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Values of Twisted Tensor L-functions of Automorphic Forms Over Imaginary Quadratic Fields

Dominic Lanphier and Howard Skogman
Appendix by Hiroyuki Ochiai

Abstract. Let K be a complex quadratic extension of $\mathbb{Q}$ and let $\mathbb{A}_K$ denote the adeles of K. We find special values at all of the critical points of twisted tensor L-functions attached to cohomological cuspforms on $GL_2(\mathbb{A}_K)$ and establish Galois equivariance of the values. To investigate the values, we determine the archimedean factors of a class of integral representations of these L-functions, thus proving a conjecture due to Ghate. We also investigate analytic properties of these L-functions, such as their functional equations.

1 Introduction

Let $D > 0$ be an integer and let $K = \mathbb{Q}(\sqrt{-D})$ be a complex quadratic extension of $\mathbb{Q}$. Let $\mathbb{A}_K$ be the adeles of K. Then $GL_2(\mathbb{A}_K)$ acts on hyperbolic 3-space $\mathfrak{H}_3$. The twisted tensor L-function $L_T(s, f)$ attached to a cohomological cuspform f on $\mathfrak{H}_3$ is essentially a subseries of the standard L-function of f. As such, it is analogous to the Asai L-function [2] of a real quadratic extension of $\mathbb{Q}$. A precise definition of $L_T(s, f)$ is given in Section 3. Analytic properties of $L_T(s, f)$ for certain cuspforms were studied by Takase [19] and Zhao [20]. Arithmetic properties of the L-functions studied here were investigated by Ghate in [6, 7]. In particular, special values at all critical integers in the right half of the critical strip were found, and these values were shown to be consistent with the conjecture of Deligne [3]. The arithmetic results were generalized to the case where K is a complex quadratic extension of a totally real number field in [8].

The special values in [6–8] were obtained by studying a certain class of integral representations of $L_T(s, f)$. The arithmetic properties of the integrals follow from ideas developed by Hida in [11]. The integrals were not completely determined as their archimedean factors were not fully computed. However, an explicit form for these factors was conjectured in [7], and the conjecture was subsequently proved by H. Ochiai, but not published. We thank the referee for pointing us to Ochiai's proof, which now appears as an appendix to this paper. In this paper we study these integral representations and determine the archimedean factors precisely, consequently giving a different proof of the conjecture of [7]. This and the functional equation for the

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twisted tensor L-function allow us to extend the special value results of [7] to the remaining critical points in the left half of the critical strip in the sense of [3]. Note that the functional equation in this setting was proved in [6] under the assumptions that w is odd, ψ_N is primitive, and $\mathfrak{n}$ is an extended ideal. Here we include the case when ψ_N is trivial, but only when K has class number 1. We also prove Galois equivariance of the values and investigate the locations of possible poles of the L-functions. Galois equivariance can be readily established following [14, 16].

Let $\mathcal{O}_K$ denote the integers of K and let $\mathfrak{n} \subseteq \mathcal{O}_K$ be a nonzero ideal. Let $\mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}), \psi_{\mathfrak{n}})$ be the space of cuspforms on $\mathfrak{H}_3$ of weight $\mathbf{n}$, level $\mathfrak{n}$ and nebentype a Hecke character $\psi_{\mathfrak{n}}$. The precise definitions of these cuspforms and their adelic versions are given in Section 2. A cuspform $f \in \mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}), \psi_{\mathfrak{n}})$ can be realized as a differential 1-form $\delta(f)$ on hyperbolic 3-space where δ is the Eichler–Shimura–Harder isomorphism. Then $\delta(f)$ takes values in a certain sheaf constructed from an irreducible $SL_2(K)$-module and the restriction $\delta(f)|_{\mathfrak{H}_1}$ to the complex upper-half plane $\mathfrak{H}_1$ decomposes into a sum of differential 1-forms on $\mathfrak{H}_1$. We denote such a summand as $\delta_{2n-2m}(f)$. An Eisenstein series $E_{2n-2m+2}$ on $\mathfrak{H}_1$ and a pairing $\langle \cdot, \cdot \rangle$ of cohomological automorphic forms are defined in Section 3, following [11]. The Dirichlet character ψ_N is defined from $\psi_{\mathfrak{n}}$ in Section 3. For i and c, the two embeddings of K into $\mathbb{C}$, let $\mathbf{n} = n_i i + n_c c$ and $\mathbf{v} = v_i i + v_c c$ where the infinity type of $\psi_{\mathfrak{n}}$ is $-\mathbf{n} - 2\mathbf{v}$. The main results of this paper are the following integral representations (first studied in [6]) and the special values of Theorem 1.4.

Theorem 1.1 *Let $f \in \mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}), \psi_{\mathfrak{n}})$ be a newform and a Hecke eigenfunction and let*

$$L_\infty(s-m, f) = \frac{(-1)^m \sqrt{\pi}\binom{n}{m}^2}{2^s} \frac{\Gamma\left(\frac{s}{2}+n-m+1\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)} \frac{\Gamma(s+n-m+1)\Gamma(s+2n-m+2)}{\Gamma(s+2n-2m+2)}.$$

For ψ_N primitive and nontrivial let $0 \le m \le n$, and for ψ_N trivial let $0 \le m < n$. For $m \equiv v_i + v_c \pmod 2$ and $w = n + 1 + v_i + v_c$, we have

$$\int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \langle \delta_{2n-2m}(f), E_{2n-2m+2}(s, \cdot, \psi_N) \rangle = \frac{\sqrt{D}^{s+n-m+w+1} i^{v_i - v_c}}{(2\pi)^{s+2n+2-m} L_N(2s+2n-2m+2, \psi_N)} \times L_\infty(s-m, f) L_T(s+n-m+w+1, f).$$

All of the results here can be generalized to CM fields following [8]. The explicit integral representations above give the following, which was also shown in [5], and the w odd case was shown in [6, Section 7.2].

Corollary 1.2 *Let $f \in \mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}), \psi_{\mathfrak{n}})$ with ψ_N primitive or trivial. Then $L_T(s, f)$ is holomorphic for w odd and has at most a simple pole at $s = w + 1$ for w even.*

Theorem 1.3 gives explicit functional equations of $L_T(s, f)$ for odd and even weights of f, where $\mathfrak{g}(\psi_N)$ is a Gauss sum defined in Section 3. This result is proved in Section 5. In the following we normalize the cuspform f so that its first Fourier coefficient is 1, as in [7], and $\overline{f}$ is the cuspidal eigenform whose Fourier coefficients are the complex conjugates of those of f. As discussed in Section 5, we take K to have class number 1 for simplicity.

Theorem 1.3 *Let f and $L_T(s, f)$ be as in Theorem 1.1 and let K have class number 1. Set $\epsilon = 0$ for w even and $\epsilon = 1$ for w odd and set*

$$\Lambda_T(s, f) = \Big(\frac{\sqrt{D}}{2\pi^2}\Big)^s \Gamma\Big(\frac{s - w + \epsilon}{2}\Big)^2 \Gamma(s - w + n + 1) L_T(s, f).$$

Then

$$\Lambda_T(s, f) = c\psi_N(-1)\, \mathfrak{g}(\psi_N) N^{2w+1+\epsilon-3s} \Lambda_T(2w + 1 - s, \overline{f}),$$

where $c \in \mathbb{Q}(f, \psi_{\mathfrak{n}})$ and $|c| = 1$.

The critical points of $L_T(s, f)$ in the sense of [3] are $k \in [w - n, w]$ for k odd and $k \in [w + 1, w + 1 + n]$ for k even. The result below for ψ_N primitive and k even is essentially the main result of [7], where $\Omega(f)$ is a canonical period attached to f. We define $\Omega(f)$ in Section 3. Here we obtain special value results at the remaining critical points of $L_T(s, f)$ for ψ_N primitive or trivial and $\mathfrak{n}$ principal. We also establish Galois equivariance for all of the values. The special values follow from Theorem 1.1, the ideas of Hida in [11], and the functional equation. In the following, $\mathbb{Q}(f, \psi_{\mathfrak{n}})$ denotes the field $\mathbb{Q}$ adjoined the values of the Fourier coefficients of f and the values of the character $\psi_{\mathfrak{n}}$. Also, $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ denotes the field automorphisms of $\mathbb{C}$, and for $a \in \mathbb{C}$ let a^σ denote the action of σ on a and for f a cuspform let f^σ denote the action of σ on the Fourier coefficients of f.

Theorem 1.4 *Let ψ_N be primitive and nontrivial and let $f \in \mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}), \psi_{\mathfrak{n}})$ be a normalized newform that is a Hecke eigenfunction. For even $k \in [w + 1, w + 1 + n]$ we have*

$$A_1(f, \psi_N) = \frac{L_T(k, f)}{\Omega(f)\, \mathfrak{g}(\psi_N)(2\pi i)^{3k-3w+n+1}} \in \mathbb{Q}(f, \psi_{\mathfrak{n}})$$

and $A_1(f, \psi_N)^\sigma = A_1(f^\sigma, \psi_N^\sigma)$ for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.

Let K have class number 1. For odd $k \in [w - n, w]$ we have

$$A_2(f) = \frac{L_T(k, \overline{f})}{\Omega(f)\sqrt{D}(2\pi i)^{k-w+n+1}} \in \mathbb{Q}(f, \psi_{\mathfrak{n}})$$

and $A_2(f)^\sigma = A_2(f^\sigma)$ for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.

In the case that ψ_N is trivial and $m \neq n$, the values hold as above with the factor $\mathfrak{g}(\psi_N)$ removed.

In Section 2 we define the class of automorphic forms that we work with, and in Section 3 we introduce the integral representations of $L_T(s, f)$ from [6] and [7]. We prove the conjecture from [7] in Section 4, and this gives Theorem 1.1. We prove Theorems 1.3 and 1.4 and Corollary 1.2 in the last section.

2 Automorphic Forms and Differential Forms

In this section we define cuspforms of a complex quadratic extension of $\mathbb{Q}$ adelically and recall the formalism of cohomological automorphic forms following [11]. To a cuspform on the adelic group we then associate a cuspform on hyperbolic 3-space, as in [7]. For integral $D > 0$ let $K = \mathbb{Q}(\sqrt{-D})$ be a CM extension of $\mathbb{Q}$ with discriminant $-D$. Let $\mathcal{O}_K$ denote the integers of K. Denote the two embeddings of K into $\mathbb{C}$ by i and c and let $\mathbf{n} = n_i i + n_c c$ and $\mathbf{v} = v_i i + v_c c$ be formal sums in $\mathbb{Z}[i, c]$, where $n_i, n_c \geq 0$. Let $\mathbb{A}_K$ denote the adeles of K and let $\mathbb{J}_K$ be the ideles of K. Fix an ideal $\mathfrak{n} \subset \mathcal{O}_K$ and let $\psi\colon K^\times \backslash \mathbb{J}_K \to \mathbb{C}$ be a Hecke character with infinity type $-\mathbf{n} - 2\mathbf{v}$ whose conductor divides $\mathfrak{n}$. Let $W(n_i + n_c + 2, \mathbb{C})$ be the space of homogeneous polynomials of degree $n_i + n_c + 2$ over $\mathbb{C}$ in variables S and T. Let $\widehat{\mathcal{O}}_K = \prod_{v \text{ prime}} \mathcal{O}_v$. Then $\psi|_{\widehat{\mathcal{O}}_K}$ can be considered to be a character on $\left(\mathcal{O}_K/(\mathfrak{n})\right)^\times$ and we denote this character by $\psi_{\mathfrak{n}}$. For a matrix m let m^T denote its transpose.

For a function $f\colon GL_2(\mathbb{A}_K) \to W(n_i + n_c + 2, \mathbb{C})$ and $g \in GL_2(\mathbb{A}_K)$ we have that $f(g)$ is a function of the variables S and T and so we can write $f(g)(S, T)$. For $\sigma = i, c$ let D_σ be the operators from [10, Section 1.3]. That is, $D_\sigma/4$ is a component of the Casimir operator in the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$. For a ring R let $U_R = \{\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \mid u \in R\}$. Let $Z_{\mathbb{A}}$ denote the center of $GL_2(\mathbb{A}_K)$, let

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathcal{O}}_K) \;\middle|\; c \equiv 0 \pmod{\mathfrak{n}} \right\},$$

and for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{n})$ let $\psi_{\mathfrak{n}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \prod_{v \mid \mathfrak{n}} \psi_v(d_v)$.

A cuspform of weight $(\mathbf{n}, \mathbf{v})$, level $\mathfrak{n}$, and central character $\psi_{\mathfrak{n}}$ is a function $f\colon GL_2(\mathbb{A}_K) \to W(n_i + n_c + 2, \mathbb{C})$ with the following properties. We have $f(\gamma g) = f(g)$ for $\gamma \in GL_2(K)$, $f(zg) = \psi_{\mathfrak{n}}(z) f(g)$ for $z \in Z_{\mathbb{A}}$, and for $\gamma_0\gamma_\infty \in \Gamma_0(\mathfrak{n}) SU_2(\mathbb{C})$ we have $f(g\gamma_0\gamma_\infty)(S, T) = \psi_{\mathfrak{n}}(\gamma_0) f(g)(\gamma_\infty(S, T))$. Let f be an eigenfunction of the operators D_σ as in [10] or [20]. In particular, $D_\sigma f = (n_\sigma^2/2 + n_\sigma) f$ as in [11]. As f is a cuspform, we have

$$\int_{U_K \backslash U_{\mathbb{A}}} f(ug)\, du = 0$$

for all $g \in GL_2(\mathbb{A}_K)$ and where du is Lebesgue measure on $\mathbb{A}_K$. Denote the space of such cuspforms by $\mathcal{S}_{(\mathbf{n},\mathbf{v})}(\mathfrak{n}, \psi_{\mathfrak{n}})$. Following [10, Section 2.3, Corollary 2.2] we can assume that $n_i = n_c$, which we denote by n.

Let

$$\mathfrak{H}_3 = \left\{ z = \begin{pmatrix} x & -y \\ y & \bar{x} \end{pmatrix} \;\middle|\; x \in \mathbb{C},\ y \in \mathbb{R},\ y > 0 \right\}$$

be upper-half hyperbolic 3-space. For $a \in \mathbb{C}$ let $\tau(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{C})$, the transitive action on $\mathfrak{H}_3$ is given by

$$\gamma(z) = \left[\tau(a)(z) + \tau(b)\right]\left[\tau(c)(z) + \tau(d)\right]^{-1}.$$

Let $j(\gamma, z) = \tau(c)(z) + \tau(d)$. There is a natural embedding $\mathfrak{H}_1 \hookrightarrow \mathfrak{H}_3$ given by $x + iy \hookrightarrow \left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right)$. The stabilizer of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is $SU_2(\mathbb{C})$, so we can identify $\mathfrak{H}_3$ with $SL_2(\mathbb{C})/SU_2(\mathbb{C})$. The embedding above is compatible with

$$SL_2(\mathbb{R})/SO_2(\mathbb{R}) \hookrightarrow SL_2(\mathbb{C})/SU_2(\mathbb{C}).$$

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A cuspform $f \in \mathcal{S}_{(\mathbf{n},\mathbf{v})}(\mathfrak{n}, \psi_\mathfrak{n})$ gives rise to a cuspform on $GL_2(\mathbb{C})$ and then on $\mathfrak{H}_3$ following [7]. Let h be the class number of K and let $\mathbb{A}_0$ denote the finite adeles of K. By strong approximation we have

$$GL_2(\mathbb{A}_K) = \bigsqcup_{j=1}^{h} GL_2(K)\left(\begin{smallmatrix} a_j & 0 \\ 0 & 1 \end{smallmatrix}\right)\Gamma_0(\mathfrak{n})GL_2(\mathbb{C})$$

for certain $a_j \in \mathbb{A}_0$, and we can assume that $a_1 = 1$. Let $\mathfrak{a}_j = a_j\mathcal{O}_K$ and let

$$\Gamma_{\mathfrak{a}_j} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathcal{O}}_K) \;\middle|\; a, d \in \mathcal{O}_K,\ b \in \mathfrak{a}_j,\ c \in \mathfrak{a}_j^{-1}\mathfrak{n},\ ad - bc = 1 \right\}.$$

For $f \in \mathcal{S}_{(\mathbf{n},\mathbf{v})}(\mathfrak{n}, \psi_\mathfrak{n})$, define a cuspform on $GL_2(\mathbb{C})$ by $F_j = f\big(\left(\begin{smallmatrix} a_j & 0 \\ 0 & 1 \end{smallmatrix}\right)g\big)$. Let

$$f_j(z)(S,T) = F_j(g)\,j\big(g, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^T\big)(S,T),$$

where $g \in GL_2(\mathbb{C})$ is chosen so that $g\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = z$. Then $f_j\colon \mathfrak{H}_3 \to W(2n+2, \mathbb{C})$ is well defined, and for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathfrak{a}_j}$ it satisfies

$$f_j(\gamma z)(S,T) = \psi_\mathfrak{n}(d) f_j(z)\, j(\gamma, z)^T(S,T).$$

Thus, f_j is a cuspform on $\mathfrak{H}_3$ of weight $\mathbf{n}$, character $\psi_\mathfrak{n}$, and with respect to $\Gamma_{\mathfrak{a}_j}$ and we denote the space of such cuspforms by $\mathcal{S}_\mathbf{n}(\Gamma_{\mathfrak{a}_j}, \psi_\mathfrak{n})$. Let δ_K be the different of K and let $d_K \in \mathbb{J}_K$ be so that $d_K\mathcal{O}_K = (\delta_K)$. Let $|\cdot|_\mathbb{J}$ denote the idele norm. Let $W\colon \mathbb{C}^\times \to W(2n+2, \mathbb{C})$ be the Whittaker function

$$W(s) = \sum_{\alpha=0}^{2n+2} \binom{2n+2}{\alpha} \frac{1}{s^{v_i}\bar{s}^{v_c}} \left(\frac{s}{i|s|}\right)^{n+1-s} K_{\alpha-n-1}(4\pi|s|) S^{2n+2-\alpha} T^\alpha,$$

where $K_\alpha(x)$ is a modified Bessel function, as in [1, Section 4.5]. Let

$$e_K = \prod_v (e_v \circ \mathrm{Tr}_{K_v/\mathbb{Q}_v}) \cdot (e_\infty \circ \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}),$$

where $e_v(\sum_j c_j v^j) = e^{-2\pi i \sum_{j<0} c_j v^j}$ and $e_\infty(x) = e^{2\pi i x}$. Then from [11, Theorem 6.1] for example, a cuspform f as above has a Fourier expansion given by

$$f\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) = |y|_\mathbb{J} \sum_{\xi \in K^\times} c_f(\xi y d_K) W(\xi y_\infty) e_K(\xi x), \tag{2.1}$$

where y_∞ is a real place of y. Note that $c_f(\cdot)$ can be considered to be a function on the fractional ideals of K that vanishes outside of the integral ideals. In the sequel we consider f to be a newform, a Hecke eigenfunction in the sense of [20, Section 4], and normalized so that $c_f(\mathcal{O}_K) = 1$.

Let A be a $\mathbb{Q}(\psi_\mathfrak{n})$-algebra and let $L(\mathbf{n}, A)$ be the space of homogeneous polynomials of degree n in $\mathbf{x} = (X, Y)$ and degree n in $\bar{\mathbf{x}} = (\overline{X}, \overline{Y})$ with coefficients in A. Then $L(\mathbf{n}, A)$ is a $\Gamma_{\mathfrak{a}_j}$-module, since for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathfrak{a}_j}$ we have $\gamma P(X, \overline{X}) = \psi_\mathfrak{n}(d) P(\gamma^T X, \bar{\gamma}^T \overline{X})$. Let $\widetilde{L(\mathbf{n}, A)}$ be the sheaf of locally constant sections of the projection $\Gamma_{\mathfrak{a}_j}\backslash\mathfrak{H}_3 \times L(2n+2, A) \to \Gamma_{\mathfrak{a}_j}\backslash\mathfrak{H}_3$. The cuspforms on $\mathfrak{H}_3$ contribute to the cohomology of $\Gamma_{\mathfrak{a}_j}\backslash\mathfrak{H}_3$ in degree 1 and let $H^1_{\mathrm{cusp}}(\Gamma_{\mathfrak{a}_j}\backslash\mathfrak{H}_3, \widetilde{L(\mathbf{n}, \mathbb{C})})$ be the cuspidal cohomology group. This is the subgroup spanned by cuspidal harmonic forms of the square integral cohomology group; see [11, Section 2] for more details. The cohomology is computed using the de Rham resolution. As K is complex quadratic, there are two isomorphisms δ_1, δ_2 that generalize the Eichler–Shimura isomorphism ([9, Section 3]). We are interested in

$$\delta_1 \colon \ \mathcal{S}_{\mathbf{n}}(\Gamma_{\mathfrak{a}_j}, \psi_{\mathfrak{n}}) \to H^1_{\text{cusp}}(\Gamma_{\mathfrak{a}_j} \backslash \mathfrak{H}_3, \widetilde{L(\mathbf{n}, \mathbb{C})}),$$

which realizes cuspforms on $\mathfrak{H}_3$ as differential 1-forms. In the sequel we simply label δ_1 by δ. The Hecke algebra acts naturally on both spaces above, and the isomorphism δ is Hecke equivariant. Thus, we can consider $\delta(f_j)$ to be a differential form on $\mathfrak{H}_3$ that takes values in the sheaf $L(\mathbf{n}, \mathbb{C})$. In [11, Section 2.5] this isomorphism is given explicitly.

By the Clebsch-Gordan formula there is an injection $\Phi \colon W(2n{+}2, \mathbb{C}) \to L(\mathbf{n}, \mathbb{C}) \otimes L(2, \mathbb{C})$ and we define $\delta(f_j)(g) := g \cdot \big(\Phi \circ f_j(g)\big)$. As in [7, Section 5.1], $\delta(f_j)$ is $SU_2(\mathbb{C})$-invariant. Note that the action of g on $L(\mathbf{n}, \mathbb{C})$ is as above whereby the action on $L(2, \mathbb{C})$ is given by $gP(A, B) = P(j(g^{-1}, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))^{T-1}(A, B))$. This notion is made completely explicit in [6] which we summarize for our applications to the special value results. From [11, Section 11] we have that, as an $SL_2(\mathbb{Z})$-module, $L(\mathbf{n}, \mathbb{C})$ is not irreducible, and so as $SL_2(\mathbb{Z})$-modules we have the decompositions

$$\begin{aligned} L(\mathbf{n}, \mathbb{C}) &\cong \bigoplus_{m=0}^{n} L(2n-2m, \mathbb{C}) \\ P(\mathbf{x}, \overline{\mathbf{x}}) &\to \bigoplus_{m=0}^{n} \frac{1}{m!^2} \Big(\frac{\partial^2}{\partial X \partial \overline{Y}} - \frac{\partial^2}{\partial \overline{X} \partial Y} \Big)^m P(\mathbf{x}, \overline{\mathbf{x}}) \Big|_{\substack{\overline{X}=X \\ \overline{Y}=Y}}. \end{aligned}$$

This gives the decomposition

$$H^1_{\text{cusp}}\big(\Gamma_0(N) \backslash \mathfrak{H}_1, \widetilde{L(\mathbf{n}, \mathbb{C})}\big) \cong \bigoplus_{m=0}^{n} H^1_{\text{cusp}}\big(\Gamma_0(N) \backslash \mathfrak{H}_1, L(\widetilde{2n-2m}, \mathbb{C})\big),$$

where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

Therefore restricting $\delta(f_j)$ to $\mathfrak{H}_1$, we have the decomposition

$$\delta(f_j)\big|_{\mathfrak{H}_1} = \bigoplus_{m=0}^{n} \delta_{2n-2m}(f_j).$$

In the sequel, we fix f_j to be f_1, and when we consider a cuspform on $\mathfrak{H}_3$ we simply write f for f_1. We let $\mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}), \psi_{\mathfrak{n}})$ denote the space of cuspforms on $\mathfrak{H}_3$ of weight $\mathbf{n}$, level $\mathfrak{n}$, and nebentype $\psi_{\mathfrak{n}}$. Note that for $j = 1$ we have $a_1 = 1$, and so $\mathfrak{a}_1 = \mathcal{O}_K$. Thus, $\Gamma_{\mathfrak{a}_1} = \Gamma_0(\mathfrak{n})$, and therefore $\Gamma_{\mathfrak{a}_1} \cap GL_2^+(\mathbb{Q}) = \Gamma_0(N)$, where $N \in \mathbb{Z}_{>0}$ is the generator of the ideal $\mathfrak{n} \cap \mathbb{Z}$.

Let f^α be a component of f as a $W(2n+2, \mathbb{C})$-valued function, so

$$f(z) = \sum_{j=0}^{2n+2} f^\alpha(z) S^{2n+2-\alpha} T^\alpha.$$

Let $\psi(X, Y, \overline{X}, \overline{Y}, A, B) = (\psi_0, \ldots, \psi_{2n+2})^T$, where

$$\psi_\alpha(X, Y, \overline{X}, \overline{Y}, A, B) = (-1)^\alpha \frac{A^2 c_\alpha - 2ABc_{\alpha-1} + B^2 c_{\alpha-2}}{\binom{2n+2}{\alpha}}$$

and

$$c_\alpha(X,Y,\overline{X},\overline{Y}) = \sum_{\substack{j,k=0\\ n-(j-k)=\alpha}}^{n} (-1)^k \binom{n}{k}\binom{n}{j} X^{n-k}\overline{X}^{n-j}Y^k\overline{Y}^j.$$

Then from [7],

$$\delta(f)|_{\mathfrak{H}_1}(z) = \sum_{\alpha=0}^{2n+2} \sqrt{y}^{2n+2} f^\alpha(z)\psi_\alpha\Big(\frac{1}{\sqrt{y}}X, \sqrt{y}Y, \frac{1}{\sqrt{y}}\overline{X}\sqrt{y}\overline{Y}, \frac{1}{\sqrt{y}}A, \frac{1}{\sqrt{y}}B\Big).$$

Let $\widetilde{\delta(f)}|_{\mathfrak{H}_1} = \left(\begin{smallmatrix}1 & -x\\ 0 & 1\end{smallmatrix}\right)\delta(f)|_{\mathfrak{H}_1}$. This differential form is simpler than $\delta(f)|_{\mathfrak{H}_1}$, as it amounts to setting $x = 0$. This occurs in the evaluation of the relevant integral representation in [7] and these terms are used to evaluate the archimedean factors, so we give this differential form explicitly. We have

$$\widetilde{\delta(f)}|_{\mathfrak{H}_1}(z) = \sum_{\alpha=0}^{2n+2} f^\alpha(z)\psi_\alpha(X, yY, \overline{X}, y\overline{Y}, A, B).$$

For $z = x + iy \in \mathfrak{H}_1$ let

$$g^\alpha(z) = \begin{cases} \dfrac{f^\alpha(z) + (-1)^{n-m+1-\alpha} f^{2n+2-\alpha}(z)}{\binom{2n+2}{\alpha}} & \alpha = 0,1,\dots,n,\\[2ex] \dfrac{f^{n+1}(z)}{\binom{2n+2}{n+1}} & \alpha = n+1,\end{cases}$$

and following [7, Section 5.2] let

(2.2)

$$a(m,\ell,\alpha) = \binom{n}{m}^2 (-1)^{\frac{n+\alpha-\ell-m}{2}} \sum_{p=0}^{m} (-1)^p \binom{m}{p}$$
$$\left(\binom{n-m}{\frac{n-m-\ell+\alpha}{2}-p}\binom{n-m}{\frac{3n-3m-\ell-\alpha}{2}+p} + \binom{n-m}{\frac{n-m-\ell+\alpha-2}{2}-p}\binom{n-m}{\frac{3n-3m-\ell-\alpha+2}{2}+p}\right)$$
$$b(m,\ell,\alpha) = \binom{n}{m}^2 (-1)^{\frac{n+\alpha-\ell-m-1}{2}} \sum_{p=0}^{m} (-1)^p \binom{m}{p}$$
$$\left(\binom{n-m}{\frac{n-m-\ell+\alpha-1}{2}-p}\binom{n-m}{\frac{3n-3m-\ell-\alpha+1}{2}+p}\right).$$

Therefore from [7, (24)], for $z \in \mathfrak{H}_1$ we have

$$\widetilde{\delta_{2n-2m}(f)}(z) = \sum_{\ell=0}^{2n-2m}\Big[\Big(\sum_{\alpha=0}^{n+1}(-1)^\alpha g^\alpha(z)a(m,\ell,\alpha)\Big)\,dx + 2\Big(\sum_{\alpha=0}^{n+1}(-1)^\alpha g^\alpha(z)b(m,\ell,\alpha)\Big)\,dy\Big]\, y^{2n-m-\ell}X^\ell Y^{2n-2m-\ell}$$

and $\widetilde{\delta_{2n-2m}(f)} \in H^1_{\text{cusp}}(\Gamma_0(N)\backslash\mathfrak{H}_1, \widetilde{L(2n-2m,\mathbb{C})})$.

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3 Integral Representations of the Twisted Tensor L-Function

In this section we recall the integral representations of $L_T(s, f)$ developed in [7] and give a brief overview of their arithmetic properties following [7,11]. We prove Galois equivariance of the values obtained in [7]. The details of the (partial) evaluations of these integrals are in [7, Sections 6 and 7].

Let $f \in \mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}), \psi_{\mathfrak{n}})$ be a normalized primitive form as in Section 2 and let

$$\psi_N = \psi_{\mathfrak{n}}|_{\mathbb{Q}} \cdot |\ |_{\mathbb{J}}^{2n+2v_i+2v_c} \colon \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times.$$

We regard ψ_N as a Dirichlet character $\psi_N \colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ and note that we define ψ_N as the inverse ψ_N^{-1} of the Dirichlet character in [7]. Let $L_N(s, \psi_N)$ denote the Dirichlet L-function attached to ψ_N. The twisted tensor L-function is defined by

$$L_T(s, f) = L_N(2s - 2w, \psi_N) \sum_{k=1}^{\infty} \frac{c_f(k)}{k^s}$$

where the coefficients $c_f(k)$ are from (2.1). The L-function has an Euler product of the form $L_T(s, f) = \prod_{p \text{ prime}} L_p(s, f)$ where

$$L_p(s, f)^{-1} = \begin{cases} (1 - \alpha_{\mathfrak{p}}\alpha_{\overline{\mathfrak{p}}}p^{-s})(1 - \alpha_{\mathfrak{p}}\beta_{\overline{\mathfrak{p}}}p^{-s})(1 - \beta_{\mathfrak{p}}\alpha_{\overline{\mathfrak{p}}}p^{-s})(1 - \beta_{\mathfrak{p}}\beta_{\overline{\mathfrak{p}}}p^{-s}) & p = \mathfrak{p}\overline{\mathfrak{p}}, \\ (1 - \alpha_{\mathfrak{p}}p^{-s})(1 - \psi_{\mathfrak{n}}|_{\mathbb{Q}}(p)p^{-2s+2})(1 - \beta_{\mathfrak{p}}p^{-s}) & p = \mathfrak{p}, \\ (1 - \alpha_{\mathfrak{p}}^2 p^{-s})(1 - \psi_{\mathfrak{n}}(\mathfrak{p})p^{-s+1})(1 - \beta_{\mathfrak{p}}^2 p^{-s}) & p = \mathfrak{p}^2. \end{cases}$$

The L-function $L_T(s, f)$ has a meromorphic continuation and functional equation from [20]. All critical values of $L_T(s, f)$ in the right half of the critical strip were treated in [7]. Further, a functional equation was proved in [6, Theorem 7.2], and a motivic interpretation of $L_T(s, f)$ was given in [7].

For $z \in \mathfrak{H}_1$ consider the Eisenstein series

$$E_{2n-2m+2}(s, z, \psi_N) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\psi_N(d)}{(cz + d)^{2n-2m+2}|cz + d|^{2s}} y^s \omega,$$

where $\omega = (X - zY)^{2n-2m} dz$ and $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \mid n \in \mathbb{Z}\}$. The relevant arithmetic properties of these Eisenstein series are summarized in [11, Section 10]. From [11, Equation 3.1b], for a $\mathbb{Q}$-algebra A we consider the pairing

$$\langle \cdot, \cdot \rangle \colon L(n, A) \otimes L(n, A) \longrightarrow A$$

defined by

$$\Big\langle \sum_{\ell=0}^{n} a_\ell X^{n-\ell} Y^\ell, \sum_{\ell=0}^{n} b_\ell X^{n-\ell} Y^\ell \Big\rangle = \sum_{\ell=0}^{n} (-1)^\ell a_\ell b_{n-\ell} \binom{n}{\ell}^{-1}.$$

From [11], the pairing is $SL_2(\mathbb{Z})$-invariant and it follows that

$$\big\langle \delta_{2n-2m}(f), E_{2n-2m+2}(s, \cdot, \psi_N) \big\rangle$$

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is an integrable function on $\Gamma_0(N)\backslash\mathfrak{H}_1$. The main objects of investigation in [6,7] are the integrals of these functions. In particular, the following integral representation was determined.

Theorem 3.1 ([7, Equation (29)]) *For $m \equiv \nu_i + \nu_c \pmod 2$, $w = n + 1 + \nu_i + \nu_c$, and ψ_N primitive or trivial we have*

$$\begin{aligned}(3.1)\quad \int_{\Gamma_0(N)\backslash\mathfrak{H}_1} &\big\langle \delta_{2n-2m}(f), E_{2n-2m+2}(s,\,\cdot\,,\psi_N)\big\rangle \\ &= \frac{\sqrt{D}^{s+n-m+w+1} i^{\nu_i-\nu_c}}{(2\pi)^{s+2n+2-m} L_N(2s+2n-2m+2,\psi_N)} \\ &\qquad \times L_\infty(s-m,f) L_T(s+n-m+w+1,f),\end{aligned}$$

where

$$(3.2)\quad L_\infty(s-m,f) = \sum_{\substack{\alpha=0\\ \alpha\equiv n+1+m \pmod 2}}^{2n+2} c(m,\alpha)\Gamma\Big(\frac{s+n+1-m+\alpha}{2}\Big)\Gamma\Big(\frac{s+3n+3-m-\alpha}{2}\Big)$$

and

$$c(m,\alpha) = \frac{(-1)^{n+1}}{2}\sum_{\substack{\ell=0\\ \ell \text{ even}}}^{2n-2m} i^\ell\big(a(m,\ell-1,\alpha)-2b(m,\ell,\alpha)\big).$$

The archimedean factors $L_\infty(s-m,f)$ were not computed in [7]. However, the following conjecture was formulated.

Conjecture 3.2 ([7, Conjecture 1]) *Let $\epsilon \in \{0,1\}$ be so that $\epsilon \equiv n+1+m \pmod 2$. Then*

$$L_\infty(s-m,f) = c_{n,m}\cdot P_{n,m}(s)\cdot \frac{\Gamma(s+2n-m+2)}{\Gamma(s+2n-2m+2)}\Gamma\Big(\frac{s+n+1-m+\epsilon}{2}\Big)^2,$$

where

$$\begin{aligned}P_{n,m}(s) &= (s+1)(s+3)(s+5)\cdots(s+n-m-\epsilon)\\ &\qquad\times\Big(\frac{s}{2}+n-m\Big)\Big(\frac{s}{2}+n-m-1\Big)\cdots\Big(\frac{s}{2}+n-m-\frac{n-m-1-\epsilon}{2}\Big)\end{aligned}$$

is a polynomial in s and

$$c_{n,m} = \frac{(-1)^m\cdot n!^2}{m!^2\cdot P_{n,m}(0)\cdot\Big(\frac{n-m-1+\epsilon}{2}\Big)!^2}.$$

Let $c_{n,n} = (-1)^n$ and $P_{n,n}(s) = 1$.

Note that a footnote on [7, p. 630] states that Conjecture 3.2 was subsequently proven by H. Ochiai. However, the proof does not appear in the literature. Therefore, we provide Ochiai's proof in the appendix as well as our alternate proof in Section 4.

Using well-known properties of the gamma function, for $m \neq n$ we can rewrite this conjecture as

$$L_\infty(s-m,f) = \frac{(-1)^m\sqrt{\pi}\binom{n}{m}^2}{2^s} \frac{\Gamma\left(\frac{s}{2}+n-m+1\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)} \frac{\Gamma(s+n-m+1)\Gamma(s+2n-m+2)}{\Gamma(s+2n-2m+2)}.$$

Note that restricting this expression to the $m = n$ case we get

$$\frac{(-1)^n\sqrt{\pi}}{2^s} \frac{\Gamma\left(\frac{s}{2}+1\right)\Gamma(s+1)\Gamma(s+n+2)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma(s+2)}.$$

Multiplying the top and bottom of the latter expression by $\Gamma\left(\frac{s}{2}+1\right)$, applying the duplication formula $\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2x}\Gamma(2x)$ in the denominator, and simplifying gives

$$L_\infty(s-n,f) = \frac{(-1)^n\Gamma(s+n+2)\Gamma\left(\frac{s}{2}+1\right)^2}{\Gamma(s+2)}.$$

These are the factors suggested by Conjecture 3.2 in the $m = n$ case. Also note that, using our notation, the $L_\infty(s-m,f)$ factor here differs by the factor $\Gamma(s+2n-2m+2)$ from the same symbol employed in [7].

Although the archimedean factors were not completely determined, the integral was explicit enough to determine some arithmetic properties of $L_T(s,f)$ in the following way. For $\mathfrak{m} \subset \mathcal{O}_K$ let $T(\mathfrak{m})$ be a Hecke operator as in [20, Section 4] and then $T(\mathfrak{m})f = \lambda_f(T(\mathfrak{m}))f$, where λ_f is the Hecke algebra character corresponding to f. Let $E = \mathbb{Q}(f,\psi_\mathfrak{n})$. From [11, Section 8] we have the rank-1 free modules over E given by

$$\mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}),\psi_\mathfrak{n})[\lambda_f] = E\cdot f$$
$$H^1_{\mathrm{cusp}}(\Gamma_0(\mathfrak{n})\backslash\mathfrak{H}_3,\widetilde{L(\mathbf{n},E)})[\lambda_f] = E\cdot\eta(f),$$

where $\eta(f)$ is a rational Hecke eigen-differential form. From the Eichler-Shimura isomorphism δ we can define a period $\Omega_{\eta(f)}(f)$ by $\delta(f) = \Omega_{\eta(f)}(f)\eta(f)$. Note that the period depends on the choice of basis of the above cohomology group. However, the period is independent of the basis up to multiplication by a nonzero element of E. As the arithmetic results of Theorem 1.4 are only determined up to multiplication by such an element, we can denote the period simply by $\Omega(f)$. As in Section 2 we have the decomposition of the restriction to $\mathfrak{H}_1$, $\eta(f)|_{\mathfrak{H}_1} = \bigoplus_{m=0}^n \eta_{2n-2m}(f)$, where we have the rational forms

$$\eta_{2n-2m}(f) \in H^1_{\mathrm{cusp}}\left(\Gamma_0(N)\backslash\mathfrak{H}_1, L(\widetilde{2n-2m},E)\right).$$

Thus, we have $\delta_{2n-2m}(f) = \Omega(f)\eta_{2n-2m}(f)$, and so

$$\begin{aligned} I_1(f,\psi_N) &= \int_{\Gamma_0(N)\backslash\mathfrak{H}_1} \langle \delta_{2n-2m}(f), E_{2n-2m+2}(0,\cdot,\psi_N)\rangle \\ &= \Omega(f)\int_{\Gamma_0(N)\backslash\mathfrak{H}_1} \langle \eta_{2n-2m}(f), E_{2n-2m+2}(0,\cdot,\psi_N)\rangle. \end{aligned}$$

The arithmeticity of the latter integral basically follows from [11, Equation 5.3]. From [7, Lemma 5], and [11, Equation 5.3], composing the cup product with the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ maps

$$H^1_{\text{cusp}}(\Gamma_0(N)\backslash\mathfrak{H}_1, L(\widetilde{2n-2m}, E)) \otimes H^1(\Gamma_0(N)\backslash\mathfrak{H}_1, L(\widetilde{2n-2m}, E)) \tag{3.3}$$

to $H^2_c(\Gamma_0(N)\backslash\mathfrak{H}_1, E)$, where H^2_c means those forms with compact support. By definition, we have $\eta_{2n-2m}(f) \in H^1_{\text{cusp}}(\Gamma_0(N)\backslash\mathfrak{H}_1, L(2n-2m, E))$ and from [7, Proposition 4] we have

$$E_{2n-2m+2}(0, z, \psi_N) \in H^1\big(\Gamma_0(N)\backslash\mathfrak{H}_1, L(\widetilde{2n-2m}, E)\big).$$

It follows that

$$\big\langle \eta_{2n-2m}(f), E_{2n-2m+2}(0,\,\cdot\,,\psi_N)\big\rangle \in H^2_c\big(\Gamma_0(N)\backslash\mathfrak{H}_1, E\big),$$

and therefore we have the arithmetic result

$$I_2(f,\psi_N) = \int_{\Gamma_0(N)\backslash\mathfrak{H}_1} \big\langle \eta_{2n-2m}(f), E_{2n-2m+2}(0,\,\cdot\,,\psi_N)\big\rangle \in E.$$

We then obtain arithmetic results on certain values of $L_T(s,f)$ from integral (3.1), assuming the nonvanishing of the integral. The nonvanishing was shown in [7] by indirect methods using an argument due to Hida. Of course, it follows trivially from Conjecture 3.2. Given $L_\infty(-m,f) \neq 0$, the special value results of [7] (and some of the results of Theorem 1.4) are obtained as follows. Recall that for k even and ψ_N a primitive even Dirichlet character we have $L_N(k,\psi_N)/\pi^k\,\mathfrak{g}(\psi_N) \in \mathbb{Q}(\psi_N)$, where $\mathfrak{g}(\psi_N) = \sum_{a \pmod N} \psi_N(a)e^{2\pi i a/N}$ is a Gauss sum. For ψ_N primitive and nontrivial, the integral representation of Theorem 1.1 gives the first special values result of Theorem 1.4 directly, and this is the main result of [7].

To obtain Galois equivariance of the values, we first establish Galois equivariance of the integrals $I_2(f,\psi_N)$. The equivariance essentially follows, because the cohomological pairing is equivariant.

Lemma 3.3 *For $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we have $I_2(f,\psi_N)^\sigma = I_2(f^\sigma,\psi_N^\sigma)$.*

Proof The pairing of (3.3) is obtained by cup product and $\langle\,\cdot\,,\,\cdot\,\rangle$, which induces Poincaré duality on cohomology. From [14] such a pairing is Galois equivariant, so we have

$$\big\langle \eta_{2n-2m}(f), E_{2n-2m+2}(0,\,\cdot\,,\psi_N)\big\rangle^\sigma = \big\langle \eta_{2n-2m}(f)^\sigma, E_{2n-2m+2}(0,\,\cdot\,,\psi_N)^\sigma\big\rangle.$$

From the definition of the Eisenstein series we have

$$E_{2n-2m+2}(0,\,\cdot\,,\psi_N)^\sigma = E_{2n-2m+2}(0,\,\cdot\,,\psi_N^\sigma),$$

and from the rational structure defining $\eta_{2n-2m}(f)$ we have

$$\eta_{2n-2m}(f)^\sigma = \eta_{2n-2m}(f^\sigma).$$

From [16, Section 3], by the definition of the action of σ on de Rham cohomology classes, for any $\omega \in H^d_c(m,\mathbb{C})$ we have $\big(\int_M \omega\big)^\sigma = \int_M \omega^\sigma$. As the integrand of

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$I_2(f,\psi_N)$ is in $H_c^2(\Gamma_0(N)\backslash \mathfrak{H}_1, E)$, we can put all of these results together and compute

$$\begin{aligned} I_2(f,\psi_N)^\sigma &= \Big(\int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \big\langle \eta_{2n-2m}(f), E_{2n-2m+2}(0,\,\cdot\,,\psi_N)\big\rangle\Big)^\sigma \\ &= \int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \big\langle \eta_{2n-2m}(f), E_{2n-2m+2}(0,\,\cdot\,,\psi_N)\big\rangle^\sigma \\ &= \int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \big\langle \eta_{2n-2m}(f)^\sigma, E_{2n-2m+2}(0,\,\cdot\,,\psi_N)^\sigma\big\rangle \\ &= \int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \big\langle \eta_{2n-2m}(f^\sigma), E_{2n-2m+2}(0,\,\cdot\,,\psi_N^\sigma)\big\rangle = I_2(f^\sigma,\psi_N^\sigma). \qquad\blacksquare \end{aligned}$$

Therefore, from Lemma 3.3,

$$\Big(\frac{I_1(f,\psi_N)}{\Omega(f)}\Big)^\sigma = I_2(f,\psi_N)^\sigma = I_2(f^\sigma,\psi_N^\sigma) = \frac{I_1(f^\sigma,\psi_N^\sigma)}{\Omega(f^\sigma)}.$$

Thus, we obtain Galois equivariance for the special values of $L_T(s,f)$ at the even critical points. For ψ_N trivial we have $L_N(s,\psi_N) = \zeta(s)\prod_{p|N}(1-p^{-s})$. As $\zeta(k)/\pi^k \in \mathbb{Q}$ for k even, then from the integral representation of Theorem 1.1 and the functional equation of Theorem 1.3 (proved in Section 5) we get the special values at the odd critical points as in Theorem 1.4. Galois equivariance follows as above. The proof of Theorem 1.1 is completed in the following section.

4 Hypergeometric Series and Archimedean Factors

In the sequel we fix n and m for a given f. The main result of this section is the following theorem.

Theorem 4.1 *Conjecture 3.2 is true.*

Applying this result to Theorem 3.1 we get Theorem 1.1. As an immediate consequence we have that

$$L_\infty(-m,f) = (-1)^m \frac{n!^2}{m!^2}\frac{(2n-m+1)!}{(2n-2m+1)!} \neq 0.$$

Although somewhat lengthy, the proof does not require deep machinery but follows from adroit use of gamma function and hypergeometric series identities. We evaluate

$$\sum_{\substack{\alpha=0\\ \alpha\equiv n+1+m \pmod 2}}^{2n+2} c(m,\alpha)\Gamma\Big(\frac{s+n-m+1+\alpha}{2}\Big)\Gamma\Big(\frac{s+3n-m+3-\alpha}{2}\Big) \tag{4.1}$$

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using notation similar to that of [7, (30)]. Note that as $\alpha \equiv n+1+m \pmod 2$, $3n-m+\alpha+1$ is even. Setting $\ell = 2j$, from (2.2) and (3.2) we can write

$$\begin{aligned} c(m,\alpha) &= \frac{(-1)^{n+1}}{2}\sum_{j=0}^{n-m}(-1)^j\big(a(m,2j-1,\alpha)-2b(m,2j,\alpha)\big)\\ &= \frac{(-1)^{\frac{3n-m+3+\alpha}{2}}}{2}\binom{n}{m}^2\sum_{j=0}^{n-m}\Bigg[\sum_{p=0}^{m}(-1)^p\binom{m}{p}\Bigg(\binom{n-m}{\frac{n-m-2j+1+\alpha}{2}-p}\binom{n-m}{\frac{3n-3m-2j+1-\alpha}{2}+p}\\ &\quad+\binom{n-m}{\frac{n-m-2j-1+\alpha}{2}-p}\binom{n-m}{\frac{3n-3m-2j+3-\alpha}{2}+p}\\ &\quad+2\binom{n-m}{\frac{n-m-2j+\alpha-1}{2}-p}\binom{n-m}{\frac{3n-3m-2j-\alpha+1}{2}+p}\Bigg)\Bigg]. \end{aligned}$$

Note that α has the same parity as $n+m+1$ and so the same as $n-m+1$, $n-m-1$, $3n-3m+1$, and $3n-3m+3$. Switching summations, applying the Vandermonde convolution $\sum_{j=0}^{N}\binom{N}{a+j}\binom{N}{b-j}=\binom{2N}{a+b}$ and then $\binom{N}{a}+\binom{N}{a-1}=\binom{N+1}{a}$ three times each to the expression for $c(m,\alpha)$, we get that (4.1) is

$$\begin{aligned} \binom{n}{m}^2 \sum_{\substack{\alpha=0\\ \alpha\equiv n+1+m \pmod 2}}^{2n+2} \frac{(-1)^{\frac{3n-m+3+\alpha}{2}}}{2}\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{2n-2m+2}{\alpha-2p}\\ \times\Gamma\Big(\frac{s+n-m+\alpha+1}{2}\Big)\Gamma\Big(\frac{s+3n-m-\alpha+3}{2}\Big). \end{aligned}$$

Note that this is [6, (6.5)].

For simplicity, we make the substitution $\alpha = 2\beta+\epsilon$. Thus, (4.1) is

$$\begin{aligned} \frac{(-1)^{\frac{3n-m+3+\epsilon}{2}}}{2}\binom{n}{m}^2\sum_{\beta=0}^{n+1-\epsilon}(-1)^\beta\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{2n-2m+2}{2\beta+\epsilon-2p}\\ \times\Gamma\Big(\frac{n-m+1+\epsilon}{2}+\frac{s}{2}+\beta\Big)\Gamma\Big(\frac{3n-m+3-\epsilon}{2}+\frac{s}{2}-\beta\Big). \end{aligned}\tag{4.2}$$

Let $s\in\mathbb{C}$ so that $\mathrm{Re}(s) < -2(n-m)-1$. From the reflection formula

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(x\pi)}$$

we can write

$$\Gamma\Big(\frac{n-m+1+\epsilon}{2}+\frac{s}{2}+\beta\Big)=\frac{\pi}{\sin\big(\big(\frac{n-m+1+\epsilon}{2}+\frac{s}{2}+\beta\big)\pi\big)}\frac{1}{\Gamma\big(-\frac{n-m-1+\epsilon}{2}-\frac{s}{2}-\beta\big)}.$$

For $a\in\mathbb{Z}$ we have $\sin((x+a)\pi)=(-1)^a\sin(x\pi)$, and thus the summation in (4.2) is

$$\frac{(-1)^{\frac{n-m+1+\epsilon}{2}}\pi}{\sin\big(\frac{s}{2}\pi\big)}\sum_{\beta=0}^{n+1-\epsilon}\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{2n-2m+2}{2\beta+\epsilon-2p}\frac{\Gamma\big(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta\big)}{\Gamma\big(-\frac{n-m-1+\epsilon}{2}-\frac{s}{2}-\beta\big)}.\tag{4.3}$$

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Note that $-\frac{n-m-1+\epsilon}{2} - \frac{\mathrm{Re}(s)}{2} \geq 0$, and so (4.3) is

(4.4)

$$\begin{aligned}
&\frac{(-1)^{\frac{n-m+1+\epsilon}{2}}\pi}{\sin\left(\frac{s}{2}\pi\right)} \sum_{p=0}^{m}(-1)^p\binom{m}{p}\sum_{\beta=0}^{n+1-\epsilon}\binom{2n-2m+2}{2\beta+\epsilon-2p}\frac{\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta\right)}{\Gamma\left(-\frac{n-m-1+\epsilon}{2}-\frac{s}{2}-\beta\right)} \\
&= \frac{(-1)^{\frac{n-m+1+\epsilon}{2}}\pi}{\sin\left(\frac{s}{2}\pi\right)} \sum_{p=0}^{m}(-1)^p\binom{m}{p}\sum_{\beta=p}^{n+1-\epsilon-m+p}\binom{2n-2m+2}{2\beta+\epsilon-2p}\frac{\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta\right)}{\Gamma\left(-\frac{n-m-1+\epsilon}{2}-\frac{s}{2}-\beta\right)} \\
&= \frac{(-1)^{\frac{n-m+1+\epsilon}{2}}\pi}{\sin\left(\frac{s}{2}\pi\right)} \sum_{p=0}^{m}(-1)^p\binom{m}{p}\sum_{\beta=0}^{n+1-\epsilon-m}\binom{2n-2m+2}{2\beta+\epsilon} \\
&\qquad\qquad \times \frac{\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta-p\right)}{\Gamma\left(-\frac{n-m-1+\epsilon}{2}-\frac{s}{2}-\beta-p\right)} \\
&= \frac{(-1)^{\frac{n-m+1+\epsilon}{2}}\pi}{\sin\left(\frac{s}{2}\pi\right)} \sum_{\beta=0}^{n+1-\epsilon-m}\binom{2n-2m+2}{2\beta+\epsilon}\sum_{p=0}^{m}(-1)^p\binom{m}{p} \\
&\qquad\qquad \times \frac{\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta-p\right)}{\Gamma\left(-\frac{n-m-1+\epsilon}{2}-\frac{s}{2}-\beta-p\right)}.
\end{aligned}$$

The inner sum of (4.4) can be written

$$\begin{aligned}
(4.5)\quad &\sum_{p=0}^{m}(-1)^p\binom{m}{p}\frac{\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta-p\right)}{\Gamma\left(-\frac{n-m-1+\epsilon}{2}-\frac{s}{2}-\beta-p\right)} = \\
&\qquad m!\sum_{p=0}^{m}(-1)^p\frac{\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta-p\right)}{p!(m-p)!\Gamma\left(-\frac{n-m-1+\epsilon}{2}-\frac{s}{2}-\beta-p\right)}.
\end{aligned}$$

Recall that for $p \in \mathbb{Z}_{\geq 0}$ the Pochhammer symbol is $(a)_p = \Gamma(a+p)/\Gamma(a)$. A hypergeometric series is defined by

$$ {}_mF_n(a_1,\ldots,a_m;b_1,\ldots,b_n;x) = \sum_{p=0}^{\infty}\frac{(a_1)_p\cdots(a_m)_p}{(b_1)_p\cdots(b_n)_p}\frac{x^p}{p!}.$$

If one of the factors in the numerators of the terms in the summand is $(-a)_p$ for $-a$ a negative integer and none of the factors in the denominators is the Pochhammer symbol of a negative integer, then the index of such a sum will range from 0 to a.

We have

$$\Gamma(a+1-p) = \frac{(-1)^p\Gamma(a+1)}{(-a)_p}.$$

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Applying this and simplifying, we can rewrite (4.5) as

$$\frac{\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta\right)}{\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}-\beta\right)}\sum_{p=0}^{m}\frac{(-m)_p\left(\frac{n-m+1+\epsilon}{2}+\frac{s}{2}+\beta\right)_p}{\left(-\frac{3n+1-m-\epsilon}{2}-\frac{s}{2}+\beta\right)_p p!}.$$

The factor $(-m)_p$ is in the numerator, and it follows that the summation above can be expressed as an ${}_2F_1$-hypergeometric series. Thus, (4.5) is

$$\frac{\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta\right)}{\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}-\beta\right)} {}_2F_1\left(-m,\frac{n-m+1+\epsilon}{2}+\frac{s}{2}+\beta;-\frac{3n+1-m-\epsilon}{2}-\frac{s}{2}+\beta;1\right). \tag{4.6}$$

Gauss's Theorem for ${}_2F_1$-hypergeometric series, from [1, Theorem 2.2.2, p. 66] for example, states that for $\mathrm{Re}(c-a-b)>0$ we have

$$ {}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{4.7}$$

Taking $a=-m$, $b=\frac{n-m+1+\epsilon}{2}+\frac{s}{2}+\beta$ and $c=-\frac{3n+1-m-\epsilon}{2}-\frac{s}{2}+\beta$ we see that $c-a-b=-2n+2m-1-\mathrm{Re}(s)>0$ by our condition on s. Thus, (4.7) applies to (4.6), and we get

$$\frac{\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta\right)\Gamma\left(-\frac{3n+1-m-\epsilon}{2}-\frac{s}{2}+\beta\right)\Gamma(-2n+2m-1-s)}{\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}-\beta\right)\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}+\beta\right)\Gamma(-2n+m-1-s)}. \tag{4.8}$$

The reflection identity gives

$$\Gamma\left(\frac{3n+3-m-\epsilon}{2}+\frac{s}{2}-\beta\right)\Gamma\left(-\frac{3n+1-m-\epsilon}{2}-\frac{s}{2}+\beta\right)=\frac{(-1)^{\frac{3n-m+3-\epsilon}{2}+\beta}\pi}{\sin\left(\frac{s}{2}\pi\right)},$$

and therefore (4.8) is

$$\frac{(-1)^{\frac{3n-m+3-\epsilon}{2}+\beta}\pi\Gamma(-2n+2m-1-s)}{\sin\left(\frac{s}{2}\pi\right)\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}-\beta\right)\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}+\beta\right)\Gamma(-2n+m-1-s)}.$$

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Applying this to (4.3) we get

(4.9)
$$\frac{\Gamma(-2n+2m-1-s)}{\Gamma(-2n+m-1-s)}\frac{(-1)^m\pi^2}{\sin^2\left(\frac{s}{2}\pi\right)}$$
$$\times\sum_{\beta=0}^{n+1-\epsilon-m}(-1)^\beta\binom{2n-2m+2}{2\beta+\epsilon}\frac{1}{\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}-\beta\right)\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}+\beta\right)}$$
$$=\frac{\Gamma(-2n+2m-1-s)}{\Gamma(-2n+m-1-s)}(2n-2m+2)!\frac{(-1)^m\pi^2}{\sin^2\left(\frac{s}{2}\pi\right)}$$
$$\times\sum_{\beta=0}^{n+1-\epsilon-m}(-1)^\beta\frac{1}{(2\beta+\epsilon)!(2n-2m+2-2\beta-\epsilon)!\times\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}-\beta\right)\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}+\beta\right)}.$$

As $\Gamma(a+\beta)=\Gamma(a)(a)_\beta$, we have $\Gamma(2\beta+\epsilon+1)=\Gamma(\epsilon+1)(\epsilon+1)_{2\beta}=(\epsilon+1)_{2\beta}$ and also
$$\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}+\beta\right)=\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)_\beta$$
$$\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}-\beta\right)=\frac{(-1)^\beta\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}\right)}{\left(\frac{n-m+1+\epsilon}{2}+\frac{s}{2}\right)_\beta}$$
$$\Gamma(2n-2m+3-\epsilon-2\beta)=\frac{(-1)^{2\beta}\Gamma(2n-2m+3-\epsilon)}{(-2n+2m-2+\epsilon)_{2\beta}}.$$

Thus, (4.9) is

$$\frac{(-1)^m\pi^2}{\sin^2\left(\frac{s}{2}\pi\right)}\frac{\Gamma(-2n+2m-1-s)\Gamma(2n-2m+3)}{\Gamma(-2n+m-1-s)\Gamma(2n-2m+3-\epsilon)\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}\right)\times\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)}$$
$$\times\sum_{\beta=0}^{n+1-\epsilon-m}\frac{(-2n+2m-2+\epsilon)_{2\beta}\left(\frac{n-m+1+\epsilon}{2}+\frac{s}{2}\right)_\beta}{(\epsilon+1)_{2\beta}\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)_\beta}. \tag{4.10}$$

As $(a)_{2\beta}=2^{2\beta}\left(\frac{a}{2}\right)_\beta\left(\frac{a+1}{2}\right)_\beta$, we have
$$(\epsilon+1)_{2\beta}=2^{2\beta}\left(\frac{\epsilon+1}{2}\right)_\beta\left(\frac{\epsilon+2}{2}\right)_\beta=2^{2\beta}\left(\epsilon+\frac{1}{2}\right)_\beta(1)_\beta$$
$$(-2n+2m-2+\epsilon)_{2\beta}=2^{2\beta}\left(-n+m-1+\frac{\epsilon}{2}\right)_\beta\left(-n+m-\frac{1}{2}+\frac{\epsilon}{2}\right)_\beta.$$

Thus, we can rewrite (4.10) as

$$\frac{(-1)^m\pi^2}{\sin^2\left(\frac{s}{2}\pi\right)}\frac{\Gamma(-2n+2m-1-s)\Gamma(2n-2m+3)}{\Gamma(-2n+m-1-s)\Gamma(2n-2m+3-\epsilon)\times\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}\right)\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)}$$
$$\times\sum_{\beta=0}^{n-m+1-\epsilon}\frac{\left(-n+m-1+\frac{\epsilon}{2}\right)_\beta\left(-n+m-\frac{1}{2}+\frac{\epsilon}{2}\right)_\beta\left(\frac{n-m+1+\epsilon}{2}+\frac{s}{2}\right)_\beta}{\left(\epsilon+\frac{1}{2}\right)_\beta\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)_\beta(1)_\beta}.$$

If $\epsilon=0$, then $(-n+m-1)_\beta=(-n+m-1+\epsilon)_\beta$ is in the numerators of the terms of the above summand, and if $\epsilon=1$, then $(-n+m)_\beta=(-n+m-1+\epsilon)_\beta$ is in the numerators. The denominators of the terms of the summand do not have factors of the form $(b_j)_\beta$ with b_j a negative integer. As the index β already ranges from 0 to $n-m+1-\epsilon$, it follows that the above sum can be written as a ${}_3F_2$-hypergeometric series. So (4.10) is

$$\frac{(-1)^m\pi^2}{\sin^2\left(\frac{s}{2}\pi\right)}\frac{\Gamma(-2n+2m-1-s)\Gamma(2n-2m+3)}{\Gamma(-2n+m-1-s)\Gamma(2n-2m+3-\epsilon)\times\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}\right)\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)}$$
$$\times{}_3F_2\Big(-n+m-1+\frac{\epsilon}{2},-n+m-\frac{1}{2}+\frac{\epsilon}{2},\frac{n-m+1+\epsilon}{2}+\frac{s}{2};\epsilon+\frac{1}{2},\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2};1\Big). \tag{4.11}$$

From [1, (2.2.11), p. 72] for example, Dixon's identity gives that for $\mathrm{Re}\left(\frac{a}{2}+b+c+1\right)>0$ we have

$$ {}_3F_2(a,-b,-c;a+b+1,a+c+1;1)=\frac{\Gamma\left(\frac{a}{2}+1\right)\Gamma(a+b+1)\Gamma(a+c+1)\Gamma\left(\frac{a}{2}+b+c+1\right)}{\Gamma(a+1)\Gamma\left(\frac{a}{2}+b+1\right)\Gamma\left(\frac{a}{2}+c+1\right)\Gamma(a+b+c+1)}. \tag{4.12}$$

Note that by the reflection identity, for $x,y>0$ we have

$$\frac{\Gamma(-x)}{\Gamma(-y)}=\frac{\Gamma(y+1)}{\Gamma(x+1)}\frac{\sin(\pi x)}{\sin(\pi y)}.$$

The latter expression can have proper values for integral x and y. Therefore in the sequel we abuse notation and write $\frac{\Gamma(-x)}{\Gamma(-y)}$ for $x,y\in\mathbb{Z}_{>0}$. We distinguish the cases $\epsilon=0$ and $\epsilon=1$. For $\epsilon=0$ let $a=-n+m-1$, $b=n-m+\frac{1}{2}$, and $c=\frac{-n+m-1}{2}-\frac{s}{2}$. Then (4.12) applies to (4.11), and in this case (4.11) becomes

$$\frac{(-1)^m\pi^2}{\sin^2\left(\frac{s}{2}\pi\right)}\frac{\Gamma(-2n+2m-1-s)\Gamma(2n-2m+3)}{\Gamma(-2n+m-1-s)\Gamma(2n-2m+3)\Gamma\left(\frac{-n+m+1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{-3n+3m-1}{2}-\frac{s}{2}\right)}$$
$$\times\frac{\Gamma\left(\frac{-n+m-1}{2}+1\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{-3n+3m-1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{\Gamma(-n+m)\Gamma\left(\frac{n}{2}-\frac{m}{2}+1\right)\Gamma\left(-n+m-\frac{s}{2}\right)\Gamma\left(\frac{-n+m}{2}-\frac{s}{2}\right)}.$$

For $\epsilon = 1$ let $a = -n + m$, $b = n - m + \frac{1}{2}$, and $c = \frac{-n+m-2}{2} - \frac{s}{2}$. As above, (4.12) applies to (4.11) in this case also. We get

$$\frac{(-1)^m \pi^2}{\sin^2\left(\frac{s}{2}\pi\right)} \frac{\Gamma(-2n+2m-1-s)\Gamma(2n-2m+3)}{\Gamma(-2n+m-1-s)\Gamma(2n-2m+2)\Gamma\left(\frac{-n+m}{2}-\frac{s}{2}\right)\Gamma\left(\frac{-3n+3m}{2}-\frac{s}{2}\right)} \times \frac{\Gamma\left(\frac{-n+m}{2}+1\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{-3n+3m}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{\Gamma(-n+m+1)\Gamma\left(\frac{n-m+3}{2}\right)\Gamma\left(-n+m-\frac{s}{2}\right)\Gamma\left(\frac{-n+m+1}{2}-\frac{s}{2}\right)}.$$

We can put these cases together and get that (4.3) is equal to

$$\frac{(-1)^m \pi^2}{\sin^2\left(\frac{s}{2}\pi\right)} \frac{\Gamma(-2n+2m-1-s)\Gamma(2n-2m+3)}{\Gamma(-2n+m-1-s)\Gamma(2n-2m+3-\epsilon) \times \Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}\right)\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)} \times \frac{\Gamma\left(\frac{-n+m+1+\epsilon}{2}\right)\Gamma\left(\frac{1}{2}+\epsilon\right)\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{\Gamma(-n+m+\epsilon)\Gamma\left(\frac{n-m+2+\epsilon}{2}\right)\Gamma\left(-n+m-\frac{s}{2}\right)\Gamma\left(\frac{-n+m+\epsilon}{2}-\frac{s}{2}\right)}. \tag{4.13}$$

It is a matter of using common gamma function identities to show that replacing the summation in (4.2) with (4.13) yields the result. For completeness, we give an outline of these simplifications. Applying the reflection identity several times as necessary gives the identities

$$\begin{aligned}
&\frac{(-1)^m \pi^2}{\sin^2\left(\frac{s}{2}\pi\right)\Gamma\left(\frac{-n+m+1-\epsilon}{2}-\frac{s}{2}\right)\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)} \\
&= (-1)^m \Gamma\left(\frac{n-m+1+\epsilon}{2}+\frac{s}{2}\right)\Gamma\left(\frac{3n-3m+3-\epsilon}{2}+\frac{s}{2}\right)\frac{\Gamma(-2n+2m-1-s)}{\Gamma(-2n+m-1-s)} \\
&= \frac{(-1)^m\Gamma(s+2n-m+2)}{\Gamma(s+2n-2m+2)} \frac{\Gamma\left(\frac{-3n+3m-1+\epsilon}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{\Gamma\left(-n+m-\frac{s}{2}\right)\Gamma\left(\frac{-n+m+\epsilon}{2}-\frac{s}{2}\right)} \\
&= \frac{\Gamma\left(\frac{s}{2}+n-m+1\right)\Gamma\left(\frac{s}{2}+\frac{n-m+2-\epsilon}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{3n-3m+3-\epsilon}{2}\right)}.
\end{aligned}$$

Also, from the reflection identity we have

$$\frac{\Gamma\left(\frac{-n+m+1+\epsilon}{2}\right)}{\Gamma(-n+m+\epsilon)} = (-1)^{\frac{n-m+1-\epsilon}{2}} 2 \frac{\Gamma(n-m+1-\epsilon)}{\Gamma\left(\frac{n-m+1-\epsilon}{2}\right)}.$$

Applying these to (4.13) and simplifying, we get

$$(-1)^{\frac{n-m+1-\epsilon}{2}} 2 \frac{\Gamma(n-m+1-\epsilon)}{\Gamma\left(\frac{n-m+1-\epsilon}{2}\right)} \frac{\Gamma\left(\frac{1}{2}+\epsilon\right)}{\Gamma\left(\frac{n-m+2+\epsilon}{2}\right)} \frac{\Gamma(2n-2m+3)}{\Gamma(2n-2m+3-\epsilon)} \times \frac{\Gamma(s+2n-m+2)\Gamma\left(\frac{s}{2}+\frac{n-m+1+\epsilon}{2}\right)\Gamma\left(\frac{s}{2}+\frac{n-m+2-\epsilon}{2}\right)}{\Gamma(s+2n-2m+2)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}. \tag{4.14}$$

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Applying the duplication formula $\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2x}\Gamma(2x)$ gives the identities

$$\Gamma\left(\frac{s}{2}+\frac{n-m+1+\epsilon}{2}\right)\Gamma\left(\frac{s}{2}+\frac{n-m+2-\epsilon}{2}\right) = \frac{\sqrt{\pi}\Gamma(s+n-m+1)}{2^{s+n-m}}$$

$$\Gamma\left(\frac{n-m+1-\epsilon}{2}\right)\Gamma\left(\frac{n-m+2+\epsilon}{2}\right) = \frac{\sqrt{\pi}\Gamma(n-m+1-\epsilon)(n-m+1)^{\epsilon}}{2^{n-m}}.$$

Putting these into (4.14) and simplifying, we get that (4.3) is

$$\frac{(-1)^{\frac{n-m+1-\epsilon}{2}}}{2^{s-1}}\frac{\Gamma\left(\frac{1}{2}+\epsilon\right)\Gamma(2n-2m+3)}{\Gamma(2n-2m+3-\epsilon)(n-m+1)^{\epsilon}} \times \frac{\Gamma(s+n-m+1)\Gamma(s+2n-m+2)}{\Gamma(s+2n-2m+2)}\frac{\Gamma\left(\frac{s}{2}+n-m+1\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}. \tag{4.15}$$

We now apply $\frac{\Gamma\left(\frac{1}{2}+\epsilon\right)\Gamma(2n-2m+3)}{(n-m+1)^{\epsilon}\Gamma(2n-2m+3-\epsilon)} = \sqrt{\pi}$ to (4.15) and get

$$L_{\infty}(s-m,f) = \frac{(-1)^m\sqrt{\pi}\binom{n}{m}^2}{2^s}\frac{\Gamma\left(\frac{s}{2}+n-m+1\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}\frac{\Gamma(s+n-m+1)\Gamma(s+2n-m+2)}{\Gamma(s+2n-2m+2)}.$$

This holds for $s \in \mathbb{C}$ with real part less than $-2(n-m)-1$. Meromorphically continuing this to all $s \in \mathbb{C}$ proves Theorem 4.1.

5 Applications to Values and Poles of $L_T(s, f)$

In this section we apply the exact integral representations of Theorem 1.1 to prove explicit functional equations of the L-functions. From the functional equations we get special value results at critical points of $L_T(s, f)$ not studied in [7]. We also study the poles of $L_T(s, f)$ and prove Corollary 1.2.

Following [18], for $\alpha > 0$ let

$$H_{\alpha}(s,z,\psi_N) = \pi^{-s}\Gamma(s)y^s \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \psi_N(n)\frac{(mNz+n)^{\alpha}}{|mNz+n|^{2s}}.$$

This series is absolutely convergent for $\mathrm{Re}(s) > \frac{\alpha}{2}+1$. The $\alpha = 0$ case of the following result is [20, Lemma 2.6].

Lemma 5.1 *If $\alpha > 0$, then $H_{\alpha}(s,z,\psi_N)$ can be continued to an entire function in $s \in \mathbb{C}$.*

If ψ_N is primitive, then

$$H_{\alpha}(\alpha+1-s,z,\psi_N) = \psi_N(-1)\,\mathfrak{g}(\psi_N)N^{3s-\alpha-2}z^{\alpha}H_{\alpha}\left(s,\frac{-1}{Nz},\overline{\psi}_N\right).$$

If $\psi_N = \mathbf{1}_N$, then

$$H_{\alpha}(\alpha+1-s,z,\mathbf{1}_N) = (-1)^{\alpha+1}N^{3s-\alpha-2}z^{\alpha}H_{\alpha}\left(s,\frac{-1}{Nz},\mathbf{1}_N\right).$$

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Proof For ψ_N primitive the result follows from [18, Lemma 3.3]. We modify that proof for the case $\psi_N = \mathbf{1}_N$. Following [18, (1)] we define

$$\begin{aligned}\xi_\alpha(t,z,(p,q)) &= \sum_{(m,n)\equiv(p,q) \pmod N} (mz+n)^\alpha e^{-\pi t|mz+n|^2/N^2 y}\\ &= (-N)^\alpha t^{-\alpha-1} \sum_{(m,n)\in\mathbb{Z}^2} e^{2\pi i(qm-pn)/N}(mz+n)^\alpha e^{-\pi|mz+n|^2/yt}\\ &= (-N)^\alpha t^{-\alpha-1} \sum_{(a,b) \pmod N} e^{2\pi i(qa-pb)/N}\xi_\alpha\big(N^2 t^{-1}, z,(a,b)\big).\end{aligned}$$

Let $\eta_\alpha(t,z,(p,q)) = \sum_{k=1}^N \psi_N(k)\xi_\alpha(t,z,k(p,q))$. Then

$$\begin{aligned}&\eta_\alpha\big(t^{-1},z,(p,q)\big)\\ &\quad= (-N)^\alpha t^{\alpha+1}\sum_{k=1}^N \psi_N(k) \sum_{(a,b) \pmod N} e^{2\pi i\left(\frac{k}{N}(qa-pb)\right)}\xi_\alpha\big(N^2 t, z,(a,b)\big)\\ &\quad= (-N)^\alpha t^{\alpha+1}\sum_{k=1}^N \psi_N(k)e^{2\pi ik/N} \sum_{(a,b) \pmod N} \overline{\psi}_N(qa-pb)\xi_\alpha\big(N^2 t, z,(a,b)\big)\end{aligned}$$

by replacing $k(qa-pb)$ with k. Now, $\sum_{k=1}^N \psi_N(k)e^{2\pi ik/N} = \mathfrak{g}(\psi_N)$ for ψ_N primitive and -1 for ψ_N trivial. In the latter case we have

$$\eta_\alpha\big(t^{-1},z,(p,q)\big) = -(-N)^\alpha t^{\alpha+1} \sum_{\substack{(a,b) \pmod N\\ (qa-pb)\not\equiv 0 \pmod N}} \xi_\alpha\big(N^2 t, z,(a,b)\big).$$

From [18, (4)], for $\alpha > 0$ or $N > 1$ we have

$$\begin{aligned}\int_0^\infty \eta_\alpha\big(t,z,(0,1)\big)t^{s-1}\,dt &= N^{2s}\pi^{-s}y^s\Gamma(s)\sum_{k=1}^{N-1}\psi_N(k)\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ m\equiv 0 \pmod N\\ n\equiv k \pmod N}} \frac{(mz+n)^\alpha}{|mz+n|^{2s}}\\ &= N^{2s}H_\alpha(s,z,\mathbf{1}_N).\end{aligned}$$

From [18, (5)] we have that $|\eta_\alpha(t,z,(p,q))|$ is less than or equal to Me^{-ct} for $t > 1$, and less than or equal to $M't^{-\alpha-1}e^{-c't}$ for $t < 1$, where M, M', c, c' depend only on z, p, q. Splitting the above integral into integrals over $(0,1)$ and $(1,\infty)$, the inequalities show that the integral above converges for all $s \in \mathbb{C}$. Thus, $H_\alpha(s,z,\mathbf{1}_N)$ is entire for $\alpha > 0$. As

$$\begin{aligned}\int_0^\infty \xi_\alpha(N^2 t,z,(a,b))t^{s-1}\,dt &= \sum_{(m,n)\equiv(a,b) \pmod N}(mz+n)^\alpha\int_0^\infty e^{-\pi t|mz+n|^2/y}t^{s-1}\,dt\\ &= \pi^{-s}\Gamma(s)y^s \sum_{(m,n)\equiv(a,b) \pmod N}\frac{(mz+n)^\alpha}{|mz+n|^{2s}},\end{aligned}$$

we have

$$
\begin{aligned}
(5.1)\qquad & N^{2(\alpha+1-s)} H_\alpha(\alpha+1-s, z, \mathbf{1}_N) \\
&= \int_0^\infty \eta_\alpha(t, z, (0,1)) t^{\alpha-s}\, dt \\
&= \int_0^\infty \eta_\alpha(t^{-1}, z, (0,1)) t^{s-\alpha-2}\, dt \\
&= -(-N)^\alpha \pi^{-s} \Gamma(s) y^s \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \mathbf{1}_N(m) \frac{(mz+n)^\alpha}{|mz+n|^{2s}}.
\end{aligned}
$$

From a direct computation,

$$
H_\alpha\Big(s, \frac{-1}{Nz}, \mathbf{1}_N\Big) = \pi^s N^{-s} \Gamma(s) y^s z^{-\alpha} \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \mathbf{1}_N(n) \frac{(nz+m)^\alpha}{|nz+m|^{2s}}.
$$

Thus, (5.1) is $(-1)^{\alpha+1} N^{\alpha+s} z^\alpha H_\alpha\big(s, \frac{-1}{Nz}, \mathbf{1}_N\big)$. ∎

Following [15, equation 7.2.62] we have

$$
\begin{aligned}
&\pi^{-(s+2n-2m+2)} \Gamma(s+2n-2m+2) 2 L_N(2s+2n-2m+2, \psi_N) E_{2n-2m+2}(s, z, \psi_N) \\
(5.2)\qquad &= \frac{1}{y^{2n-2m+2}} H_{2n-2m+2}(s+2n-2m+2, \bar{z}, \psi_N) \cdot \omega,
\end{aligned}
$$

where $\omega = (X - zY)^{2n-2m}\, dz$ as in Section 3. From Theorem 1.1 and Lemma 5.1 it follows that

$$
\frac{\Gamma\big(\frac{s}{2}+n-m+1\big)\Gamma(s+n-m+1)\Gamma(s+2n-m+2)}{\Gamma\big(\frac{s}{2}+\frac{1}{2}\big)} L_T(s+w+n-m+1, f)
$$

is holomorphic for all $s \in \mathbb{C}$, where $w = n+1+\nu_i+\nu_c$. Therefore the only possible poles of $L_T(s+w+n-m+1, f)$ occur where $s \le -1$ is an odd integer. Because of the gamma functions in the numerator above, any pole must also satisfy $s > -(n-m+1)$. For w odd, as m and $\nu_i+\nu_c$ have the same parity, we can put $m = n$, and so the conditions $s \le -1$ and $s > -(n-m+1)$ imply that $L_T(s+w+1, f)$ is holomorphic. For w even we put $m = n-1$ and this gives the conditions $s \le -1$ and $s > -2$. So the only possible pole of $L_T(s+w+2, f)$ is at $s = -1$. This gives Corollary 1.2.

The functional equation for the w odd case is also obtained in [6, Section 7.2]. Note that from Section 2 we are considering only one component f_1 (which we label f), arising from strong approximation, of the cuspform on $\mathfrak{H}_3$. So we need to obtain a functional equation for each component. Therefore, for simplicity, we assume that there is only one such component, and so we require K to have class number 1. For $m = n$, from (5.2) and the integral representations of Theorem 1.1,

$$
\begin{aligned}
(5.3)\qquad & \int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \Big\langle \delta_0(f), \frac{1}{y^2} H_2(s+2, \bar{z}, \psi_N)\, dz \Big\rangle = \\
&\frac{(-1)^n 2\sqrt{D}^{s+w+1} i^{\nu_i-\nu_c}}{(2\pi)^{s+n+2}\pi^{s+2}} \times \Gamma\Big(\frac{s}{2}+1\Big)^2 \Gamma(s+n+2) L_T(s+w+1, f),
\end{aligned}
$$

and for $m = n - 1$,

$$(5.4)\quad \int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \Big\langle \delta_2(f), \frac{1}{y^4} H_4(s+4, \overline{z}, \psi_N)(X - zY)^2 \, dz \Big\rangle = \frac{(-1)^{n-1} n^2 \sqrt{D}^{s+w+2} i^{\nu_i - \nu_c}}{(2\pi)^{s+n+3} \pi^{s+4}} \times \frac{\Gamma\big(\frac{s}{2}+1\big)^2 \Gamma(s+3)\Gamma(s+n+1)}{\Gamma(s+1)} L_T(s+w+2, f).$$

For $f \in \mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}), \psi_{\mathfrak{n}})$ let $f'(g) = \overline{\psi}_{\mathfrak{n}}(\det g) f\big(g\big(\begin{smallmatrix} 0 & -1 \\ \nu & 0 \end{smallmatrix}\big)\big)$, where ν is a finite idele so that $\nu \mathcal{O}_K = \mathfrak{n}$. As in [6, Proposition 3] we have $f' \in \mathcal{S}_{\mathbf{n}}(\Gamma_0(\mathfrak{n}), \overline{\psi}_{\mathfrak{n}})$. Following [6], for f a normalized newform that is an eigenfunction of the Hecke operators (as in [20, Section 4]), we have $f' = c\overline{f}$, where $|c| = 1$, $c \in E$, and $\overline{f}$ has Fourier coefficients $c(\mathfrak{m}, \overline{f}) = \overline{c(\mathfrak{m}, f)}$. Following [6, Section 3] for $\gamma = \big(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\big)$ we have

$$\gamma^* \delta_k(f) = \delta_k(f \circ \gamma) = N^{-(n+k+\nu_i+\nu_c)} z^{-k} \delta_k(f') = cN^{-(n+k+\nu_i+\nu_c)} z^{-k} \delta_k(\overline{f}).$$

From Lemma 3.3, for ψ_N primitive or trivial we have the functional equations

$$(5.5)\qquad H_2(s+2, \overline{z}, \psi_N) = \psi_N(-1)\, \mathfrak{g}(\psi_N) N^{-3s-1} \overline{z}^2 H_2\Big(1-s, \frac{-1}{N\overline{z}}, \overline{\psi}_N\Big),$$

$$(5.6)\qquad H_4(s+4, \overline{z}, \psi_N) = \psi_N(-1)\, \mathfrak{g}(\psi_N) N^{-3s-3} \overline{z}^4 H_4\Big(1-s, \frac{-1}{N\overline{z}}, \overline{\psi}_N\Big),$$

$$(5.7)\qquad H_2(s+2, \overline{z}, \mathbf{1}_N) = -N^{-3s-1} \overline{z}^2 H_2\Big(1-s, \frac{-1}{N\overline{z}}, \mathbf{1}_N\Big),$$

$$(5.8)\qquad H_4(s+4, \overline{z}, \mathbf{1}_N) = -N^{-3s-3} \overline{z}^4 H_4\Big(1-s, \frac{-1}{N\overline{z}}, \mathbf{1}_N\Big).$$

From the functional equation (5.5) we have

$$(5.9)\quad \int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \Big\langle \delta_0(f), \frac{1}{y^2} H_2(s+2, \overline{\cdot}, \psi_N)\, dz \Big\rangle = \psi_N(-1)\, \mathfrak{g}(\psi_N) N^{-3s-1} \times \int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \Big\langle \delta_0(f), \frac{\overline{z}^2}{y^2} H_2(1-s, \frac{-1}{N\overline{\cdot}}, \overline{\psi}_N)\, dz \Big\rangle.$$

For $\gamma = \big(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\big)$ we make the substitution $\gamma z = \frac{-1}{Nz}$ for z, and (5.9) becomes

$$\begin{aligned}
&\psi_N(-1)\, \mathfrak{g}(\psi_N) N^{-3s-2} \int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \Big\langle \gamma^* \delta_0(f), \frac{1}{y^2} H_2(1-s, \overline{z}, \overline{\psi}_N)\, dz \Big\rangle \\
&\quad = c\psi_N(-1)\, \mathfrak{g}(\psi_N) N^{-3s-w-1} \int_{\Gamma_0(N)\backslash \mathfrak{H}_1} \Big\langle \delta_0(\overline{f}), \frac{1}{y^2} H_2(1-s, \overline{z}, \overline{\psi}_N)\, dz \Big\rangle \\
&\quad = c\psi_N(-1)\, \mathfrak{g}(\psi_N) N^{-3s-w-1} \frac{(-1)^n 2 \sqrt{D}^{w-s} i^{\nu_i - \nu_c}}{(2\pi)^{1+n-s} \pi^{1-s}} \\
&\qquad\qquad \times \Gamma\Big(\frac{1-s}{2}\Big)^2 \Gamma(n+1-s) L_T(w-s, \overline{f}).
\end{aligned}$$

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Applying this to (5.3) and substituting s with $s-w-1$ gives the functional equation of Theorem 1.3 for w odd. In a similar way we apply (5.6) to (5.4) and get

$$
\begin{aligned}
&c\psi_N(-1)\,\mathfrak{g}(\psi_N)N^{-3s-w-5}\int_{\Gamma_0(N)\backslash\,\mathfrak{H}_1}\Big\langle \delta_2(\overline{f}),\frac{1}{y^4}H_4(1-s,\overline{z},\overline{\psi}_N)(X-zY)^2\,dz\Big\rangle\\
&\quad= c\psi_N(-1)\,\mathfrak{g}(\psi_N)N^{-3s-w-5}\frac{(-1)^{n-1}n^2\sqrt{D}^{w-s-1}i^{v_i-v_c}}{(2\pi)^{n-s}\pi^{1-s}}\\
&\qquad\times\frac{\Gamma\big(\frac{-1-s}{2}\big)^2\Gamma(-s)\Gamma(n-2-s)}{\Gamma(-s-2)}L_T(w-1-s,\overline{f}).
\end{aligned}
$$

Making the substitution s with $s-w-2$ and simplifying by applying the reflection identity twice gives the functional equation of Theorem 1.3 for w even. Note that normalizing the functional equations to be with respect to the transformation $s\leftrightarrow 1-s$, the Langlands parameters for the archimedean factors are $\epsilon/2$, $\epsilon/2$, $n/2$, $(n+1)/2$; see [13].

For $\psi_N=\mathbf{1}_N$, we apply (5.7) and (5.8) to integral representations that are entirely analogous to (5.3) and (5.4) for that case. The functional equations are of the same form as that of Theorem 1.3 but with the factors $\overline{\psi}_N(-1)\,\mathfrak{g}(\psi_N)$ removed. A direct application of Theorem 1.3 to the special values of $L_T(s,f)$ at the even critical points gives the special values at the odd critical points. Galois equivariance of the values follows as in Section 3. This proves Theorem 1.4.

Appendix A

A Product Formula for the Gamma Factor

Hiroyuki Ochiai

Abstract. We give a product formula of the Gamma factor arising from the study of the twisted tensor L-function of a cuspidal automorphic form on the imaginary quadratic field.

A.1 Introduction

The twisted tensor L-function [2, 4, 20]

$$G(s,f)=L(s)\sum_{m=1}^{\infty}C(m,f)m^{-s}$$

is a Dirichlet series, up to an explicit normalizing factor $L(s)$, attached to a cuspidal automorphic form f over K, a quadratic extension of a number field. It is important to study the arithmetic property of critical values of the L-function $G(s,f)$. In the case of Hilbert modular forms, this property has been discussed by Shimura [17]. On the other hand, E. Ghate considers the arithmeticity of the L-function $G(s,f)$ in imaginary quadratic case. He introduces a function $G_\infty(s,f)$ defined by

$$G_\infty(s,f)=\Gamma(s+2n-2m+2)G'_\infty(s,f),$$

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with

(A.1)
$$G'_\infty(s,f)=\sum_{\substack{\alpha=0,\\ \alpha\equiv n+1+m(2)}}^{2n+2} c(m,\alpha)\Gamma\Big(\frac{s+n-m+1+\alpha}{2}\Big)\Gamma\Big(\frac{s+3n+3-m-\alpha}{2}\Big),$$

where the coefficients $c(m,\alpha)$ are given in (A.2). It has been shown that the product function $G_\infty(s,f)G(s+j_0,f)$ is expressed by means of the Rankin–Selberg integral, where the integer j_0 is explicitly given by the cusp form f. Hence, in order to prove the arithmetic property that the critical value $G(j_0,f)$ is an algebraic number times a "period" of the form f, it is sufficient to prove the non-vanishing of $G_\infty(s,f)$ at $s=0$.

In this appendix, we give a product representation of the function $G_\infty(s,f)$. This representation completely describes the zeros of $G_\infty(s,f)$. Using Theorem A.1, we also show that $G_\infty(0,f)$ does not vanish and that [7, Conjecture 1] by Ghate holds.

A.2 Main Result

We recall the definition of the constant $c(m,\alpha)$ appearing in (A.1),

$$\text{(A.2)}\quad c(m,\alpha)=\frac{(-1)^{n+1}}{2}\sum_{l=0,\text{even}}^{2n-2m} i^l\big(a(m,l-1-\alpha)-2b(m,l,\alpha)\big)$$
$$=\frac{(-1)^{n+1}}{2}\Big[\sum_{k=1}^{n-m}(-1)^k a(m,2k-1,\alpha)-2\sum_{k=0}^{n-m}(-1)^k b(m,2k,\alpha)\Big],$$

$$\text{(A.3)}\quad a(m,l,\alpha)=\binom{n}{m}^2(-1)^{\frac12(n-m-l+\alpha)}\sum_{j=0}^{m}\binom{m}{j}\Big[\binom{n-m}{\frac{n-m-l+\alpha}{2}-j}$$
$$\times\binom{n-m}{\frac{3n-3m-l-\alpha}{2}+j}+\binom{n-m}{\frac{n-m-l+\alpha-2}{2}-j}\binom{n-m}{\frac{3n-3m-l-\alpha+2}{2}+j}\Big],$$

$$\text{(A.4)}\quad b(m,l,\alpha)=\binom{n}{m}^2(-1)^{\frac12(n-m-l+\alpha-1)}$$
$$\times\sum_{j=0}^{m}\binom{m}{j}\binom{n-m}{\frac{n-m-l+\alpha-1}{2}-j}\binom{n-m}{\frac{3n-3m-l-\alpha+1}{2}+j}.$$

Then our main theorem follows.

Theorem A.1 *Let m,n be integers with $0\le m\le n$. We take $\epsilon=0$ or 1 so that $n-m-1-\epsilon\in 2\mathbb{Z}$. Then we have*

$$G_\infty(s,f)=(-1)^m 2^{\epsilon-1}\binom{n}{m}^2\Gamma(s+2n-m+2)\cdot\Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big)^2$$
$$\times\prod_{i=0}^{(n-m-1-\epsilon)/2}(s+1+2i)(s+2n-2m-2i).$$

The zeros of $G_\infty(s,f)$ are described completely in Theorem A.1. For example, as a corollary of Theorem A.1, we have the following statement, which is proved under some condition in [7, Proposition 5].

Corollary $G_\infty(s,f) \neq 0$.

We hope that Theorem A.1 will be helpful in analyzing further properties of $G(s+j_0,f)$. The proof of Theorem A.1 is given in the next section, where we use several kinds of identities on binomial coefficients.

Next we show [7, Conjecture 1].

Theorem A.2 *We have*

$$G_\infty(s,f) = c_{n,m}P_{n,m}(s)\cdot\Gamma(s+2n-m+2)\cdot\Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big)^2,$$

where

$$P_{n,m}(s) = \prod_{i=0}^{(n-m-1-\epsilon)/2}(s+1+2i)\Big(\frac{s}{2}+n-m-i\Big)$$

and

$$c_{n,m} = \frac{(-1)^m\cdot n!^2}{m!^2\cdot P_{n,m}(0)\cdot(\frac{n-m-1+\epsilon}{2})!^2}.$$

Proof It is enough to show that

$$c_{n,m}P_{n,m}(s) = (-1)^m 2^{\epsilon-1}\binom{n}{m}^2\times\prod_{i=0}^{(n-m-1-\epsilon)/2}(s+1+2i)(s+2n-2m-2i).$$

Since

$$\begin{aligned}P_{n,m}(s) &= (s+1)(s+3)(s+5)\cdots(s+n-m-\epsilon)\\ &\quad\times\Big(\frac{s}{2}+n-m\Big)\Big(\frac{s}{2}+n-m-1\Big)\cdots\Big(\frac{s}{2}+n-m-\frac{n-m-1-\epsilon}{2}\Big),\end{aligned}$$

we see that

$$P_{n,m}(0) = (n-m-\epsilon)!!\times\frac{(n-m)!}{(\frac{n-m+\epsilon-1}{2})!}.$$

Hence

$$\begin{aligned}c_{n,m} &= (-1)^m\binom{n}{m}^2\frac{(n-m)!}{(n-m-\epsilon)!!(\frac{n-m+\epsilon-1}{2})!}\\ &= (-1)^m\binom{n}{m}^2\frac{(n-m+\epsilon-1)!!}{(\frac{n-m+\epsilon-1}{2})!} = (-1)^m\binom{n}{m}^2 2^{(n-m+\epsilon-1)/2}\\ &= (-1)^m 2^{\epsilon-1}\binom{n}{m}^2 2^{(n-m+1-\epsilon)/2}.\end{aligned}$$

Here the second equality follows from the formula $k!/(k-\epsilon)!! = (k+\epsilon-1)!!$ for $\epsilon = 0,1$, and the third equality from the formula $(2k)!!/k! = 2^k$. This completes the proof. ∎

A.3 Proof of Theorem A.1

We begin by calculating the constant $c(m,\alpha)$. We refer the reader to [12] for several kinds of identities for binomial coefficients.

Lemma A.3

$$c(m,\alpha)=(-1)^m\binom{n}{m}^2\cdot\frac{1}{2}(-1)^{\frac{1}{2}(n-m+1-\alpha)}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{2n-2m+2}{\alpha-2j}. \tag{A.5}$$

Proof Definitions (A.2), (A.3), and (A.4) lead to

$$\begin{aligned}
c(m,\alpha)&=\frac{(-1)^{n+1}}{2}\Big(\sum_{k=1}^{n-m}(-1)^k a(m,2k-1,\alpha)-2\sum_{k=0}^{n-m}(-1)^k b(m,2k,\alpha)\Big)\\
&=\frac{(-1)^{n+1}}{2}\binom{n}{m}^2(-1)^{\frac{1}{2}(n-m+1+\alpha)}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\\
&\quad\times\Bigg[\sum_{k=1}^{n-m}\Bigg[\binom{n-m}{\frac{n-m+\alpha+1}{2}-k-j}\binom{n-m}{\frac{3n-3m-\alpha+1}{2}-k+j}\\
&\qquad+\binom{n-m}{\frac{n-m+\alpha-1}{2}-k-j}\binom{n-m}{\frac{3n-3m-\alpha+3}{2}-k+j}\Bigg]\\
&\qquad+2\sum_{k=0}^{n-m}\binom{n-m}{\frac{n-m+\alpha-1}{2}-k-j}\binom{n-m}{\frac{3n-3m-\alpha+1}{2}-k+j}\Bigg].
\end{aligned}$$

Thus, deriving (A.5) is equivalent to proving

$$\begin{aligned}
&\sum_{k=1}^{n-m}\binom{n-m}{\frac{n-m+\alpha+1}{2}-j-k}\binom{n-m}{\frac{3n-3m-\alpha+1}{2}+j-k}\\
&\quad+\sum_{k=1}^{n-m}\binom{n-m}{\frac{n-m+\alpha-1}{2}-j-k}\binom{n-m}{\frac{3n-3m-\alpha+3}{2}+j-k}\\
&\quad+2\sum_{k=1}^{n-m}\binom{n-m}{\frac{n-m+\alpha-1}{2}-j-k}\binom{n-m}{\frac{3n-3m-\alpha+1}{2}+j-k}\\
&=\binom{2n-2m+2}{\alpha-2j}.
\end{aligned} \tag{A.6}$$

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The first sum of the left-hand side of equality (A.6) is calculated as

$$\begin{aligned}
&\sum_{k=1}^{n-m}\binom{n-m}{\frac{n-m+\alpha+1}{2}-j-k}\binom{n-m}{\frac{3\alpha-3m-\alpha+1}{2}+j-k}\\
&\quad=\sum_{k=1}^{n-m}\binom{n-m}{\frac{n-m+\alpha+1}{2}-j-k}\binom{n-m}{\frac{-n+m+\alpha-1}{2}-j+k}\\
&\quad=\binom{2n-2m}{\alpha-2j},
\end{aligned}$$

where the first equality follows from the identity $\binom{a}{b}=\binom{a}{a-b}$, and the second equality from the identity on the binomial coefficients

$$\sum_{k=\max(c-a,-d)}^{\min(b-d,c)}\binom{a}{c-k}\binom{b}{d+k}=\binom{a+b}{c+d}.$$

Similarly, the second and the third sums of the left-hand side of (A.6) are calculated as

$$\begin{aligned}
&\sum_{k=1}^{n-m}\binom{n-m}{\frac{n-m+\alpha-1}{2}-k-j}\binom{n-m}{\frac{3n-3m-\alpha+3}{2}-k+j}=\binom{2n-2m}{\alpha-2j-2},\\
&\sum_{k=0}^{n-m}\binom{n-m}{\frac{n-m+\alpha-1}{2}-k-j}\binom{n-m}{\frac{3n-3m-\alpha+1}{2}-k+j}=\binom{2n-2m}{\alpha-2j-1}.
\end{aligned}$$

Hence the left-hand side of (A.6) is reduced to

$$\binom{2n-2m}{\alpha-2j}+2\binom{2n-2m}{\alpha-2j-1}+\binom{2n-2m}{\alpha-2j-2},$$

which is easily shown to be the right-hand side of (A.6). ■

In what follows, we use the Pochhammer symbol defined by

$$(a;k)=\Gamma(a+k)/\Gamma(a)=a(a+1)(a+2)\cdots(a+k-1),\quad \text{for } a\in\mathbb{C}, k\in\mathbb{Z}_{\geq 0}.$$

With the help of Lemma A.3, we deduce the following expression of $G'_\infty(s,f)$.

Lemma A.4

$$\begin{aligned}
\text{(A.7)}\quad G'_\infty(s,f)&=(-1)^m\binom{n}{m}^2\cdot\Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big)^2(s+2n-2m+2;m)\\
&\quad\times\frac{1}{2}(-1)^{\frac{1}{2}(n-m+1-\epsilon)}\sum_{i=0}^{n-m+1-\epsilon}(-1)^i\binom{2n-2m+2}{2i+\epsilon}\Big(\frac{s+n-m+1+\epsilon}{2};i\Big)\\
&\qquad\times\Big(\frac{s+n-m+1+\epsilon}{2};n-m+1-\epsilon-i\Big).
\end{aligned}$$

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Proof By substituting the identities

$$\Gamma\Big(\frac{s+n-m+1+\alpha}{2}\Big) = \Big(\frac{s+n-m+1+\epsilon}{2};\frac{\alpha-\epsilon}{2}\Big)\Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big),$$
$$\Gamma\Big(\frac{s+3n+3-m-\alpha}{2}\Big) = \Big(\frac{s+n-m+1+\epsilon}{2};n+1-\epsilon-\frac{\alpha-\epsilon}{2}\Big) \times\Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big),$$

into definition (A.1), we have

$$G'_\infty(s,f) = \Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big)^2 \sum_{\substack{\alpha=0,\\ \alpha\equiv n+1+m(2)}}^{2n+2} c(m,\alpha) \times\Big(\frac{s+n-m+1+\epsilon}{2};\frac{\alpha-\epsilon}{2}\Big)\Big(\frac{s+n-m+1+\epsilon}{2};n+1-\epsilon-\frac{\alpha-\epsilon}{2}\Big),$$

which leads to the following by Lemma A.3.

$$G'_\infty(s,f) = (-1)^m\binom{n}{m}^2\Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big)^2\cdot\frac{1}{2}\sum_{\substack{\alpha=0,\\ \alpha\equiv\epsilon(2)}}^{2n+2}(-1)^{\frac{1}{2}(n-m+1-\alpha)} \times\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{2n-2m+2}{\alpha-2j}\Big(\frac{s+n-m+1+\epsilon}{2};\frac{\alpha-\epsilon}{2}\Big) \times\Big(\frac{s+n-m+1+\epsilon}{2};n+1-\epsilon-\frac{\alpha-\epsilon}{2}\Big).$$

The change of the variable α into i such that $\alpha=2i+(\epsilon+2j)$ results in

$$G'_\infty(s,f) = (-1)^m\binom{n}{m}^2\Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big)^2\times\frac{1}{2}(-1)^{\frac{1}{2}(n-m+1-\epsilon)} \times\sum_{i=0}^{n-m+1-\epsilon}(-1)^i\binom{2n-2m+2}{2i+\epsilon}\sum_{j=0}^{m}\binom{m}{j}\Big(\frac{s+n-m+1+\epsilon}{2};j+i\Big) \times\Big(\frac{s+n-m+1+\epsilon}{2};n+1-\epsilon-j-i\Big).$$

The desired formula follows from the identity

$$\sum_{j=0}^{m}\binom{m}{j}(x;i+j)(y;b-j) = m!(x;i)(y;b-m)\sum_{j=0}^{m}\frac{(x+i;j)}{j!}\frac{(y+b-m;m-j)}{(m-j)!} = m!(x;i)(y;b-m)\frac{(x+i+y+b-m;m)}{m!}.$$

Here the quantity

$$\sum_{j=0}^{m}\frac{(x;j)}{j!}\frac{(y;m-j)}{(m-j)!} = \frac{(x+y;m)}{m!}$$

is the coefficient of the term t^m of the generating function $(1-t)^{-x}(1-t)^{-y} = (1-t)^{-x-y}$ with $b=n+1-\epsilon-i$ and $x=y=(s+n-m+1+\epsilon)/2$. ■

In what follows, we fix a non-negative integer N. In order to evaluate the sum over i in the right-hand side of equality (A.7), we introduce a polynomial $F(t;x,\epsilon)$ in t with a complex parameter x and a parameter $\epsilon=0,1$:

$$F(t;x,\epsilon)=\sum_{i=0}^{2N}x^i\binom{4N+2\epsilon}{2i+\epsilon}(t;i)(t;2N-i). \tag{A.8}$$

The polynomial $F(t;x,\epsilon)$ has the following properties.

Lemma A.5 (i) *The polynomial $F(\frac{1}{2}+\epsilon;x,\epsilon)$ in x has zero at $x=-1$ with the multiplicity $2N$.*

(ii) *For a non-negative integer j, we have*

$$F(t+j;x,\epsilon)=\frac{(t+\theta_x;j)(t+2N-\theta_x;j)}{(t;j)^2}F(t;x,\epsilon), \tag{A.9}$$

where the Euler operator in x is denoted by $\theta_x=x\frac{\partial}{\partial x}$.

(iii) $F(\frac{1}{2}+\epsilon+j;-1,\epsilon)=0$ *for* $j=0,1,\dots,N-1$.

Proof (i) We consider the two cases, $\epsilon=0$ and $\epsilon=1$, separately.

When $\epsilon=1$, we have

$$\begin{aligned}F\Big(\frac{3}{2};x,1\Big)&=\sum_{i=0}^{2N}x^i\binom{4N+2}{2i+1}\frac{(2i+1)!}{2^{2i}i!}\frac{(4N-2i+1)!}{2^{4N-2i}(2N-i)!}\\&=\frac{(4N+2)!}{2^{4N}(2N)!}\sum_{i=0}^{2N}\binom{2N}{i}x^i=\frac{(4N+2)!}{2^{4N}(2N)!}(1+x)^{2N},\end{aligned}$$

where the first equality follows from the identity $(\frac{3}{2};a)=(2a+1)(\frac{1}{2};a)=\frac{(2a+1)!}{2^{2a}a!}$. This shows the desired property of $F(\frac{1}{2}+1;x,1)$.

When $\epsilon=0$, the similar calculation leads to

$$F(\frac{1}{2};x,0)=\frac{(4N)!}{2^{4N}(2N)!}(1+x)^{2N},$$

which proves the desired property of $F(\frac{1}{2};x,0)$. This completes the proof of (i).

(ii) It is seen that

$$\begin{aligned}F(t+j;x,\epsilon)&=\sum_{i=0}^{2N}x^i\binom{4N+2\epsilon}{2i+\epsilon}(t;i)(t;2N-i)\cdot\frac{(t+i;j)(t+2N-i;j)}{(t;j)^2}\\&=\sum_{i=0}^{2N}\binom{4N+2\epsilon}{2i+\epsilon}(t;i)(t;2N-i)\cdot\frac{(t+\theta_x;j)(t+2N-\theta_x;j)}{(t;j)^2}x^i,\end{aligned}$$

where we have used the identity $(t;j)(t+j;i)=(t;i)(t+i;j)$ for the first identity. By using the definition (A.8), we obtain the desired equality (A.9).

(iii) Let j be an integer with $0\le j<N$. As a special case $t=\frac{1}{2}+\epsilon$ of formula (A.9), we have

$$F\Big(\frac{1}{2}+\epsilon+j;x,\epsilon\Big)=\frac{(\frac{1}{2}+\epsilon+\theta_x;j)(\frac{1}{2}+\epsilon+2N-\theta_x;j)}{(\frac{1}{2}+\epsilon;j)^2}F\Big(\frac{1}{2}+\epsilon;x,\epsilon\Big). \tag{A.10}$$

By (i), the order $2j$ of the differential operator in the right-hand side of (A.10) is strictly less than the multiplicity $2N$ of the zero of the polynomial $F(\frac{1}{2}+\epsilon; x, \epsilon)$ at $x=-1$. Hence we know that the right-hand side of (A.10) has zero at $x=-1$. This means that $F(\frac{1}{2}+\epsilon+j; -1, \epsilon)=0$. ∎

Proposition A.6 *The polynomial $F(t;-1,\epsilon)$ is expressed by*

$$F(t;-1,\epsilon)=(-1)^N 2^{2N+\epsilon}(t;N)\prod_{j=0}^{N-1}(t-\frac{1}{2}-\epsilon-j). \tag{A.11}$$

Proof The polynomial $F(t;-1,\epsilon)$ can be divided by the polynomial $P_1(t)=(t;N)$, because the polynomial $(t,i)(t;2N-i)$ for $0\le i\le 2N$ can be divided by $P_1(t)$. On the other hand, Lemma A.5(iii) implies that the polynomial $F(t;-1,\epsilon)$ can be divided by

$$P_2(t)=\prod_{j=0}^{N-1}(t-\frac{1}{2}-\epsilon-j).$$

Therefore the polynomial $F(t;-1,\epsilon)$ can be divided by the product $P_1(t)P_2(t)$, since two polynomials $P_1(t)$ and $P_2(t)$ have no common zeros.

Note that the degree of the polynomial $F(t;-1,\epsilon)$ in t is equal to $2N$. Indeed, the coefficient of t^{2N} in $F(t;-1,\epsilon)$ is

$$\begin{aligned}\sum_{i=0}^{2N}(-1)^i\binom{4N+2\epsilon}{2i+\epsilon}&=\operatorname{Re}\sum_{j=0}^{4N+2\epsilon}(\sqrt{-1})^{j-\epsilon}\binom{4N+2\epsilon}{j}\\&=\operatorname{Re}(\sqrt{-1})^{-\epsilon}(1+\sqrt{-1})^{4N+2\epsilon}=(-1)^N2^{2N+\epsilon}.\end{aligned}$$

Since the polynomial $P_1(t)P_2(t)$ is monic and of degree $2N$, this proves that

$$F(t;-1,\epsilon)=(-1)^N2^{2N+\epsilon}P_1(t)P_2(t).$$ ∎

By using Lemma A.4 and Proposition A.6, we give a proof of Theorem A.1. We consider $F(t;-1,\epsilon)$ with

$$N=\frac{n-m+1-\epsilon}{2},\qquad t=\frac{s+n-m+1+\epsilon}{2}.$$

Formula (A.11) leads to

$$F(t;-1,\epsilon)=(-1)^{\frac{1}{2}(n-m+1-\epsilon)}2^{\epsilon}\prod_{i=0}^{(n-m-1-\epsilon)/2}(s+1+2i)(s+2n-2m-2i). \tag{A.12}$$

On the other hand, formula (A.7) implies

$$\begin{aligned}G'_\infty(s,f)=(-1)^m\binom{n}{m}^2\cdot\Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big)^2(s+2n-2m+2;m)\\ \times\frac{1}{2}(-1)^{\frac{1}{2}(n-m+1-\epsilon)}F(t;-1,\epsilon).\end{aligned} \tag{A.13}$$

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Combining equations (A.12) and (A.13), we obtain

$$G'_\infty(s,f) = (-1)^m \binom{n}{m}^2 \cdot \Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big)^2 (s+2n-2m+2;m) \\ \times 2^{\epsilon-1} \prod_{i=0}^{(n-m-1-\epsilon)/2} (s+1+2i)(s+2n-2m-2i).$$

Since $(s+2n-2m+2;m)\Gamma(s+2n-2m+2) = \Gamma(s+2n-m+2)$, we have obtained

$$G_\infty(s,f) = (-1)^m \binom{n}{m}^2 \cdot \Gamma\Big(\frac{s+n-m+1+\epsilon}{2}\Big)^2 \Gamma(s+2n-m+2) \\ \times 2^{\epsilon-1} \prod_{i=0}^{(n-m-1-\epsilon)/2} (s+1+2i)(s+2n-2m-2i).$$

This completes the proof of Theorem A.1. ∎

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Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101
e-mail: dominic.lanphier@wku.edu

Department of Mathematics, SUNY Brockport, Brockport, NY 14420
e-mail: hskogman@brockport.edu

Institute of Mathematics for Industry, Kyushu University, Motooka, Fukuoka, 819-0395, Japan
e-mail: ochiai@imi.kyushu-u.ac.jp