bar{\iota}^{\boldsymbol{B'}}\nabla_{\boldsymbol{QB'}}\Psi_{\boldsymbol{ABCE}} \simeq 0. \tag{10.14}$$

Thus,

$$\iota^{\boldsymbol{Q}}\nabla_{\boldsymbol{QE'}}\Psi_{\boldsymbol{ABCE}} \simeq \alpha\,\bar{\iota}_{\boldsymbol{E'}}\zeta_{\boldsymbol{ABCE}}$$

for some scalar α and a spinor $\zeta_{\boldsymbol{ABCE}} \neq 0$. Substituting back into (10.14) one concludes that $\alpha = 0$ so that one has

$$\iota^{\boldsymbol{Q}}\nabla_{\boldsymbol{QE'}}\Psi_{\boldsymbol{ABCE}} \simeq 0. \tag{10.15}$$

In order to bring this last result into a more convenient form one completes the spinor ι^A to a spin basis $\{\epsilon_{\boldsymbol{A}}{}^A\} = \{o^A, \iota^A\}$ with $o_A\iota^A = 1$ so that $\iota^{\boldsymbol{A}} = \delta_{\boldsymbol{1}}{}^{\boldsymbol{A}}$ and $o^{\boldsymbol{A}} = \delta_{\boldsymbol{0}}{}^{\boldsymbol{A}}$. Thus, contracting Equation (10.15) with $\bar{o}^{\boldsymbol{E'}}$ and substituting (10.12) into Equation (10.15) one obtains

$$\iota^{\boldsymbol{Q}}\bar{o}^{\boldsymbol{E'}}\nabla_{\boldsymbol{QE'}}(\psi\iota_{\boldsymbol{A}}\iota_{\boldsymbol{B}}\iota_{\boldsymbol{C}}\iota_{\boldsymbol{E}}) = \nabla_{\boldsymbol{10'}}(\psi\iota_{\boldsymbol{A}}\iota_{\boldsymbol{B}}\iota_{\boldsymbol{C}}\iota_{\boldsymbol{E}}) \simeq 0. \tag{10.16}$$

The above expression is to be regarded as a differential equation for ψ over the cuts of $\mathscr{I}^+$. To conclude the argument one makes use of the ***formalism of the*** $\eth$ ***and*** $\bar{\eth}$ ***operators*** as discussed in the Appendix to this chapter. Accordingly, in what follows it is assumed that one has a conformal representation for which the cuts are metric unit spheres $\mathbb{S}^2$. Contracting (10.16) with $o^{\boldsymbol{A}}o^{\boldsymbol{B}}o^{\boldsymbol{C}}o^{\boldsymbol{E}}$ one obtains

$$\bar{\eth}\psi \simeq 0.$$

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Now, from $\psi = \Psi_{\boldsymbol{ABCD}} o^{\boldsymbol{A}} o^{\boldsymbol{B}} o^{\boldsymbol{C}} o^{\boldsymbol{D}}$ it follows that ψ has spin-weight 2. Hence, using Lemma 10.1 in the Appendix to this chapter it follows that $\psi \simeq 0$ and thus $\Psi_{\boldsymbol{ABCD}}$ vanishes at $\mathscr{I}$. □

Remark. The above result strongly depends on the fact that for an asymptotically simple spacetime with $\lambda = 0$ one has that $\mathscr{I} \approx \mathbb{R} \times \mathbb{S}^2$. For the spacetimes with *toroidal* null infinities considered in Schmidt (1996), the crucial Lemma 10.1 does not hold – see Frauendiener and Szabados (2001) – and the desired conclusion cannot be obtained.

A more detailed description

To obtain a more detailed description of the *peeling behaviour*, it is necessary to introduce further structure. In what follows, consider a null geodesic γ in $(\mathcal{M}, \boldsymbol{g})$ reaching $\mathscr{I}$ at a point p and let $\tilde{\gamma}$ denote the corresponding null geodesic on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. At a point $q \in \tilde{\gamma}$ one can choose a spin dyad $\{\tilde{\boldsymbol{o}}, \tilde{\boldsymbol{\iota}}\}$ such that the tangent to $\tilde{\gamma}$ is given by the vector $\tilde{\boldsymbol{l}}$ with spinorial counterpart $\tilde{l}^{AA'} = \tilde{o}^A \bar{\tilde{o}}^{A'}$. The spin dyad can be naturally propagated along $\tilde{\gamma}$ by requiring

$$\tilde{D}\tilde{o}^A = 0, \qquad \tilde{D}\tilde{\iota}^A = 0, \tag{10.17}$$

where $\tilde{D} \equiv \tilde{l}^a \tilde{\nabla}_a = \tilde{o}^A \bar{\tilde{o}}^{A'} \tilde{\nabla}_{AA'}$ in standard Newman-Penrose (NP) notation. Now, let $\tilde{r}$ denote an affine parameter along $\tilde{\gamma}$. It follows that $\tilde{D} = \mathrm{d}/\mathrm{d}\tilde{r}$. In order to rewrite the above expressions in terms of quantities defined on the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ it is convenient to consider the transformation

$$o_A = \tilde{o}_A, \qquad o^A = \Xi^{-1}\tilde{o}^A, \qquad \iota_A = \Xi\tilde{\iota}_A, \qquad \iota^A = \tilde{\iota}^A; \tag{10.18}$$

compare Equations (5.31a)–(5.31c) in Chapter 5. Using the transformation laws under conformal transformations for the covariant derivatives it follows from (10.17) that

$$Do^A = 0, \qquad D\iota^A = \big(\Xi^{-1}\bar{\delta}\Xi\big)o^A,$$

where $\bar{\delta} \equiv \bar{m}^a \nabla_a = \iota^A \bar{o}^{A'} \nabla_{AA'}$. The second of the above expressions is potentially singular at $\mathscr{I}$ – observe, however, that as $\Xi \simeq 0$, it follows that $\bar{\delta}\Xi \simeq 0$ as $\bar{\boldsymbol{m}}$ is intrinsic to $\mathscr{I}$. Thus, the spin dyad $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ is well defined and regular at $\mathscr{I}$. Now, from $Do^A = 0$ it follows that the null geodesic γ is affinely parametrised. Let r denote a possible affine parameter. Its origin and scaling can be chosen so that

$$r = 0, \qquad \text{and} \qquad D\Xi = \frac{\mathrm{d}\Xi}{\mathrm{d}r} = -1 \qquad \text{at} \quad p \in \mathscr{I}.$$

From Remark (c) in Section 7.1 it follows that

$$\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r} = \frac{1}{\Xi^2}$$

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where $\tilde{r}$ is an affine parameter in the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Hence, one concludes that

$$\tilde{r} = O(\Xi^{-1}) \qquad \text{near } \mathscr{I}. \tag{10.19}$$

Making use of the above relations one obtains the following, more detailed, version of the peeling behaviour:

Theorem 10.4 (***Peeling theorem***) *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote an asymptotically simple spacetime with $\lambda = 0$ for which the hypotheses of Theorem 10.3 hold. Moreover, let*

$$\tilde{\psi}_0 \equiv \Psi_{ABCD}\tilde{o}^A\tilde{o}^B\tilde{o}^C\tilde{o}^D, \quad \tilde{\psi}_1 \equiv \Psi_{ABCD}\tilde{\iota}^A\tilde{o}^B\tilde{o}^C\tilde{o}^D, \quad \tilde{\psi}_2 \equiv \Psi_{ABCD}\tilde{\iota}^A\tilde{\iota}^B\tilde{o}^C\tilde{o}^D,$$
$$\tilde{\psi}_3 \equiv \Psi_{ABCD}\tilde{\iota}^A\tilde{\iota}^B\tilde{\iota}^C\tilde{o}^D, \qquad \tilde{\psi}_4 \equiv \Psi_{ABCD}\tilde{\iota}^A\tilde{\iota}^B\tilde{\iota}^C\tilde{\iota}^D,$$

then

$$\tilde{\psi}_0 = O(\tilde{r}^{-5}), \qquad \tilde{\psi}_1 = O(\tilde{r}^{-4}), \qquad \tilde{\psi}_2 = O(\tilde{r}^{-3})$$
$$\tilde{\psi}_3 = O(\tilde{r}^{-2}), \qquad \tilde{\psi}_4 = O(\tilde{r}^{-1}).$$

Proof Let

$$\psi_0 \equiv \Psi_{ABCD}o^Ao^Bo^Co^D, \quad \ldots \quad \psi_4 \equiv \Psi_{ABCD}\iota^A\iota^B\iota^C\iota^D.$$

It follows from Theorem 10.3 that $\psi_k = O(\Xi)$. Now, using the transformation rules (10.18) one has that

$$\tilde{\psi}_k = \Xi^{4-k}\psi_k.$$

Thus, recalling (10.19), one finds the desired result. □

Combining the definitions of the fields $\tilde{\psi}_k$ with the corresponding decays given by Theorem 10.4 one obtains a detailed expression for the asymptotic behaviour of the Weyl spinor. It can be written schematically as

$$\Psi_{ABCD} = \frac{[N]_{ABCD}}{\tilde{r}} + \frac{[III]_{ABCD}}{\tilde{r}^2} + \frac{[II]_{ABCD}}{\tilde{r}^3} + \frac{[I]_{ABCD}}{\tilde{r}^4} + O(\tilde{r}^{-5}), \tag{10.20}$$

where $[N]_{ABCD}$, $[III]_{ABCD}$, $[II]_{ABCD}$ and $[I]_{ABCD}$ represent, respectively, totally symmetric spinors of ***Petrov type*** N, III, II and I; for a concise discussion of the ***Petrov classification of the Weyl tensor*** using spinors, see Stewart (1991). For Petrov type N Weyl tensors the spinor Ψ_{ABCD} has four repeated principal null directions. They are associated to gravitational plane waves. Similarly, a spacetime with a Weyl spinor of Petrov type III has three repeated principal null directions; one of Petrov type II has two principal directions, while one of Petrov type I is algebraically general. The observation that a repeated principal null direction is lost at each order in the expansion (10.20) justifies the name of *peeling* in analogy to the peeling of a fruit; see Figure 10.1.

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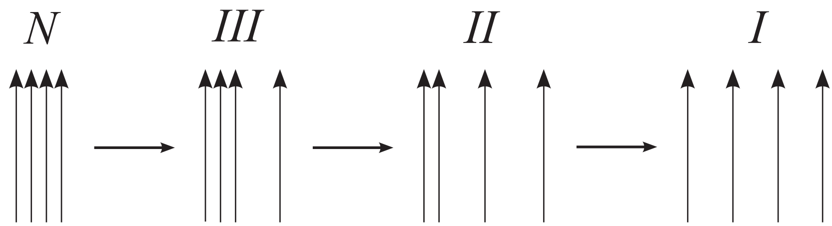

Figure 10.1 Schematic representation of the Peeling theorem: the leading behaviour of the Weyl tensor corresponds to that of a plane wave (Petrov type N). More general behaviour is observed as one looks into higher order terms.

Remark. The key assumption in the derivation of the peeling behaviour is the smoothness of the Weyl tensor at $\mathscr{I}^+$. A careful inspection of the arguments in the previous sections shows that the smoothness requirement can be relaxed and that the conclusions of Theorems 10.3 and 10.4 can be recovered if it is assumed that Ψ_{ABCD} is of class C^{k_*} at $\mathscr{I}^+$ for some positive integer k_*. A determination of a *sharp value* of k_* will not be pursued here. One of the challenges in the construction of spacetimes satisfying the peeling behaviour or, more generally, spacetimes which are asymptotically simple is to ensure that their Weyl tensor has the required regularity at the conformal boundary. The latter will be a recurrent idea in the remainder of this book. The analysis of the *non-linear stability of the Minkowski spacetime* in Christodoulou and Klainerman (1993) renders a Weyl tensor with a limited regularity at $\mathscr{I}^+$ for which only a *partial peeling behaviour* of the form

$$\tilde{\psi}_0 = O(\tilde{r}^{-1}), \qquad \tilde{\psi}_1 = O(\tilde{r}^{-2}), \qquad \tilde{\psi}_2 = O(\tilde{r}^{-3}),$$
$$\tilde{\psi}_3 = O(\tilde{r}^{-7/2}), \qquad \tilde{\psi}_4 = O(\tilde{r}^{-7/2}),$$

can be recovered; see, for example, Friedrich (1992) for a discussion.

10.3 The Newman-Penrose gauge

The analysis leading to the Peeling theorem shows the advantages of using a gauge which is adapted to the geometry of null infinity. In this section this idea is further elaborated. The resulting ***Newman-Penrose gauge*** allows one to obtain further insights into the properties of asymptotically simple spacetimes.

10.3.1 The construction of the gauge

As in the previous section let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ denote a conformal extension of an asymptotically simple spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ with $\lambda = 0$. For conciseness, the subsequent discussion will be restricted to future null infinity $\mathscr{I}^+$. An analogous discussion can be readily adapted for $\mathscr{I}^-$.

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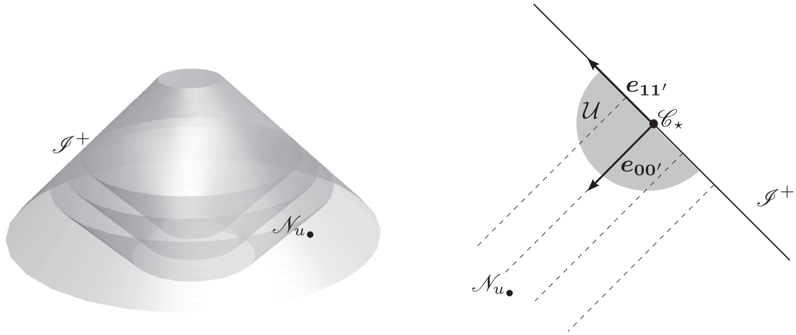

Figure 10.2 Schematic representation of the setting for the construction of the NP gauge. The NP gauge is based on a fiduciary cut $\mathscr{C}_\star$ on $\mathscr{I}^+$ and is valid in a neighbourhood $\mathcal{U}$ of the conformal boundary. The vector $\boldsymbol{e}_{\mathbf{11'}}$ is tangent to the generators of null infinity, while $\boldsymbol{e}_{\mathbf{00'}}$ generates the outgoing null hypersurfaces $\mathscr{N}_{u_\bullet}$. See the main text for further details.

In what follows, let $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ be a frame satisfying $\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{AA'}}, \boldsymbol{e}_{\boldsymbol{BB'}}) = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}$ defined in a neighbourhood $\mathcal{U}$ of $\mathscr{I}^+$. The frame will be said to be ***adapted to*** $\mathscr{I}^+$ if – see Figure 10.2:

(i) The vector $\boldsymbol{e}_{\mathbf{11'}}$ is tangent to $\mathscr{I}^+$ and is parallely propagated along its generators; that is, one has

$$\nabla_{\mathbf{11'}}\boldsymbol{e}_{\mathbf{11'}} \simeq 0.$$

(ii) On $\mathcal{U}$ there exists a function u (a ***retarded time***) which can be regarded as an affine parameter of the generators of $\mathscr{I}^+$ such that $\boldsymbol{e}_{\mathbf{11'}}(u) \simeq 1$. The retarded time is constant on null hypersurfaces transverse to $\mathscr{I}^+$ and satisfies $\boldsymbol{e}_{\mathbf{00}} = \boldsymbol{g}^\sharp(\mathbf{d}u, \cdot)$. It follows that $\boldsymbol{e}_{\mathbf{00}}$ is tangent to the hypersurfaces

$$\mathscr{N}_{u_\bullet} = \{p \in \mathcal{U} \mid u(p) = u_\bullet\},$$

where $u_\bullet$ is a constant. Moreover, $\boldsymbol{e}_{\mathbf{00'}}$ is tangent to the null generators of $\mathscr{N}_{u_\bullet}$.

(iii) The fields $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ are tangent to the cuts $\mathscr{C}_{u_\bullet} \equiv \mathscr{N}_{u_\bullet} \cap \mathscr{I}^+$ and parallely propagated along the direction of $\boldsymbol{e}_{\mathbf{00'}}$. That is, one has

$$\nabla_{\mathbf{00'}}\boldsymbol{e}_{\boldsymbol{AA'}} = 0 \qquad \text{on} \qquad \mathscr{N}_{u_\bullet}.$$

Using the definition of the spin-connection coefficients it follows from the above requirements that

$$\Gamma_{\mathbf{10'11}} \simeq 0, \qquad \Gamma_{\mathbf{11'11}} \simeq 0, \tag{10.21a}$$

$$\Gamma_{\mathbf{10'00}} = \bar{\Gamma}_{\mathbf{1'00'0'}}, \qquad \Gamma_{\mathbf{11'00}} = \bar{\Gamma}_{\mathbf{1'00'1'}} + \Gamma_{\mathbf{01'01}} \qquad \text{on } \mathcal{U}, \tag{10.21b}$$

$$\Gamma_{\mathbf{00'}\boldsymbol{AB}} = 0 \qquad \text{on } \mathcal{U}. \tag{10.21c}$$

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The condition $\Gamma_{\boldsymbol{10'11}} \simeq 0$ is, in fact, another way of expressing the fact that the congruence of null generators of $\mathscr{I}^+$ is shear free. This can be seen by evaluating the conformal field Equation (8.35c)

$$\nabla_{\boldsymbol{AA'}}\nabla_{\boldsymbol{BB'}}\Xi = -\Xi L_{\boldsymbol{AA'BB'}} + s\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}} \tag{10.22}$$

at $\mathscr{I}^+$ for ${}_{\boldsymbol{AA'}\ \boldsymbol{BB'}} = {}_{\boldsymbol{10'10'}}$. It follows that $\Gamma_{\boldsymbol{10'11}}\boldsymbol{e_{00'}}(\Xi) \simeq 0$, but $\boldsymbol{e_{00'}}(\Xi) \not\simeq 0$ so that one concludes $\Gamma_{\boldsymbol{10'11}} \simeq 0$ as claimed.

Remark. The discussion of the previous sections shows that an adapted frame can always be obtained in a neighbourhood $\mathcal{U}$ of $\mathscr{I}^+$. The key observation is that $\boldsymbol{N} = \boldsymbol{g}^\sharp(\mathbf{d}\Xi, \cdot)$ is tangent to the null generators of $\mathscr{I}^+$ so that one can set $\boldsymbol{e_{11'}}$ proportional to $\boldsymbol{N}$. A suitable choice of affine parameter for $\boldsymbol{N}$ renders the retarded time u and hence the frame vector $\boldsymbol{e_{00'}}$. The rest of the frame is then naturally completed by looking at a basis on the tangent bundle of the cuts $\mathscr{C}_{u_\bullet}$.

Following the ideas of Section 10.1.2, the gauge can be further specialised by considering a suitable conformal rescaling. Accordingly, consider

$$\boldsymbol{g} \mapsto \boldsymbol{g}' = \vartheta^2\boldsymbol{g}, \qquad \Xi \mapsto \Xi' = \vartheta\Xi. \tag{10.23}$$

The above rescaling will be used to obtain an *improved adapted frame* $\{\boldsymbol{e}'_{\boldsymbol{AA'}}\}$. For an arbitrary conformal factor $\vartheta > 0$ and an arbitrary function $\varkappa > 0$ which is constant along the generators of $\mathscr{I}^+$ set

$$\boldsymbol{e}'_{\boldsymbol{11'}} \simeq \vartheta^{-2}\varkappa\boldsymbol{e_{11'}}; \tag{10.24}$$

compare the discussion in Section 10.1.2. In addition, define a further parameter $u' = u'(u)$ such that $\mathrm{d}u'/\mathrm{d}u = \varkappa^{-1}\vartheta^2$. Integrating along the generators of null infinity one finds that

$$u' = \frac{1}{\varkappa}\int_{u_\star}^{u} \vartheta^2(\mathrm{s})\mathrm{ds} + u'_\star.$$

The real constants $u_\star$ and $u'_\star$ are fixed so that they identify a certain *fiduciary cut* $\mathscr{C}_\star \equiv \mathscr{C}_{u_\star}$. In what follows, for convenience, *the symbol* $\overset{\star}{\simeq}$ *is used to denote equality at* $\mathscr{C}_\star$. It can be verified that $\boldsymbol{e}'_{\boldsymbol{11'}}$ is parallely propagated and that $\boldsymbol{e}'_{\boldsymbol{11'}}(u') = 1$. The transformation rule (10.24) is supplemented at $\mathscr{C}_\star$ by

$$\boldsymbol{e}'_{\boldsymbol{00'}} \overset{\star}{\simeq} \varkappa^{-1}\boldsymbol{e_{00'}}, \qquad \boldsymbol{e}'_{\boldsymbol{01'}} \overset{\star}{\simeq} \vartheta^{-1}\boldsymbol{e_{01'}}. \tag{10.25}$$

It can be verified that $\boldsymbol{g}'(\boldsymbol{e}'_{\boldsymbol{AA'}}, \boldsymbol{e}'_{\boldsymbol{BB'}}) = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}$ on $\mathscr{C}_\star$.

As seen in Section 10.1.2, $\mathscr{C}_\star \approx \mathbb{S}^2$ so that the metric $\boldsymbol{k}_\star$ induced by $\boldsymbol{g}'$ on $\mathscr{C}_\star$ is conformal to the standard metric $\boldsymbol{\sigma}$ of $\mathbb{S}^2$. Accordingly, the conformal factor ϑ can be chosen on $\mathscr{C}_\star$ so that $\boldsymbol{k}_\star \overset{\star}{\simeq} \boldsymbol{\sigma}$. A calculation using the transformation laws of Chapter 5 shows that the rescaling (10.23) and the conditions (10.24) and (10.25) imply on $\mathscr{C}_\star$

$$\Gamma'_{\boldsymbol{10'00}} = \varkappa^{-1}\big(\Gamma_{\boldsymbol{10'00}} - \vartheta^{-1}\boldsymbol{e_{00'}}(\vartheta)\big), \tag{10.26a}$$

$$\Gamma'_{\boldsymbol{01'11}} = \varkappa\vartheta^{-2}\big(\Gamma_{\boldsymbol{01'11}} + \vartheta^{-1}\boldsymbol{e_{11'}}(\vartheta)\big). \tag{10.26b}$$

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Hence, by a suitable choice of $\mathbf{d}\vartheta$ and $\varkappa$ it is possible to ensure that

$$\Gamma'_{\mathbf{1}\mathbf{0}'\mathbf{0}\mathbf{0}} \overset{\star}{\simeq} 0, \quad \Gamma'_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}} \overset{\star}{\simeq} 0, \qquad \boldsymbol{e}'_{\mathbf{0}\mathbf{0}'}(\Xi') \overset{\star}{\simeq} \text{constant} \neq 0. \tag{10.27}$$

A convenient way of prescribing the conformal factor ϑ off $\mathscr{C}_\star$ follows from the transformation law for the trace-free part of the Ricci tensor Φ_{ab} under the rescaling (10.23):

$$\Phi'_{ab} - \Phi_{ab} = -\,2\vartheta^{-1}\Big((\nabla_a\nabla_b\vartheta - 2\vartheta^{-1}\nabla_a\vartheta\nabla_b\vartheta) - \frac{1}{4}g_{ab}(\nabla_c\nabla^c)\vartheta - 2\vartheta^{-1}\nabla_c\vartheta\nabla^c\vartheta)\Big);$$

see Equation (5.6a). Transvecting this last equation with $\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}\otimes\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}$ it follows that if ϑ satisfies the equation

$$\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta)) - 2\vartheta^{-1}\big(\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta)\big)^2 \simeq \vartheta\Phi_{22}, \tag{10.28}$$

then $\Phi'_{22} \simeq 0$. By means of the substitution $z = \vartheta^{-1}$, Equation (10.28) can be read as a second-order linear ordinary differential equation for ϑ^{-1} along the generators of $\mathscr{I}^+$. Thus, this equation can always be solved, at least in a neighbourhood of $\mathscr{C}_\star$ on $\mathscr{I}^+$ to ensure that

$$\Phi'_{22} \simeq 0. \tag{10.29}$$

This last construction also fixes the value of $e'_{\mathbf{0}\mathbf{1}'}(\vartheta)$ on $\mathscr{I}^+$.

The initial data for Equation (10.28) on the fiduciary cut $\mathscr{C}_\star$ is chosen so that $\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta) \overset{\star}{\simeq} -\Gamma_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}}$ consistent with Equation (10.27); compare Equation (10.26b).

Now, taking into account Equations (10.21a) and (10.29), one has that the Ricci identity – compare the conformal field Equation (8.35b) of Chapter 8 – gives for the values $\boldsymbol{AA'} = \mathbf{1}\mathbf{1}'$, $\boldsymbol{BB'} = \mathbf{0}\mathbf{1}'$ and $\boldsymbol{CD} = \mathbf{1}\mathbf{1}$ that

$$\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\Gamma'_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}}) + \big(\Gamma'_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}}\big)^2 + \Gamma'_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}}\bar{\Gamma}'_{\mathbf{1}'\mathbf{1}\mathbf{0}'\mathbf{1}'} = 0.$$

The latter equation can be interpreted as a homogeneous differential equation along the generators of $\mathscr{I}^+$ for the reduced spin connection coefficient $\Gamma'_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}}$. As a consequence of the initial condition (10.27) on $\mathscr{C}_\star$, it follows that $\Gamma'_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}} \simeq 0$.

The construction described in the previous paragraphs provides a specification of the conformal factor ϑ and of the function $\varkappa$ which fixes the frame vector $\boldsymbol{e}'_{\mathbf{1}\mathbf{1}'}$ completely on $\mathscr{I}^+$. Notice, however, that the vectors $\boldsymbol{e}'_{\mathbf{0}\mathbf{1}'}$ and $\boldsymbol{e}'_{\mathbf{1}\mathbf{0}'}$ (tangent to the cuts $\mathscr{C}_{u_\bullet}$) are determined up to a rotation of the form

$$\boldsymbol{e}'_{\mathbf{0}\mathbf{1}'} \mapsto e^{\mathrm{i}c}\boldsymbol{e}'_{\mathbf{0}\mathbf{1}'}, \qquad \boldsymbol{e}'_{\mathbf{1}\mathbf{0}'} \mapsto e^{-\mathrm{i}c}\boldsymbol{e}'_{\mathbf{1}\mathbf{0}'}, \tag{10.30}$$

with c a *real phase* on $\mathscr{I}^+$. A rotation on $T(\mathscr{C}_{u_\bullet})$ can be exploited to obtain additional simplifications in the spin connection coefficients. A calculation using the definition of the spin connection coefficients and taking into account that $\nabla_{\mathbf{1}\mathbf{1}'}\boldsymbol{e}'_{\mathbf{1}\mathbf{1}} \simeq 0$ gives that

$$\Gamma'_{\mathbf{1}\mathbf{1}'\mathbf{0}\mathbf{1}} \simeq -\frac{1}{2}\langle\boldsymbol{\omega}'^{\mathbf{1}\mathbf{0}}, \nabla'_{\mathbf{1}\mathbf{1}'}\boldsymbol{e}'_{\mathbf{1}\mathbf{0}'}\rangle.$$

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Under the rotation (10.30) the above relation transforms as

$$\Gamma'_{\mathbf{1}\mathbf{1}'\mathbf{0}\mathbf{1}} \mapsto \frac{\mathrm{i}}{2}\boldsymbol{e}'_{\mathbf{1}\mathbf{1}'}(c) - \frac{1}{2}\Gamma'_{\mathbf{1}\mathbf{1}'\mathbf{0}\mathbf{1}}, \qquad \text{on } \mathscr{I}^+.$$

Thus, given a particular choice of vectors $\boldsymbol{e}'_{\mathbf{0}\mathbf{1}'}$ and $\boldsymbol{e}'_{\mathbf{1}\mathbf{0}'}$ on $\mathscr{I}^+$, by solving the equation

$$\boldsymbol{e}'_{\mathbf{1}\mathbf{1}'}(c) \simeq -\frac{\mathrm{i}}{2}\Gamma'_{\mathbf{1}\mathbf{1}'\mathbf{0}\mathbf{1}}, \qquad \text{with } c \overset{\star}{\simeq} 0,$$

along the generators of $\mathscr{I}^+$, it is always possible to rotate the basis according to (10.30) so as to ensure that $\Gamma'_{\mathbf{1}\mathbf{1}'\mathbf{0}\mathbf{1}} \simeq 0$. In the following, it will be assumed that $\boldsymbol{e}'_{\mathbf{0}\mathbf{1}'}$ and $\boldsymbol{e}'_{\mathbf{1}\mathbf{0}'}$ have been chosen so that the latter is the case.

The choice of vectors $\boldsymbol{e}'_{\mathbf{0}\mathbf{1}'}$ and $\boldsymbol{e}'_{\mathbf{1}\mathbf{0}'}$ has some further consequences. Evaluating the primed version of Equation (10.22) at $\mathscr{I}^+$ for ${}_{\boldsymbol{AA}'} = {}_{\mathbf{0}\mathbf{1}'}$ and ${}_{\boldsymbol{BB}'} = {}_{\mathbf{1}\mathbf{0}'}$ one finds that $\nabla'_{\mathbf{0}\mathbf{1}'}\nabla'_{\mathbf{1}\mathbf{0}'}\Xi' \simeq -s'$. Now, as $\Xi' = 0$ on $\mathscr{I}^+$ and $\boldsymbol{e}'_{\mathbf{0}\mathbf{1}}$ is tangent to $\mathscr{I}^+$, it follows from $\nabla'_{\mathbf{0}\mathbf{1}'}\nabla'_{\mathbf{1}\mathbf{0}'}\Xi' = \nabla'_{\mathbf{0}\mathbf{1}'}\boldsymbol{e}'_{\mathbf{1}\mathbf{0}'}(\Xi') \simeq 0$ that $s' \simeq 0$ and that

$$\nabla'_{\boldsymbol{AA}'}\nabla'_{\boldsymbol{BB}'}\Xi' \simeq 0. \tag{10.31}$$

This last expression can be regarded as a strengthened version of the asymptotic Einstein condition (10.4). In particular, for ${}_{\boldsymbol{AA}'} = {}_{\mathbf{1}\mathbf{1}'}$ and ${}_{\boldsymbol{BB}'} = {}_{\mathbf{0}\mathbf{0}'}$ Equation (10.31) implies that $\nabla'_{\mathbf{1}\mathbf{1}'}\big(\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\Xi')\big) \simeq 0$ so that $\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\Xi')$ is constant along the generators of $\mathscr{I}^+$. Moreover, setting ${}_{\boldsymbol{AA}'} = {}_{\mathbf{0}\mathbf{0}'}$ and ${}_{\boldsymbol{BB}'} = {}_{\mathbf{0}\mathbf{1}'}$ and using that $\boldsymbol{e}_{\mathbf{0}\mathbf{1}'}(\Xi') \simeq 0$ one finds that

$$\Gamma'_{\mathbf{0}\mathbf{1}'}{}^{\boldsymbol{Q}}{}_{\mathbf{0}}\boldsymbol{e}'_{\boldsymbol{Q}\mathbf{0}'}(\Xi') + \bar{\Gamma}'_{\mathbf{1}'\mathbf{0}}{}^{\boldsymbol{Q}'}{}_{\mathbf{0}'}\boldsymbol{e}'_{\mathbf{0}\boldsymbol{Q}'}(\Xi') \simeq 0.$$

Expanding and using, again, that $\boldsymbol{e}_{\mathbf{0}\mathbf{1}'}(\Xi') \simeq 0$ and recalling (10.21b) one finds that $\Gamma'_{\mathbf{1}\mathbf{1}'\mathbf{0}\mathbf{0}}\boldsymbol{e}'_{\mathbf{0}\mathbf{0}'}(\Xi') \simeq 0$. However, $\boldsymbol{e}'_{\mathbf{0}\mathbf{0}'}(\Xi') \not\simeq 0$ so that one concludes that $\Gamma'_{\mathbf{1}\mathbf{1}'\mathbf{0}\mathbf{0}} \simeq 0$.

To conclude, it is observed that although Equation (10.28) fixed the derivative $\boldsymbol{e}'_{\mathbf{1}\mathbf{1}'}(\vartheta)$ along $\mathscr{I}^+$, the derivative $\boldsymbol{e}'_{\mathbf{0}\mathbf{0}'}(\vartheta)$ still remains free. A convenient way of fixing $\boldsymbol{e}'_{\mathbf{0}\mathbf{0}'}(\vartheta)$ can be obtained from the transformation law for the Ricci scalar – see Equation (5.6c) – which, in the present context, takes the form

$$R[\boldsymbol{g}'] = \vartheta^{-2}R[\boldsymbol{g}] + 12\vartheta^{-2}\nabla'_a\vartheta\nabla'^a\vartheta - 6\vartheta^{-1}\nabla'_a\nabla'^a\vartheta.$$

A natural requirement is to set $R[\boldsymbol{g}'] = 0$ on $\mathscr{I}^+$ so that along the generators of $\mathscr{I}^+$ one obtains the equation

$$\boldsymbol{e}'_{\mathbf{1}\mathbf{1}'}\big(\boldsymbol{e}'_{\mathbf{0}\mathbf{0}'}(\vartheta)\big) - 2\vartheta^{-1}\boldsymbol{e}'_{\mathbf{1}\mathbf{1}'}(\vartheta)\boldsymbol{e}'_{\mathbf{0}\mathbf{0}'}(\vartheta) \simeq F', \tag{10.32}$$

where

$$F' \equiv \mathrm{Re}\bigg(\boldsymbol{e}'_{\mathbf{0}\mathbf{1}'}\big(\boldsymbol{e}'_{\mathbf{1}\mathbf{0}'}(\vartheta)\big) - 2\Gamma'_{\mathbf{0}\mathbf{1}'\mathbf{0}\mathbf{1}}\boldsymbol{e}'_{\mathbf{1}\mathbf{0}}(\vartheta) - 2\vartheta^{-1}\boldsymbol{e}'_{\mathbf{0}\mathbf{1}'}(\vartheta)\boldsymbol{e}'_{\mathbf{1}\mathbf{0}}(\vartheta) + \frac{1}{12}\vartheta^{-1}R[\boldsymbol{g}]\bigg).$$

Equation (10.32) can be regarded as a linear differential equation for $\boldsymbol{e}'_{\mathbf{0}\mathbf{0}'}(\vartheta)$ along the generators of $\mathscr{I}^+$ with a non-homogeneous term F' which consists of

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quantities which are already known along $\mathscr{I}^+$. Equation (10.32) is supplemented by the condition $\boldsymbol{e}'_{\mathbf{00'}}(\vartheta) = \varkappa^{-1}\vartheta\Gamma_{\mathbf{10'00}} \overset{\star}{\simeq} 0$ consistent with Equation (10.27). It follows that

$$R[\boldsymbol{g}'] \simeq 0. \tag{10.33}$$

Using the Ricci identity, Equation (8.35b), taking into account the conformal gauge condition (10.33) and the conditions on the spin connection coefficients, gives for the values $\boldsymbol{AA'} = \mathbf{11'}$, $\boldsymbol{BB'} = \mathbf{10'}$ and $\boldsymbol{CD} = \mathbf{00}$ a *homogeneous ordinary differential equation* for $\Gamma'_{\mathbf{10'00}}$ along the generators of $\mathscr{I}^+$. Observing the initial condition (10.27) the latter implies that $\Gamma'_{\mathbf{10'00}} \simeq 0$. Finally, a further use of the Ricci identities gives $\Phi'_{12} = \Phi_{21} \simeq 0$.

The construction of the previous paragraphs is rounded up with the introduction of adapted coordinates. On the fiduciary cut $\mathscr{C}_\star \approx \mathbb{S}^2$ one chooses some coordinates $\theta = (\theta^{\mathcal{A}})$ $\mathcal{A} = 2, 3$ and extends them along $\mathscr{I}^+$ by requiring them to be constant along the null generators. On the hypersurfaces $\mathcal{N}_{u'}$ transverse to $\mathscr{I}^+$ it is natural to identify an affine parameter r' of the null generators of these hypersurfaces in such a way that $\boldsymbol{e}'_{\mathbf{00'}}(r') = 1$ and $r' \simeq 0$. The coordinates $\theta = (\theta^{\mathcal{A}})$ are propagated off $\mathscr{I}^+$ in such a way that they are constant along the generators of $\mathcal{N}_{u'}$. As a result of this construction one obtains ***Bondi coordinates*** $x = (u', r', \theta^{\mathcal{A}})$ in the neighbourhood $\mathcal{U}$ of $\mathscr{I}^+$.

Summary of the construction

The lengthy construction in this section can be summarised in the following proposition (for ease of presentation the $'$ in the objects associated to the *improved adapted frame* has been dropped from the expressions):

Proposition 10.1 (***the NP gauge at*** $\mathscr{I}^+$) *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote an asymptotically simple spacetime. Locally, it is always possible to find a conformal extension $(\mathcal{M}, \boldsymbol{g}, \Xi)$ for which*

$$R[\boldsymbol{g}] \simeq 0$$

and an adapted frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ such that the associated spin connection coefficients $\Gamma_{\boldsymbol{AA'BC}}$ satisfy

$$\Gamma_{\mathbf{00'}\boldsymbol{BC}} \simeq 0, \qquad \Gamma_{\mathbf{11'}\boldsymbol{BC}} \simeq 0,$$
$$\Gamma_{\mathbf{01'11}} \simeq 0, \qquad \Gamma_{\mathbf{10'00}} \simeq 0, \qquad \Gamma_{\mathbf{10'11}} \simeq 0$$
$$\bar{\Gamma}_{\mathbf{1'00'1'}} + \Gamma_{\mathbf{01'01}} \simeq 0.$$

In addition, one has that

$$\Phi_{12} \simeq 0, \qquad \Phi_{22} \simeq 0,$$

and $\boldsymbol{e}_{\mathbf{00'}}(\Xi)$ is constant on $\mathscr{I}^+$.

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A quick inspection reveals that in the gauge associated to Proposition 10.1 the only non-zero spin connection coefficients on $\mathscr{I}^+$ are given by $\Gamma_{\mathbf{0}\mathbf{1}'\mathbf{0}\mathbf{0}}$, $\Gamma_{\mathbf{0}\mathbf{0}'\mathbf{0}\mathbf{1}}$ and $\Gamma_{\mathbf{1}\mathbf{0}'\mathbf{0}\mathbf{1}}$ which in standard NP notation correspond, respectively, to σ, α, β. On $\mathscr{I}^+$ the connection coefficients α and β satisfy $\alpha + \bar{\beta} \simeq 0$ and describe, essentially, *the connection of the intrinsic metric of the cuts of* $\mathscr{I}^+$; that is, the connection of the standard metric of $\mathbb{S}^2$, $\boldsymbol{\sigma}$. The remaining spin connection coefficient, $\sigma = \Gamma_{\mathbf{0}\mathbf{1}'\mathbf{0}\mathbf{0}}$, encodes the (non-trivial) dynamical degrees of freedom in the set up. Its relation with the notion of ***gravitational radiation*** will be briefly explored in the next subsection.

10.3.2 The radiation field and the news function

To explore the relation between the spin connection coefficient σ and the notion of gravitational radiation it is convenient to expand the Ricci, Cotton and Bianchi identities – that is, the conformal field Equations (8.35b), (8.37a) and (8.37b) – in terms of the gauge given by Proposition 10.1. An inspection of the components of the Ricci identity not used in the derivation of the NP gauge, taking into account that $\Psi_{\boldsymbol{ABCD}} \simeq 0$, provides the relations

$$\Phi_{00} \simeq -\sigma\bar{\sigma}, \qquad \Phi_{01} \simeq -\bar{\eth}\sigma, \qquad \Phi_{02} \simeq -\dot{\sigma},$$

where $\dot{}$ denotes differentiation with respect to the retarded time u. In addition, one also finds

$$\Phi_{11} \simeq \delta\alpha - \bar{\delta}\beta + 4\alpha\beta.$$

As α and β describe the Levi-Civita connection of the standard metric of $\mathbb{S}^2$, it can be readily verified that Φ_{11} corresponds, essentially, to the curvature of $\mathbb{S}^2$ – recall that in two-dimensional manifolds the curvature is encoded in the Ricci scalar.

The relation between σ and the components of the rescaled Weyl tensor can be established by inspection of the Bianchi identity (8.37b) at $\mathscr{I}^+$. Choosing, for convenience Ξ so that $\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\Xi) \simeq -1$, one finds that

$$\phi_4 \simeq -\ddot{\bar{\sigma}}, \qquad \phi_3 \simeq -\eth\dot{\bar{\sigma}}.$$

Moreover, one also obtains the *constraint*

$$\phi_2 + \sigma\dot{\bar{\sigma}} + \eth^2\bar{\sigma} \simeq \bar{\phi}_2 + \bar{\sigma}\dot{\sigma} + \bar{\eth}^2\sigma.$$

In view of the *Peeling theorem*, Theorem 10.4, the component ϕ_4 describes the leading term of the gravitational field – the so-called ***radiation field*** or ***outgoing field***. In particular, if $\dot{\bar{\sigma}}$ is constant along $\mathscr{I}^+$ one has that $\phi_4 \simeq 0$, $\phi_3 \simeq 0$, and one interprets this situation as describing the absence of gravitational radiation – that is why $\dot{\bar{\sigma}}$ is sometimes called the ***news function***. The component ϕ_2 is interpreted as describing the *Coulomb part* of the gravitational field while ϕ_1 and ϕ_0 are associated with ***incoming radiation***; see Szekeres (1965).

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10.4 Other aspects of asymptotics

The present chapter provides a minimalistic account of the theory of asymptotics of the gravitational field. A detailed account would go beyond the scope of this book. It is, nevertheless, of interest to briefly highlight certain topics.

10.4.1 The Bondi mass

The analysis of the asymptotics of the gravitational field allows one to describe in a rigorous manner the loss of energy of an isolated system due to gravitational radiation. This physical process is described in terms of the so-called Bondi mass; see Trautman (1958), Bondi et al. (1962), Sachs (1962b) and also Penrose (1965). In terms of the notation introduced in this chapter, the ***Bondi mass*** $m_{\mathscr{B}}$ over a cut $\mathscr{C}$ of $\mathscr{I}^+$ is given by the surface integral

$$m_{\mathscr{B}} \equiv -\frac{1}{2}\int_{\mathscr{C}} \big(\phi_2 + \sigma\dot{\bar{\sigma}}\big)\mathrm{d}S.$$

A concise deduction of the above expression can be found in Stewart (1991). Moreover, it can be shown that under suitable assumptions $m_{\mathscr{B}} \geq 0$; see Ludvigsen and Vickers (1981, 1982). A further calculation renders that

$$\dot{m}_{\mathscr{B}} = -\frac{1}{2}\int_{\mathscr{C}} |\dot{\sigma}|^2\mathrm{d}S \leq 0.$$

The above inequality is called the ***Bondi mass-loss*** formula and encodes the loss of mass of an isolated system due to the energy that is carried away by (outgoing) gravitational radiation.

10.4.2 The Bondi-Metzner-Sachs group

As already mentioned, one of the central objectives of the theory of asymptotics of the gravitational field is to identify universal structures in a wide class of spacetimes and, in turn, use these to extract physical insight into the behaviour of isolated systems in general relativity. An example of this type of universal structures is given by the so-called ***Bondi-Metzner-Sachs (BMS) group***; see Sachs (1962a), Bondi et al. (1962)and Newman and Penrose (1966).

In what follows let $(u, r, \theta^{\mathcal{A}})$ denote a Bondi coordinate system defined in a neighbourhood of the future null infinity $\mathscr{I}^+$ of an asymptotically simple spacetime. The BMS group is defined by the following transformations on the u and $\theta = (\theta^{\mathcal{A}})$ coordinates:

$$u' = K(\theta)\big(u - \alpha(\theta)\big), \tag{10.34a}$$

$$\theta'^{\mathcal{A}} = \theta'^{\mathcal{A}}(\theta^2, \theta^3), \tag{10.34b}$$

where the map $(\theta^{\mathcal{A}}) \mapsto (\theta'^{\mathcal{A}})$ is a conformal transformation of $\mathbb{S}^2$ onto itself, and $K(\theta)$ is the associated conformal factor so that

$$\boldsymbol{\sigma}' = K^2\boldsymbol{\sigma},$$

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and where $\alpha(\theta)$ is an arbitrary smooth real function on $\mathbb{S}^2$. The particular BMS transformations for which $\theta'^{\mathcal{A}} = \theta^{\mathcal{A}}$ are called ***supertranslations***. Under a supertranslation, the system of null hypersurfaces $\mathscr{N}_{u_\bullet}$ with $u_\bullet$ constant is transformed into a different system $\mathscr{N}_{u'_\bullet}$. Expanding the function $\alpha(\theta)$ in terms of spherical harmonics Y_{lm} – see the Appendix to this chapter – one finds that

$$\alpha(\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm},$$

with $a_{lm} \in \mathbb{C}$. Thus, the supertranslations are an *infinite-dimensional subgroup* of the BMS group. The particular (four-dimensional) case for which $a_{lm} = 0$ for $l > 2$ is called the ***translations subgroup***.

Generic asymptotically simple spacetimes do not possess Killing vectors – in the conformal picture Killing vectors of the physical spacetime correspond to *conformal Killing vectors*. The BMS group arises from a notion of ***asymptotic symmetries*** which ensures the existence of non-trivial solutions for generic spacetimes, that is, a diffeomorphism $\varphi : \mathscr{I}^+ \to \mathscr{I}^+$ satisfying the conditions

$$\varphi^* \boldsymbol{q} = \vartheta^2 \boldsymbol{q}, \qquad \varphi_* \boldsymbol{N} = \vartheta^{-1} \boldsymbol{N}, \tag{10.35}$$

for some function $\vartheta > 0$ and where the tensor fields $\boldsymbol{q}$ and $\boldsymbol{N}$ are as given in Section 10.1.2. It can be verified that the BMS transformations (10.34a) and (10.34b) satisfy the conditions in (10.35) with $K = \vartheta$. A particular type of asymptotic symmetries corresponds to those generated by an ***asymptotic Killing vector***, that is, a field $\boldsymbol{\xi}$ on $\mathscr{I}^+$ satisfying the conditions

$$\pounds_{\boldsymbol{\xi}} \boldsymbol{q} = 2\vartheta \boldsymbol{q}, \qquad \pounds_{\boldsymbol{\xi}} \boldsymbol{N} = -\vartheta \boldsymbol{N}.$$

Given an asymptotically simple spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ endowed with a Killing vector $\tilde{\boldsymbol{\xi}}$, let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ denote a conformal extension thereof. Given that $0 = \pounds_{\tilde{\boldsymbol{\xi}}} \tilde{\boldsymbol{g}} = \pounds_{\tilde{\boldsymbol{\xi}}} (\Xi^{-2} \boldsymbol{g})$, it follows that

$$\pounds_{\tilde{\boldsymbol{\xi}}} \boldsymbol{g} = 2\big(\Xi^{-1} \tilde{\boldsymbol{\xi}}(\Xi)\big) \boldsymbol{g}, \tag{10.36}$$

for $\Xi \neq 0$, so that $\tilde{\boldsymbol{\xi}}$ is a conformal Killing vector of $\boldsymbol{g}$ on $\tilde{\mathcal{M}}$. Since this vector is determined by the smooth metric $\boldsymbol{g}$, it extends smoothly to $\mathscr{I}^+$ as a vector $\boldsymbol{\xi}$. Now, the left-hand side of Equation (10.36) extends smoothly to $\mathscr{I}^+$, and, therefore, the right-hand side does so too. It follows that

$$\boldsymbol{\xi}(\Xi) = \alpha' \Xi, \tag{10.37}$$

with α' a smooth function such that $\alpha = O(\Xi^0)$ so that $\boldsymbol{\xi}$ is tangent to $\mathscr{I}^+$. From Equation (10.36) one concludes that

$$\pounds_{\boldsymbol{\xi}} \boldsymbol{q} = 2\alpha' \boldsymbol{q}, \qquad \pounds_{\boldsymbol{\xi}} \boldsymbol{N} = -\alpha' \boldsymbol{N}.$$

Accordingly, *any Killing vector of* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *admits a unique extension to a vector on* $\mathscr{I}^+$ *and which defines an asymptotic Killing vector*. The maximum

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number of linearly independent Killing vectors in a four-dimensional manifold is 10. Accordingly, Killing vectors can give rise, at most, to 10 asymptotic Killing vectors. By means of a direct calculation, it is possible to show that the function α' in Equation (10.37) and the function α appearing in (10.34a) are the same. Thus, the BMS transformations (10.34a) and (10.34b) are asymptotic symmetries. In particular, the translations subgroup can be put in correspondence with the asymptotic Killing vectors arising from translations in the Minkowski spacetime.

For further details on the structure and properties of the BMS group, see, for example, Penrose and Rindler (1986) and Schmidt et al. (1975). A discussion of the properties of Killing vectors in asymptotically simple spacetimes can be found in Ashtekar and Xanthopoulos (1978) and Ashtekar and Schmidt (1980).

10.4.3 Newman-Penrose constants

In Newman and Penrose (1965) – see also Newman and Penrose (1968) and Penrose and Rindler (1986) – it has been shown that in an asymptotically simple spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ there exists a set of ten quantities defined as integrals over cuts of null infinity which are *absolutely conserved* in the sense that their value is independent of the particular cut $\mathscr{C}$ on which they are evaluated – the so-called ***Newman-Penrose constants***. In terms of the adapted frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ of Proposition 10.1 these constants are given by

$$G_m \equiv \int_{\mathscr{C}} {}_2\bar{Y}_{2m} \boldsymbol{e}_{\boldsymbol{00}}(\phi_0)\mathrm{d}S, \qquad m = -2, \ldots, 2,$$

where ${}_2Y_{2m}$ is a ***spin-weighted spherical harmonic***; see the Appendix to this chapter. A discussion of the relation between the above expression and the original formula of Newman and Penrose can be found in Friedrich and Kánnár (2000a).

There exists no general consensus about the physical meaning or interpretation of the Newman-Penrose constants. An explicit computation for stationary spacetimes shows that they are of the form

$$(\text{mass}) \times (\text{quadrupole}) - (\text{dipole})^2;$$

see, for example, Bäckdahl (2009). Evaluations of the Newman-Penrose spacetimes for dynamic spacetimes can be found in Friedrich and Schmidt (1987) and Friedrich and Kánnár (2000a). In particular, in the former reference it is shown that for spacetimes possessing a conformal extension which includes the points i^+ and i^- the Newman-Penrose constants correspond, essentially, to the value of the rescaled Weyl spinor $\phi_{\boldsymbol{ABCD}}$ at those points. Electrovacuum asymptotically simple spacetimes have a suitable generalisation of these absolutely conserved constants; see Exton et al. (1969).

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10.5 Further reading

An excellent introduction to the theory of asymptotics of the gravitational field is given in Stewart (1991) where the subject is called "asymptopia". A related account can be found in Penrose and Rindler (1986). A convenient entry point to the extensive literature on the subject can be found in the review of Frauendiener (2004). A detailed discussion of the ideas and general philosophy behind the treatment of the asymptotics of the gravitational field by means of conformal methods can be found in Geroch (1976). Accounts similar in spirit to the latter can be found in Ashtekar (1980, 1987). A slightly different perspective on the subject can be found in Friedrich (1992); see also Friedrich (1998a, 1999). A recent review on the subject of asymptotics is given in Ashtekar (2014).

Appendix: spin-weighted functions

Let $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ denote a spinorial dyad defined on a spacetime $(\mathcal{M}, \boldsymbol{g})$ and let $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ denote the associated null tetrad. As discussed in Section 3.1.10, the null vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ span a spacelike subspace of $T(\mathcal{M})$ which is orthogonal to both $\boldsymbol{l}$ and $\boldsymbol{n}$. Of particular interest is the case when this subspace corresponds to the tangent bundle of a compact two-dimensional submanifold $\mathscr{C}$ of $\mathcal{M}$. *In the following it is assumed that this is the case.* From the expression

$$\boldsymbol{g} = \boldsymbol{l} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{l} - \boldsymbol{m} \otimes \bar{\boldsymbol{m}} - \bar{\boldsymbol{m}} \otimes \boldsymbol{m}$$

of the metric $\boldsymbol{g}$ in terms of the null tetrad, it follows that the intrinsic metric $\boldsymbol{\sigma}$ induced by $\boldsymbol{g}$ on $\mathscr{C}$ is given by

$$\boldsymbol{\sigma} = -\boldsymbol{m} \otimes \bar{\boldsymbol{m}} - \bar{\boldsymbol{m}} \otimes \boldsymbol{m}.$$

There is a certain *gauge freedom* in the above expression since ***spin-boosts*** of the form

$$\boldsymbol{o} \mapsto e^{\frac{1}{2}\mathrm{i}c}\boldsymbol{o}, \qquad \boldsymbol{\iota} \mapsto e^{-\frac{1}{2}\mathrm{i}c}\boldsymbol{\iota}, \tag{10.38}$$

with arbitrary $c \in \mathbb{R}$ which imply the transition

$$\boldsymbol{m} \mapsto e^{\mathrm{i}c}\boldsymbol{m}, \qquad \bar{\boldsymbol{m}} \mapsto e^{-\mathrm{i}c}\bar{\boldsymbol{m}},$$

leave the metric $\boldsymbol{\sigma}$ unchanged.

Given a spinor $\eta_{A_1\cdots A_n A'_1\cdots A'_m}$ of valence $n+m$, it is natural to consider the behaviour of its components with respect to the dyad $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ under the spin boost (10.38). For example, given p, q, r, $t \in \mathbb{N}$ such that $p+q=n$, $r+t=m$, the scalar

$$\eta \equiv \eta_{A_1\cdots A_p B_1\cdots B_q A'_1\cdots A'_r B'_1\cdots B'_t} o^{A_1}\cdots o^{A_p}\iota^{B_1}\cdots\iota^{B_q}\bar{o}^{A'_1}\cdots\bar{o}^{A'_r}\bar{\iota}^{B'_1}\cdots\bar{\iota}^{B'_t} \tag{10.39}$$

has a transformation given by

$$\eta \mapsto e^{\frac{1}{2}\mathrm{i}(p+t-q-r)\vartheta}\eta.$$

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One says, then, that η has ***spin weight*** $s = \frac{1}{2}(p+t-q-r)$. The spin weight of all the possible components of $\eta_{A_1\cdots A_n A'_1\cdots A'_m}$ lies in the range $-m-n \leq s \leq m+n$.

In what follows, we adopt the standard Newman-Penrose conventions to denote the directional covariant derivatives with respect to $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ and let $\delta \equiv m^a\nabla_a$, $\bar{\delta} \equiv \bar{m}^a\nabla_a$. Generically, the directional derivatives δ and $\bar{\delta}$ acting on a spin-weighted scalar do not give rise to scalars with a well-defined spin weight. To amend this deficiency it is convenient to define operators $\eth$ and $\bar{\eth}$ which, acting on scalars with a given spin weight, give rise to new scalars with a well-defined spin weight. Given the spin-weighted scalar η of Equation (10.39), the action of $\eth$ and $\bar{\eth}$ is defined to be

$$
\begin{aligned}
\eth\eta \equiv\, & o^{A_1}\cdots o^{A_p}\iota^{B_1}\cdots\iota^{B_q}\bar{o}^{A'_1}\cdots\bar{o}^{A'_r}\bar{\iota}^{B'_1}\cdots\bar{\iota}^{B'_t} \\
& \times\delta(\eta\,\iota_{A_1}\cdots\iota_{A_p}o_{B_1}\cdots o_{B_q}\bar{\iota}_{A'_1}\cdots\bar{\iota}_{A'_r}\bar{o}_{B'_1}\cdots\bar{o}_{B'_t}), \qquad (10.40a)\\
\bar{\eth}\eta \equiv\, & o^{A_1}\cdots o^{A_p}\iota^{B_1}\cdots\iota^{B_q}\bar{o}^{A'_1}\cdots\bar{o}^{A'_r}\bar{\iota}^{B'_1}\cdots\bar{\iota}^{B'_t} \\
& \times\bar{\delta}(\eta\,\iota_{A_1}\cdots\iota_{A_p}o_{B_1}\cdots o_{B_q}\bar{\iota}_{A'_1}\cdots\bar{\iota}_{A'_r}\bar{o}_{B'_1}\cdots\bar{o}_{B'_t}). \qquad (10.40b)
\end{aligned}
$$

The operators $\eth$ and $\bar{\eth}$ are complex conjugates of each other in the sense that $\overline{\eth\eta} = \bar{\eth}\bar{\eta}$. If the scalar η has spin weight s, one can verify that $\eth\eta$ and $\bar{\eth}\eta$ have, respectively, spin weights $s+1$ and $s-1$. Furthermore, $\eth\eta$ and $\bar{\eth}\eta$ satisfy the Leibnitz rule. In order to obtain alternative expressions for $\eth\eta$ and $\bar{\eth}\eta$ let

$$\alpha \equiv o^A\bar{\delta}\iota_A = \iota^A\bar{\delta}o_A, \qquad \beta \equiv o^A\delta\iota_A = \iota^A\delta o_A,$$

consistent with standard Newman-Penrose notation. Expanding (10.40a) and (10.40b) and using the above definitions one obtains

$$
\begin{aligned}
\eth\eta &= (-1)^{p+r}\left(\delta\eta + ((q-p)\beta + (t-r)\bar{\alpha})\,\eta\right),\\
\bar{\eth}\eta &= (-1)^{p+r}\left(\bar{\delta}\eta + \left((q-p)\bar{\beta} + (t-r)\alpha\right)\eta\right).
\end{aligned}
$$

A computation with the above expressions shows that

$$(\bar{\eth}\eth - \eth\bar{\eth})\eta = s\eta.$$

The above expressions are convenient for the discussion of ***spin-weighted harmonics***. In terms of standard spherical harmonics Y_{lm}, these are given by

$${}_0Y_{lm} \equiv Y_{lm},$$

and for $s \neq 0$

$$
{}_sY_{lm} \equiv \begin{cases}
(-1)^s\sqrt{\dfrac{2^s(l-s)!}{(l+s)!}}\,\eth^s Y_{lm} & 0 < s \leq l\\[2ex]
\sqrt{\dfrac{(l+s)!}{2^s(l-s)!}}\,\bar{\eth}^{-s} Y_{lm} & -l \leq s < 0\\[2ex]
0 & \text{otherwise;}
\end{cases}
$$

see, for example, Stewart (1991) for further discussion.

Of special relevance for Theorem 10.3 is the following result:

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Lemma 10.1 *Assume $\mathscr{C}$ to be diffeomorphic to $\mathbb{S}^2$ and let η denote a smooth scalar on $\mathscr{C}$ having spin weight s. If $\eth\eta = 0$ and $s < 0$, then $\eta = 0$. Similarly, if $\bar{\eth}\eta = 0$ and $s > 0$, then $\eta = 0$.*

Proofs of this result can be found in Penrose and Rindler (1984) and Stewart (1991). Remarkably, *the result depends on the topology (genus) of $\mathscr{C}$*; see Frauendiener and Szabados (2001). For example, the above result is not valid for surfaces diffeomorphic to the 2-torus $\mathbb{S} \times \mathbb{S}$. It is of interest to point out that Lemma 10.1 is equivalent to the statement that there exist no non-zero symmetric trace-free, divergence-free, rank 2 tensor fields on $\mathbb{S}^2$; see Beig (1985) and Frauendiener and Szabados (2001).

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