Annals of Actuarial Science (2023), **17**, pp. 63–82
doi:10.1017/S1748499522000124

ORIGINAL RESEARCH PAPER

The moments of the time of ruin in Sparre Andersen risk models

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(Received 29 December 2021; revised 25 May 2022; accepted 11 July 2022; first published online 25 August 2022)

Abstract

We derive formulae for the moments of the time of ruin in both ordinary and modified Sparre Andersen risk models without specifying either the inter-claim time distribution or the individual claim amount distribution. We illustrate the application of our results in the special case of exponentially distributed claims, as well as for the following ordinary models: the classical risk model, phase-type(2) risk models, and the Erlang(n) risk model. We also show how the key quantities for modified models can be found.

Keywords: Sparre Andersen model; Time of ruin; Moments

1. Introduction

We start with a description of an ordinary Sparre Andersen (SA) model, sometimes called a renewal risk model, which was introduced by Sparre Andersen (1957). Claims occur according to a renewal process $\{N(t)\}_{t\geq 0}$, with inter-claim times being a sequence of independent and identically distributed (i.i.d.) random variables. Specifically, let V_1 be the time until the first claim, and for $j = 2, 3, 4, \ldots$ let V_j be the time between claims $j-1$ and j. We assume that $\{V_j\}_{j=1}^{\infty}$ are positive random variables with distribution function $K = 1 - \bar{K}$, density function k, and a finite mean μ_V. Individual claim amounts are modelled as a sequence of i.i.d. random variables $\{X_j\}_{j=1}^{\infty}$ which are also positive, with a density function which we denote f, and a finite mean μ_X.

The insurer receives premiums continuously at rate $c = (1+\theta)\mu_X/\mu_V$ where $\theta > 0$ is a loading factor. The insurer's surplus at time t, given initial surplus u, is

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

and the time of ruin is defined as $T_u = \inf\{t\colon U(t) < 0\}$ with $T_u = \infty$ if $U(t) \geq 0$ for all $t > 0$. The deficit at ruin is $Y_u = |U(T_u)|$. The (defective) density of the time of ruin is defined for $t > 0$ as

$$w(u,t) = \frac{d}{dt}\,\Pr(T_u \leq t).$$

Defining $W(u,y,t) = \Pr(T_u \leq t,\ Y_u \leq y)$, the (defective) joint density of the time of ruin and the deficit at ruin is defined for $t > 0$ and $y > 0$ as

$$w(u,y,t) = \frac{d^2}{dt\,dy}\,W(u,y,t)\,.$$

Further, we define $\psi(u) = 1 - \phi(u) = \Pr(T_u < \infty)$ to be the probability of ultimate ruin, and $G(u, y) = \Pr(T_u < \infty, Y_u \le y)$, with (defective) density $g(u, y) = \frac{d}{dy} G(u, y)$ for $y > 0$. For $n = 0, 1, 2, \ldots$ define

$$\psi_n(u) = \mathrm{E}[T^n \, I(T_u < \infty)] = \int_0^\infty t^n \, w(u, t) \, dt$$

to be the nth moment of the time of ruin, with $\psi_0(u) = \psi(u)$.

For $\delta \ge 0$, Lundberg's fundamental equation for an ordinary SA risk model is

$$\mathrm{E}\left[e^{-sX_1 - (\delta - cs)V_1}\right] = 1 \,.$$

See, for example, Willmot & Woo (2017). The solutions of this equation are key in most of our applications.

The modified SA model differs in only one respect from the ordinary model – the distribution of V_1 is different to that of $\{V_j\}_{j=2}^{\infty}$ with density function $k_1 \neq k$. In the special case when the distribution of V_1 is the equilibrium distribution of K, so that $k_1 = k^e = \bar{K}/\mu_V$, we have the equilibrium SA model. For the modified and equilibrium models, we introduce superscripts m and e, respectively, to functions defined for the ordinary model, so, for example, $\psi^e(u)$ is the probability of ultimate ruin for the equilibrium model.

Moments of the time of ruin have been studied for a large number of years. Gerber (1979) gives the first moment of the time of ruin in the classical risk model with exponential claims, while Drekic & Willmot (2003) found an easily implemented explicit formula for any moment of the time of ruin in the same setting. A recursion formula for moments of the time of ruin in the classical risk model was first given by Lin & Willmot (2000) who derived their result through analysis of a defective renewal equation; a much simpler proof of their result was later given by Albrecher & Boxma (2005).

Dickson & Hipp (2001) considered moments of the time of ruin in an ordinary SA model with Erlang(2) inter-claim times. They showed how moments of T_u could be found from explicit solutions for a Gerber-Shiu function defined as the Laplace transform of T_u, and they also derived general expressions for the mean and variance of T_0. In a pair of papers, Lee & Willmot (2014, 2016) find expressions satisfied by the moments of the time of ruin in dependent SA models with particular assumptions. Independent models can be retrieved as special cases, and their results provide an alternative method of obtaining some of the explicit results obtained below for ordinary SA models, but not for modified models. Yu *et al.* (2010) obtain results for the moments of the time of ruin for a Markovian risk model. Kim & Willmot (2016) show that the Laplace transform of the time of ruin in a modified SA model satisfies a defective renewal equation, but they do not give any results on moments of the time of ruin.

In this contribution, we take a completely different approach to the works mentioned above. Without making any distributional assumptions, we obtain new general expressions for the moments of the time of ruin in ordinary and modified SA models, and we apply these results in a number of settings. The results presented for modified SA models cannot be obtained from existing results in the literature, and although the results for ordinary SA models can, we believe that our approach is more straightforward to apply than that of Lee & Willmot (2016). Our proofs are certainly simpler. We also identify and implement an approach to finding moments of the time of ruin in an equilibrium SA model for which k is the Erlang(n) density.

In the next section, we outline some preliminaries and then give the main results in section 3. We then consider the application of these results, starting with the special case of exponentially distributed individual claim amounts in section 4. The classical risk model is considered in section 5, the phase-type(2) risk model in section 6, and the Erlang(n) risk model in section 8. In section 7, we illustrate an approach to modified SA models.

https://doi.org/10.1017/S1748499522000124 Published online by Cambridge University Press

2. Preliminaries

In this section, we obtain some results for Laplace transforms which we will apply in the proofs of Theorems 3.1 and 3.2, and it is convenient to deal with the ordinary and modified models separately. Throughout, we use the notation $\tilde{a}$ to denote the Laplace transform of a function a, with a second tilde when we consider a bivariate Laplace transform.

2.1. The ordinary SA model

By considering whether or not the surplus level ever falls below u, we have

$$\phi(u) = \phi(0) + \int_0^u g(0, y)\, \phi(u-y)\, dy$$

(see, e.g. Willmot & Lin., 2001), so that

$$\tilde{\phi}(s) = \frac{\phi(0)}{s} + \tilde{g}(0, s)\, \tilde{\phi}(s) \tag{1}$$

giving

$$\frac{1}{1-\tilde{g}(0, s)} = \frac{s\tilde{\phi}(s)}{\phi(0)} = \frac{\tilde{\phi}_d(s)}{\phi(0)} + 1 \tag{2}$$

where $\phi_d(u) = \frac{d}{du}\, \phi(u)$.

Second, the density of the time of ruin satisfies

$$w(u, t) = \int_0^t \int_0^u w(0, y, \tau)\, w(u-y, t-\tau)\, dy\, d\tau + \int_u^\infty w(0, y, t)\, dy \tag{3}$$

(see Dickson, 2008), and we define the bivariate Laplace transform of $w(u, t)$ as

$$\tilde{\tilde{w}}(s, \delta) = \int_0^\infty \int_0^\infty e^{-su-\delta t}\, w(u, t)\, dt\, du\, .$$

Further defining

$$\tilde{\tilde{w}}(0, s, \delta) = \int_0^\infty \int_0^\infty e^{-sy-\delta t}\, w(0, y, t)\, dy\, dt$$

to be the bivariate Laplace transform of $w(0, y, t)$ and defining

$$\tilde{\tilde{w}}_1(s, \delta) = \int_0^\infty \int_0^\infty e^{-su-\delta t} \int_u^\infty w(0, y, t)\, dy\, dt\, du\, ,$$

we see that

$$\tilde{\tilde{w}}(s, \delta) = \tilde{\tilde{w}}(0, s, \delta)\, \tilde{\tilde{w}}(s, \delta) + \tilde{\tilde{w}}_1(s, \delta)\, . \tag{4}$$

We next consider derivatives of these transforms with respect to δ as well as the transforms evaluated at $\delta = 0$. First,

$$\frac{d^n}{d\delta^n}\, \tilde{\tilde{w}}(s, \delta)\bigg|_{\delta=0} = (-1)^n \int_0^\infty e^{-su} \int_0^\infty t^n\, w(u, t)\, dt\, du = (-1)^n\, \tilde{\psi}_n(s)\, .$$

Next we have

$$\tilde{\tilde{w}}(0, s, \delta)\bigg|_{\delta=0} = \int_0^\infty e^{-sy} \int_0^\infty w(0, y, t)\, dt\, dy = \int_0^\infty e^{-sy}\, g(0, y)\, dy = \tilde{g}(0, s)$$

and

$$\frac{d^n}{d\delta^n}\, \tilde{\tilde{w}}(0, s, \delta)\bigg|_{\delta=0} = (-1)^n \int_0^\infty e^{-sy} \int_0^\infty t^n\, w(0, y, t)\, dt\, dy = (-1)^n\, \tilde{\nu}_n(s)$$

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where for $n = 0, 1, 2, \ldots$ we define

$$v_n(y) = \int_0^\infty t^n \, w(0, y, t) \, dt \,. \tag{5}$$

We note that $v_0(y) = g(0, y)$.

Finally,

$$\tilde{\tilde{w}}_1(s, \delta)\Big|_{\delta=0} = \int_0^\infty e^{-su} \int_u^\infty \int_0^\infty w(0, y, t) \, dt \, dy \, du = \tilde{\bar{G}}(0, s)$$

where

$$\bar{G}(0, u) = \int_u^\infty g(0, y) \, dy$$

and

$$\frac{d^n}{d\delta^n} \tilde{\tilde{w}}_1(s, \delta) = (-1)^n \int_0^\infty e^{-su} \int_u^\infty \int_0^\infty t^n \, w(0, y, t) \, dt \, dy \, du = (-1)^n \, \tilde{\bar{N}}_n(s)$$

where for $n = 1, 2, 3, \ldots$ we define

$$\bar{N}_n(u) = \int_u^\infty v_n(x) \, dx \,.$$

2.2. The modified SA model

For the modified SA model, we need a smaller number of preliminary results. Using the arguments that underpin equation (3), we find that

$$w^m(u, t) = \int_0^t \int_0^u w^m(0, y, \tau) \, w(u - y, t - \tau) \, dy \, d\tau + \int_u^\infty w^m(0, y, t) \, dy \tag{6}$$

and we define

$$\tilde{\tilde{w}}^m(0, s, \delta) = \int_0^\infty \int_0^\infty e^{-sy-\delta t} \, w^m(0, y, t) \, dy \, dt$$

to be the bivariate Laplace transform of $w^m(0, y, t)$. Next define

$$\tilde{\tilde{w}}_1^m(s, \delta) = \int_0^\infty \int_0^\infty e^{-su-\delta t} \int_u^\infty w^m(0, y, t) \, dy \, dt \, du \,,$$

so that

$$\tilde{\tilde{w}}^m(s, \delta) = \tilde{\tilde{w}}^m(0, s, \delta) \, \tilde{\tilde{w}}(s, \delta) + \tilde{\tilde{w}}_1^m(s, \delta) \,. \tag{7}$$

We then have

$$\frac{d^n}{d\delta^n} \tilde{\tilde{w}}^m(s, \delta)\Big|_{\delta=0} = (-1)^n \int_0^\infty e^{-su} \int_0^\infty t^n \, w^m(u, t) \, dt \, du = (-1)^n \, \tilde{\psi}_n^m(s) \,,$$

$$\frac{d^n}{d\delta^n} \tilde{\tilde{w}}^m(0, s, \delta)\Big|_{\delta=0} = (-1)^n \int_0^\infty e^{-sy} \int_0^\infty t^n \, w^m(0, y, t) \, dt \, dy = (-1)^n \, \tilde{v}_n^m(s)$$

where for $n = 0, 1, 2, \ldots$ we define

$$v_n^m(y) = \int_0^\infty t^n \, w^m(0, y, t) \, dt \,,$$

and

$$\frac{d^n}{d\delta^n} \tilde{\tilde{w}}_1^m(s, \delta) = (-1)^n \int_0^\infty e^{-su} \int_u^\infty \int_0^\infty t^n \, w^m(0, y, t) \, dt \, dy \, du = (-1)^n \, \tilde{\bar{N}}_n^m(s)$$

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where for $n = 0, 1, 2, \ldots$ we define

$$\bar{N}_n^m(u) = \int_u^\infty \nu_n^m(x)\, dx\,.$$

3. The Main Results

We now derive expressions for moments of the time of ruin for ordinary and modified SA models in Theorems 3.1 and 3.2, respectively.

Theorem 3.1. *For $n = 1, 2, 3, \ldots$, the nth moment of the time of ruin for the ordinary SA risk model is given by*

$$\psi_n(u) = A_n(u) + \frac{1}{\phi(0)} \int_0^u A_n(x)\, \phi_d(u-x)\, dx \tag{8}$$

where

$$A_n(u) = \bar{N}_n(u) + \sum_{j=1}^{n} \binom{n}{j} \int_0^u \nu_j(x)\, \psi_{n-j}(u-x)\, dx\,. \tag{9}$$

Proof. Differentiate equation (4) n times with respect to δ using Liebniz rule (e.g. Abramowitz & Stegun, 1965) to get

$$\frac{d^n}{d\delta^n}\, \tilde{\tilde{w}}(s,\delta) = \tilde{\tilde{w}}(0,s,\delta)\, \frac{d^n}{d\delta^n}\, \tilde{\tilde{w}}(s,\delta) + \sum_{j=1}^{n} \binom{n}{j} \frac{d^j}{d\delta^j}\, \tilde{\tilde{w}}(0,s,\delta)\, \frac{d^{n-j}}{d\delta^{n-j}}\, \tilde{\tilde{w}}(s,\delta) + \frac{d^n}{d\delta^n}\, \tilde{\tilde{w}}_1(s,\delta)\,,$$

and setting $\delta = 0$ we obtain

$$(-1)^n\, \tilde{\psi}_n(s) = (-1)^n\, \tilde{g}(0,s)\, \tilde{\psi}_n(s) + \sum_{j=1}^{n} \binom{n}{j} (-1)^j\, \tilde{\nu}_j(s)\, (-1)^{n-j}\, \tilde{\psi}_{n-j}(s) + (-1)^n\, \tilde{\bar{N}}_n(s)$$

giving

$$\tilde{\psi}_n(s) = \frac{\tilde{A}_n(s)}{1 - \tilde{g}(0,s)} \tag{10}$$

where $A_n(u)$ is as defined in the statement of the theorem. Then, using (2) to replace $1/(1 - \tilde{g}(0,s))$ in (10) we have for $n = 1, 2, 3, \ldots$

$$\psi_n(u) = A_n(u) + \frac{1}{\phi(0)} \int_0^u A_n(x)\, \phi_d(u-x)\, dx\,.$$

□

Corollary 3.1. *When the initial surplus is 0, for $n = 0, 1, 2, \ldots$ we have*

$$\psi_n(0) = A_n(0) = \int_0^\infty \int_0^\infty t^n\, w(0,y,t)\, dt\, dy \; = \int_0^\infty t^n\, w(0,t)\, dt.$$

Theorem 3.2. *For $n = 1, 2, 3, \ldots$, the nth moment of the time of ruin for the modified SA risk model is given by*

$$\psi_n^m(u) = \sum_{j=0}^{n} \binom{n}{j} \int_0^u \nu_j^m(x)\, \psi_{n-j}(u-x)\, dx + \bar{N}_n^m(u)\,. \tag{11}$$

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Proof. Differentiate equation (7) n times with respect to δ using Liebniz rule to get

$$\frac{d^n}{d\delta^n}\,\tilde{\tilde{w}}^m(s,\delta) = \sum_{j=0}^{n}\binom{n}{j}\frac{d^j}{d\delta^j}\,\tilde{\tilde{w}}^m(0,s,\delta)\,\frac{d^{n-j}}{d\delta^{n-j}}\,\tilde{\tilde{w}}(s,\delta) + \frac{d^n}{d\delta^n}\,\tilde{\tilde{w}}_1^m(s,\delta)\,,$$

and setting $\delta = 0$ we get

$$(-1)^n\,\tilde{\psi}_n^m(s) = \sum_{j=0}^{n}\binom{n}{j}(-1)^j\,\tilde{\nu}_j^m(s)\,(-1)^{n-j}\,\tilde{\psi}_{n-j}(s) + (-1)^n\,\tilde{\bar{N}}_n^m(s)$$

giving

$$\psi_n^m(u) = \sum_{j=0}^{n}\binom{n}{j}\int_0^u \nu_j^m(x)\,\psi_{n-j}(u-x)\,dx + \bar{N}_n^m(u)\,.$$

□

To prove Theorems 3.1 and 3.2, we did not need to make any distributional assumptions. However, our third result applies only when the equilibrium distribution of K is of a particular form.

Theorem 3.3. *Suppose that $k^e(t) = \sum_{i=1}^r p_i\,\kappa_i(t)$ where $\sum_{i=1}^r p_i = 1$, and $\{\kappa_i\}_{i=1}^r$ are density functions. Let $\psi_n^{m_i}(u)$ denote the nth moment of the time of ruin for a modified SA process which has κ_i as the density function of the time to the first claim for $i = 1, 2, 3, \ldots, r$. Then for $n = 0, 1, 2, \ldots$*

$$\psi_n^e(u) = \sum_{i=1}^{r} p_i\,\psi_n^{m_i}(u)\,.$$

If there is value of i such that $\kappa_i(t) = k(t)$, then we replace $\psi_n^{m_i}(u)$ with $\psi_n(u)$.

We remark that we do not require positive values for each p_i.

Proof. By conditioning on the time and the amount of the first claim, we have

$$W^{m_i}(0,y,t) = \int_0^t \kappa_i(s)\int_0^{cs} f(x)\,W(cs-x,y,t-s)\,dx\,ds + \int_0^t \kappa_i(s)\int_{cs}^{cs+y} f(x)\;dx\,ds$$

for $i = 1, 2, 3, \ldots, r$. Similarly,

$$\begin{aligned} W^e(0,y,t) &= \int_0^t k^e(s)\int_0^{cs} f(x)\,W(cs-x,y,t-s)\,dx\,ds + \int_0^t k^e(s)\int_{cs}^{cs+y} f(x)\;dx\,ds \\ &= \sum_{i=1}^{r} p_i\int_0^t \kappa_i(s)\int_0^{cs} f(x)\,W(cs-x,y,t-s)\,dx\,ds \\ &\quad + \sum_{i=1}^{r} p_i\int_0^t \kappa_i(s)\int_{cs}^{cs+y} f(x)\;dx\,ds \\ &= \sum_{i=1}^{r} p_i\,W^{m_i}(0,y,t) \end{aligned}$$

so that

$$w^e(0,y,t) = \sum_{i=1}^{r} p_i\,w^{m_i}(0,y,t)$$

and hence by (6)

$$w^e(u,t) = \sum_{i=1}^{r} p_i \int_0^t \int_0^u w^{m_i}(0,y,\tau)\, w(u-y,t-\tau)\, dy\, d\tau + \sum_{i=1}^{r} p_i \int_u^\infty w^{m_i}(0,y,t)\, dy$$

$$= \sum_{i=1}^{r} p_i\, w^{m_i}(u,t)\,.$$

Thus, for $n = 0, 1, 2, \ldots$

$$\psi_n^e(u) = \int_0^\infty t^n\, w^e(u,t)\, dt = \sum_{i=1}^{r} p_i \int_0^\infty t^n\, w^{m_i}(u,t)\, dt = \sum_{i=1}^{r} p_i\, \psi_n^{m_i}(u)\,.$$

□

In order to obtain explicit solutions from these results for particular distributions for inter-claim times and individual claim amounts, it is clear that we need to know something about $w(0, y, t)$ and $w^m(0, y, t)$, but as we shall see in the following sections, we do not need explicit formulae.

4. SA Models with Exponential Claims

We first consider any ordinary SA model with exponential individual claims, that is, with $f(x) = \alpha e^{-\alpha x}$, $x > 0$, since in this case we have the same general form of both the Laplace transform of T_u and $\psi(u)$ regardless of the form of k whenever the negative solution of Lundberg's fundamental equation exists. Specifically,

$$\mathrm{E}\left[e^{-\delta T_u}\, I(T_u < \infty)\right] = \left(1 - \frac{R(\delta)}{\alpha}\right) e^{-R(\delta)u} \tag{12}$$

where $-R(\delta)$ is the negative solution of Lundberg's fundamental equation. See Willmot & Woo (2017). The special case $\delta = 0$ gives $\psi(u) = \psi(0)e^{-Ru}$ where $R(= R(0))$ is the adjustment coefficient.

As $w(0, y, t) = w(0, t)\, \alpha e^{-\alpha y}$ by the memory-less property of the exponential distribution,

$$\nu_n(y) = \int_0^\infty t^n\, w(0,t)\, \alpha e^{-\alpha y}\, dt = \psi_n(0)\, \alpha e^{-\alpha y}$$

and

$$\bar{N}_n(u) = \int_u^\infty \nu_n(y)\, dy = \psi_n(0)\, e^{-\alpha u}\,.$$

Setting $n = 1$ in equation (9) gives

$$A_1(u) = \bar{N}_1(u) + \int_0^u \nu_1(x)\, \psi(u-x)\, dx = \psi_1(0)\, e^{-Ru}\,,$$

and then equation (8) gives

$$\psi_1(u) = A_1(u) + \frac{1}{\phi(0)} \int_0^u A_1(x)\, \phi_d(u-x)\, dx$$

$$= \psi_1(0)\, e^{-Ru}\, (1 + \alpha u \psi(0))\ .$$

One way to find the moments of T_0 is to set $u = 0$ in (12) and to note that

$$\psi_n(0) = (-1)^{n+1} \frac{d^n}{d\delta^n} \frac{R(\delta)}{\alpha}\bigg|_{\delta=0}\,,$$

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and we can find the derivatives of $R(\delta)$ from Lundberg's fundamental equation given the distribution of V_1. For example, using formulae given by Dickson & Hipp (2001) for the first two derivatives of $R(\delta)$ evaluated when $\delta = 0$ for the Erlang(2) risk model, we find that $\psi_1(0) = 4.0744$ and $\psi_2(0) = 187.4743$ when $c = 1.2$, $\alpha = 1$ and the Erlang scale parameter is $\beta = 2$.

It is easily verified that when $n = 2$, equations (9) and (8) give

$$A_2(u) = \psi_2(0)\, e^{-Ru} + 2\psi_1(0)^2\, \alpha u\, e^{-Ru}$$

and

$$\psi_2(u) = e^{-Ru}\Big(\psi_2(0) + \big(2\psi_1(0)^2 + \psi_2(0)\psi(0)\big)\, \alpha u + \psi(0)(\psi_1(0)\alpha u)^2\Big).$$

We remark that Borovkov & Dickson (2008) obtained a general expression for $w(u, t)$ in terms of the convolutions of the density k, and that in some cases, for example Erlang inter-claim times, an explicit solution for $w(u, t)$ exists. It is not an easy task to find the moments of the time of ruin directly from such solutions when $u > 0$; this point is amply demonstrated by Drekic & Willmot (2003) who consider the case of the classical risk model.

For a modified SA model, we have $w^m(0, y, t) = w^m(0, t)\, \alpha e^{-\alpha y}$ so that $v_n^m(y) = \psi_n^m(0)\, \alpha e^{-\alpha y}$ and $\bar{N}_n^m(u) = \psi_n^m(0)\, e^{-\alpha u}$. Setting $n = 1$ in equation (11) gives

$$\psi_1^m(u) = \bar{N}_1^m(u) + \int_0^u v_0^m(x)\, \psi_1(u-x)\, dx + \int_0^u v_1^m(x)\, \psi_0(u-x)\, dx$$

leading to

$$\psi_1^m(u) = \big(\psi_1^m(0) + \psi_0^m(0)\psi_1(0)\alpha u\big)\, e^{-Ru}.$$

Similarly, setting $n = 2$ in equation (11) gives the final result as

$$\psi_2^m(u) = \big(\psi_2^m(0) + \big(\psi_0^m(0)\psi_2(0) + 2\psi_1^m(0)\psi_1(0)\big)\, \alpha u + \psi_0^m(0)(\psi_1(0)\alpha u)^2\big)\, e^{-Ru}.$$

We discuss an approach to finding $\psi_n^m(0)$ in section 7, and in section 8, we show how $\psi_n^m(0)$ can be obtained for some modified Erlang risk models.

5. The Classical Risk Model

We now consider the classical risk model, so that $\{N(t)\}_{t\geq 0}$ is a Poisson process with parameter λ. For this model, we can apply formula (8) for individual claim amount distributions for which the factorisation of f introduced by Willmot (2007) holds, namely

$$f(x+y) = \sum_{j=1}^{m} \eta_j(x)\, \tau_j(y) \tag{13}$$

for functions $\{\eta_j, \tau_j\}_{j=1}^m$. Willmot (2007) shows that if the individual claim amount distribution is an infinite mixture of Erlang distributions with the same scale parameter then this factorisation applies. See also Willmot & Woo (2007) for further applications.

The bivariate Laplace transform of $w(0, y, t)$ is

$$\int_0^\infty \int_0^\infty e^{-\delta t - sy}\, w(0, y, t)\, dy\, dt = \frac{\lambda}{c} \int_0^\infty \int_0^\infty e^{-\rho t - sy} f(t+y)\, dy\, dt$$

where $\rho = \rho(\delta)$ is the unique positive solution of $\lambda + \delta - cs = \lambda \tilde{f}(s)$, which is Lundberg's fundamental equation for this model. See Gerber & Shiu (1998) for details. When (13)

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applies

$$\int_0^\infty \int_0^\infty e^{-\delta t - sy} w(0, y, t)\, dy\, dt = \frac{\lambda}{c} \int_0^\infty \int_0^\infty e^{-\rho t - sy} \sum_{j=1}^{m} \eta_j(t)\, \tau_j(y)\, dy\, dt$$

$$= \frac{\lambda}{c} \sum_{j=1}^{m} \tilde{\eta}_j(\rho)\, \tilde{\tau}_j(s)$$

and so $w(0, y, t)$ is of the form $w(0, y, t) = \sum_{j=1}^{m} h_j(t)\, \tau_j(y)$ where $h_j(t) = \lambda \eta_j(t)/c$.

To evaluate $v_n(y)$ from (5), we require quantities H_j^n defined as

$$H_j^n = \int_0^\infty t^n\, h_j(t)\, dt$$

for $n = 1, 2, 3, \ldots$ and $j = 1, 2, 3, \ldots, m$, and we can find these quantities without identifying $\{h_j\}_{j=1}^m$. We have

$$\int_0^\infty t^n\, h_j(t)\, dt = (-1)^n \frac{d^n}{d\delta^n} \tilde{h}_j(\rho)\Big|_{\delta=0} \tag{14}$$

since $\rho = 0$ when $\delta = 0$; see Gerber & Shiu (1998). Further, the derivatives of ρ when $\delta = 0$ are easily found, and the first two are given in Dickson (2016).

Example 5.1. Let $\lambda = 1$, $c = 1.2$ and $f(x) = \frac{4}{3} e^{-2x} + \frac{1}{6} e^{-x/2}$. Then

$$f(x + y) = \tfrac{2}{3} e^{-2x}\, 2e^{-2y} + \tfrac{1}{3} e^{-x/2}\, \tfrac{1}{2} e^{-y/2}$$

and

$$w(0, y, t) = h_1(t)\, 2e^{-2y} + h_2(t)\, \tfrac{1}{2} e^{-y/2},$$

where

$$\tilde{h}_1(\delta) = \frac{5}{9(2+\rho)} \quad \text{and} \quad \tilde{h}_2(\delta) = \frac{5}{9(1+2\rho)}.$$

Using (14) these lead to $H_1^1 = \frac{25}{36}$ and $H_2^1 = \frac{50}{9}$. Further,

$$\psi(u) = 0.7990\, e^{-R_1 u} + 0.0343\, e^{-R_2 u}$$

where $R_1 = 0.1069$ and $R_2 = 1.5598$, leading to

$$A_1(u) = 6.2317\, e^{-R_1 u} + 0.0183\, e^{-R_2 u}$$

and

$$\psi_1(u) = (7.6152 + 3.1922u)\, e^{-R_1 u} - (1.3652 - 0.0059u)\, e^{-R_2 u},$$

in agreement with Example 6.1 of Lin & Willmot (2000). Similarly we find that $H_1^2 = 55\frac{5}{9}$ and $H_2^2 = 527\frac{7}{9}$ leading to

$$A_2(u) = (584.79 + 49.793u)\, e^{-R_1 u} - (1.4547 - 0.0063)\, e^{-R_2 u}$$

and

$$\psi_2(u) = (705.93 + 360.36u + 12.753u^2)\, e^{-R_1 u} - (122.60 + 0.4630u - 0.0010u^2)\, e^{-R_2 u}.$$

6. Phase-Type(2) Risk Models

We can follow a process very similar to that in the previous section when the distribution of times between claims is phase-type(2) (also called Coxian(2)). However, we must now impose

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extra conditions on the individual claim amount distribution beyond the factorisation (13), but these conditions are not particularly restrictive. Following Dickson & Li (2010), we consider the situation when the functions $\{\eta_j\}_{j=1}^m$ satisfy the factorisation

$$\eta_j(x+y) = \sum_{i=1}^{n} \xi_{ij}(x)\, \zeta_{ij}(y)\,. \tag{15}$$

In the previous section, we made use of the unique positive solution ρ of Lundberg's fundamental equation for the classical risk model. We know from Ji & Zhang (2012) that in the phase-type(2) risk model, there are two distinct positive solutions to Lundberg's fundamental equation, and we can use these in a similar way.

If a density k belongs to the phase-type(2) class, its Laplace transform is of the form

$$\tilde{k}(s) = \frac{\lambda_1(1-p)s + \lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)} \tag{16}$$

where $0 < p \le 1$, $\lambda_i > 0$, for $i = 1, 2$ and $\lambda_2 \ne \lambda_1(1-p)$. (See, e.g. Willmot & Woo, 2017). From Li & Garrido (2005), we know that

$$\int_0^\infty \int_0^\infty e^{-\delta t - sy}\, w(0, y, t)\, dy\, dt = \sum_{j=1}^{2} b_j \int_0^\infty \int_0^\infty e^{-r_j t - sy} f(t+y)\, dy\, dt$$

where $r_1 = r_1(\delta)$ and $r_2 = r_2(\delta)$ are the positive solutions of Lundberg's fundamental equation,

$$b_1 = \frac{\lambda^* + (\delta - cr_1)\chi}{c^2(r_2 - r_1)} \quad \text{and} \quad b_2 = \frac{\lambda^* + (\delta - cr_2)\chi}{c^2(r_1 - r_2)}\,,$$

$\lambda^* = \lambda_1\lambda_2$ and $\chi = \lambda_1(1-p)$.

Using the factorisation (13) of f, we get

$$\int_0^\infty \int_0^\infty e^{-\delta t - sy}\, w(0, y, t)\, dy\, dt = \sum_{j=1}^{2} b_j \sum_{l=1}^{m} \tilde{\eta}_l(r_j)\, \tilde{\tau}_l(s) = \sum_{l=1}^{m} \sum_{j=1}^{2} b_j\, \tilde{\eta}_l(r_j)\, \tilde{\tau}_l(s)$$

and so $w(0, y, t) = \sum_{l=1}^m \eta_l^*(t)\, \tau_l(y)$ where

$$\begin{aligned}
\tilde{\eta}_l^*(\delta) &= \sum_{j=1}^{2} b_j\, \tilde{\eta}_l(r_j) \\
&= \frac{\lambda^* + (\delta - cr_1)\chi}{c^2(r_2 - r_1)}\, \tilde{\eta}_l(r_1) - \frac{\lambda^* + (\delta - cr_2)\chi}{c^2(r_2 - r_1)}\, \tilde{\eta}_l(r_2) \\
&= \frac{\lambda^* + \delta\chi}{c^2}\, \frac{\tilde{\eta}_l(r_1) - \tilde{\eta}_l(r_2)}{r_2 - r_1} - \frac{\chi}{c}\, \frac{r_1\, \tilde{\eta}_l(r_1) - r_2\, \tilde{\eta}_l(r_2)}{r_2 - r_1} \\
&= \frac{\lambda^* + \delta\chi}{c^2}\, \frac{\tilde{\eta}_l(r_1) - \tilde{\eta}_l(r_2)}{r_2 - r_1} - \frac{\chi}{c}\, \frac{\tilde{\eta}_l'(r_1) - \tilde{\eta}_l'(r_2)}{r_2 - r_1}\,.
\end{aligned}$$

We know that for a function α,

$$\int_0^\infty \int_0^\infty e^{-rx - sy}\, \alpha(x+y)\, dy\, dx = \frac{\tilde{\alpha}(r) - \tilde{\alpha}(s)}{s - r}$$

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(e.g. Dickson & Hipp, 2001) so when factorisation (15) applies

$$\begin{aligned}\frac{\tilde{\eta}_l(r_1)-\tilde{\eta}_l(r_2)}{r_2-r_1} &= \int_0^\infty \int_0^\infty e^{-r_1x-r_2y}\,\eta_l(x+y)\,dy\,dx \\ &= \int_0^\infty \int_0^\infty e^{-r_1x-r_2y}\sum_{i=1}^n \xi_{il}(x)\,\zeta_{il}(y)\,dy\,dx \\ &= \sum_{i=1}^n \tilde{\xi}_{il}(r_1)\,\tilde{\zeta}_{il}(r_2)\,.\end{aligned}$$

Further, if a similar factorisation applies to η_l', say

$$\eta_l'(x+y)=\sum_{i=1}^k \vartheta_{il}(x)\,\varphi_{il}(y)\,,$$

then

$$\tilde{\eta}_l^*(\delta)=\frac{\lambda^*+\delta\chi}{c^2}\sum_{i=1}^n \tilde{\xi}_{il}(r_1)\,\tilde{\zeta}_{il}(r_2)-\frac{\chi}{c}\sum_{i=1}^k \tilde{\vartheta}_{il}(r_1)\,\tilde{\varphi}_{il}(r_2)\,,$$

and we can differentiate this k times with respect to δ and then set $\delta=0$ to obtain quantities H_l^n which are now defined as

$$H_l^n=\int_0^\infty t^n\,\eta_l^*(t)\,dt\,.$$

We now illustrate ideas using the Erlang (2) distribution as our example. Although this is in some sense the simplest case (as we set $\lambda_1=\lambda_2=\beta$ and $p=1$ in (16), giving $\chi=0$), it is actually the case which yields most, as we can also obtain $\psi_n^e(u)$ in this case.

Example 6.1. We consider the same set-up as in Example 5.1, except that the claim inter-arrival times are Erlang(2) distributed and we set $\beta=2$. The factorisation of f gave $\eta_1(x)=\frac{2}{3}\,e^{-2x}$ and $\eta_2(x)=\frac{1}{3}\,e^{-x/2}$ so we can set

$$\xi_{11}(x)=\tfrac{2}{3}\,e^{-2x},\quad \zeta_{11}(y)=e^{-2y},\quad \xi_{12}(x)=\tfrac{1}{3}\,e^{-x/2},\quad \zeta_{12}(y)=e^{-y/2}\,,$$

resulting in $w(0,y,t)=\eta_1(t)\,2e^{-2y}+\eta_2(t)\,\frac{1}{2}\,e^{-y/2}$ where

$$\tilde{\eta}_1(\delta)=\tilde{\xi}_{11}(r_1)\,\tilde{\zeta}_{11}(r_2)=\frac{\beta^2}{c^2}\,\frac{2}{3(2+r_1)(2+r_2)}$$

and

$$\tilde{\eta}_2(\delta)=\tilde{\xi}_{12}(r_1)\,\tilde{\zeta}_{12}(r_2)=\frac{\beta^2}{c^2}\,\frac{4}{3(1+2r_1)(1+2r_2)}\,.$$

Differentiating $\tilde{\eta}_i(\delta)$ with respect to δ for $i=1,2$, then setting $\delta=0$, we find that $H_1^1=0.5312$ and $H_2^1=6.0399$. We also have

$$\psi(u)=0.7591\,e^{-R_1u}+0.0304\,e^{-R_2u}$$

where $R_1=0.1253$ and $R_2=1.6806$, leading to

$$A_1(u)=6.5477\,e^{-R_1u}+0.0233\,e^{-R_2u}$$

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and

$$\psi_1(u) = (7.5751 + 2.9576u)\, e^{-R_1 u} - (1.0040 - 0.0057u)\, e^{-R_2 u}\,.$$

Taking the second derivatives of $\tilde{\eta}_i(\delta)$ for $i = 1, 2$, we then find that $H_1^2 = 33.8510$ and $H_2^2 = 489.5552$, giving

$$A_2(u) = (524.95 + 51.020u)\, e^{-R_1 u} - (1.5443 - 0.0087u)\, e^{-R_2 u}$$

and

$$\psi_2(u) = (601.21 + 296.09u + 11.523u^2)\, e^{-R_1 u} - (77.807 + 0.3682u - 0.0011u^2)\, e^{-R_2 u}\,.$$

To find $\psi_n^e(u)$, we apply results from Dickson & Li (2012) who consider modified Erlang(n) risk models under which the distribution of the time to the first claim is Erlang(j) for $j = 1, 2, 3, \ldots, n-1$. In the case of the modified Erlang(2) risk model under which the distribution of the time to the first claim is Erlang(1) with the same scale parameter β, we find from formula (4.2) of Dickson & Li (2012) that $w^m(0, y, t) = \eta_1^m(t)\, 2e^{-2y} + \eta_2^m(t)\, \frac{1}{2}e^{-y/2}$ where

$$\tilde{\eta}_1^m(\delta) = \frac{2\beta}{3c}\left(\frac{1}{2 + r_1} - \frac{\beta}{c}\,\frac{\sqrt{\frac{4}{3(2+r_1)} + \frac{1}{3(1+2r_1)}}}{(2 + r_1)(2 + r_2)}\right)$$

and

$$\tilde{\eta}_2^m(\delta) = \frac{\beta}{3c}\left(\frac{2}{1 + 2r_1} - \frac{4\beta}{c}\,\frac{\sqrt{\frac{4}{3(2+r_1)} + \frac{1}{3(1+2r_1)}}}{(1 + 2r_1)(1 + 2r_2)}\right)\,.$$

Extending the previous notation we find that $H_1^{0,m} = \tilde{\eta}_1^m(0) = 0.3560$ and $H_2^{0,m} = \tilde{\eta}_2^m(0) = 0.5212$ giving

$$v_0^m(x) = 0.7119\, e^{-2x} + 1.0424\, e^{-x/2}\,.$$

We next find that $H_1^{1,m} = 0.3587$ and $H_2^{1,m} = 3.5965$, and from (11) we then find that

$$\psi_1^m(u) = (5.9902 + 3.1799u)\, e^{-R_1 u} - (2.0350 - 0.0114u)\, e^{-R_2 u}\,.$$

As the equilibrium distribution of the Erlang(2,β) distribution is an equally weighted average of the Erlang (1,β) and Erlang(2,β) distributions, Theorem 3.3 gives

$$\begin{aligned}\psi_1^e(u) &= \tfrac{1}{2}\left(\psi_1(u) + \psi_1^m(u)\right)\\ &= (6.7826 + 3.0687u)\, e^{-R_1 u} - (1.5195 - 0.0085u)\, e^{-R_2 u}\,.\end{aligned}$$

We then find $H_1^{2,m} = 19.8204$ and $H_2^{2,m} = 286.3038$ leading to

$$\psi_2^m(u) = (461.17 + 301.56u + 12.389u^2)\, e^{-R_1 u} - (155.04 + 0.7464u - 0.0021u^2)\, e^{-R_2 u}$$

and

$$\psi_2^e(u) = (531.19 + 298.83u + 11.956u^2)\, e^{-R_1 u} - (116.42 + 0.5573u - 0.0016u^2)\, e^{-R_2 u}.$$

We remark that the differentiation of $\tilde{\eta}_i(\delta)$ and $\tilde{\eta}_i^m(\delta)$ is easily done with mathematical software, and the derivatives of $r_i(\delta)$ when $\delta = 0$ are easily computed – see Dickson & Hipp (2001).

The arguments that allow us to obtain the functions $\tilde{\eta}_i^m(\delta)$ for $i = 1, 2$ in the above example are given in section 4 of Dickson & Li (2012). Unfortunately, similar arguments do not seem to apply

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for other phase-type(2) risk models. However, in the next section we show how the negative solutions of Lundberg's fundamental equation can be used to obtain the bivariate Laplace transform of $w^m(0, y, t)$, giving us what we need to find $\psi_n^m(u)$.

7. Modified SA Models

We now illustrate how we can find the Laplace transform of $w^m(0, y, t)$ (leading to the functions $v_n^m(u)$) using the negative solutions of Lundberg's fundamental equation. We can do this by considering the Gerber-Shiu function with initial surplus 0 for a modified SA model of the form

$$\begin{aligned}\varphi^m(0) &= \mathrm{E}[\,\exp\{-\delta T_0^m - sY_0^m\}\, I(T_0^m < \infty)] \\ &= \int_0^\infty e^{-\delta t}\, k_1(t) \int_0^{ct} f(x)\, \varphi(ct-x)\, dx\, dt + \int_0^\infty e^{-\delta t}\, k_1(t) \int_{ct}^\infty f(x)\, e^{-s(x-ct)}\, dx\, dt\end{aligned}$$

where $\varphi(u) = \mathrm{E}[\,\exp\{-\delta T_u - sY_u\}\, I(T_u < \infty)]$ is the equivalent Gerber-Shiu function for the ordinary SA model. The objective is to first solve for $\varphi(u)$ and then use this to express $\varphi^m(0)$ as a sum of products of Laplace transforms in δ and s. We illustrate the idea in the next example. It seems difficult to give general results as we require an assumption for k_1 in order to solve for $\varphi^m(0)$.

Example 7.1. Suppose that in an ordinary SA model, the claim inter-arrival times are distributed as the sum of two independent exponential random variables with parameters λ_1 and λ_2 respectively where $\lambda_1 < \lambda_2$, and $f(x) = \sum_{i=1}^2 q_i\, \alpha_i\, e^{-\alpha_i x}$ where $q_1 + q_2 = 1$. Gerber & Shiu (2005) show that for this model

$$\varphi(u) = C_1\, e^{-R_1 u} + C_2\, e^{-R_2 u}$$

where $-R_i(=-R_i(\delta))$ for $i = 1, 2$ are the negative solutions of Lundberg's fundamental equation

$$(\lambda_1 + \delta - cs)(\lambda_2 + \delta - cs) = \lambda_1\, \lambda_2\, \tilde{f}(s)\,, \tag{17}$$

and

$$\sum_{k=1}^{2} \frac{C_k}{\alpha_i - R_k} = \frac{1}{\alpha_i + s} \tag{18}$$

for $i = 1, 2$. Equations (18) are exactly the same equations that are solved in the context of the classical risk model by Dickson & Drekic (2006), so we have

$$C_j = \sum_{k=1}^{2} \gamma_{j,k}(\delta)\, \frac{\alpha_k}{\alpha_k + s}$$

where

$$\gamma_{1,1}(\delta) = \frac{(\alpha_1 - R_1)(\alpha_1 - R_2)(\alpha_2 - R_1)}{\alpha_1(R_2 - R_1)(\alpha_1 - \alpha_2)}\,, \quad \gamma_{1,2}(\delta) = \frac{-(\alpha_1 - R_1)(\alpha_2 - R_1)(\alpha_2 - R_2)}{\alpha_1(R_2 - R_1)(\alpha_1 - \alpha_2)}\,,$$

$$\gamma_{2,1}(\delta) = \frac{-(\alpha_1 - R_1)(\alpha_1 - R_2)(\alpha_2 - R_2)}{\alpha_2(R_2 - R_1)(\alpha_1 - \alpha_2)}\,, \quad \gamma_{2,2}(\delta) = \frac{(\alpha_1 - R_2)(\alpha_2 - R_1)(\alpha_2 - R_2)}{\alpha_2(R_2 - R_1)(\alpha_1 - \alpha_2)}\,.$$

Suppose that we want to find the moments $\psi_n^e(u)$. As both k and k^e are combinations of exponentials, it is sufficient to consider the case when $k_1(t) = \lambda e^{-\lambda t}$ for $t > 0$, as we can then find the

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required quantities using Theorems 3.2 and 3.3. We get

$$\varphi^m(0) = \sum_{i=1}^{2}\sum_{j=1}^{2}\int_0^\infty \lambda\, e^{-(\lambda+\delta)t}\int_0^{ct} q_i\,\alpha_i\, e^{-\alpha_i x}\, C_j\, e^{-R_j(ct-x)}\,dx\,dt$$

$$+\sum_{i=1}^{2}\int_0^\infty \lambda\, e^{-(\lambda+\delta)t}\int_{ct}^\infty q_i\,\alpha_i\, e^{-\alpha_i x}\, e^{-s(x-ct)}\,dx\,dt$$

$$=\lambda\sum_{i=1}^{2}\sum_{j=1}^{2} q_i\,\alpha_i\, C_j\int_0^\infty e^{-(\lambda+\delta+cR_j)t}\,\frac{1-e^{-(\alpha_i-R_j)ct}}{\alpha_i-R_j}\,dt$$

$$+\lambda\sum_{i=1}^{2} q_i\,\alpha_i\int_0^\infty \frac{e^{-(\lambda+\delta+\alpha_i c)t}}{\alpha_i+s}\,dt$$

$$=\lambda\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{q_i\,\alpha_i\,C_j}{(\lambda+\delta+cR_j)(\alpha_i-R_j)}-\lambda\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{q_i\,\alpha_i\,C_j}{(\lambda+\delta+\alpha_i c)(\alpha_i-R_j)}$$

$$+\lambda\sum_{i=1}^{2}\frac{q_i\,\alpha_i}{(\lambda+\delta+\alpha_i c)(\alpha_i+s)}$$

$$=\lambda\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{q_i\,\alpha_i\,C_j\,c}{(\lambda+\delta+cR_j)(\lambda+\delta+\alpha_i c)}+\lambda\sum_{i=1}^{2}\frac{q_i\,\alpha_i}{(\lambda+\delta+\alpha_i c)(\alpha_i+s)}$$

$$=\lambda\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{q_i\,\alpha_i\,c}{(\lambda+\delta+cR_j)(\lambda+\delta+\alpha_i c)}\sum_{k=1}^{2}\gamma_{j,k}(\delta)\,\frac{\alpha_k}{\alpha_k+s}$$

$$+\lambda\sum_{i=1}^{2}\frac{q_i\,\alpha_i}{(\lambda+\delta+\alpha_i c)(\alpha_i+s)}\,.$$

So

$$w^m(0,y,t)=\sum_{l=1}^{2}\eta_l(t)\,\alpha_l e^{-\alpha_l y}$$

where

$$\tilde{\eta}_l(\delta)=\lambda\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{q_i\,\alpha_i\,c\,\gamma_{j,l}(\delta)}{(\lambda+\delta+cR_j)(\lambda+\delta+\alpha_i c)}+\frac{\lambda\, q_l}{(\lambda+\delta+\alpha_l c)}\,.$$

We can evaluate $\tilde{\eta}_l(0)$ and by differentiating Lundberg's fundamental equation (17) we can find the derivatives of $R_i(\delta)$ when $\delta=0$, and we can insert these into the derivatives of $\tilde{\eta}_l(\delta)$ to evaluate the quantities $H_l^{i,m}$ for $l=1,2$ and $i=1,2,3,\ldots$. All other details required to find $\psi_n(u)$ for this example can be found in Gerber & Shiu (2005).

We remark that this approach can also be used to find the bivariate transform of $w(0,y,t)$. For this, we simply require to find $\varphi(u)$. However, as in the above example, the transforms with respect

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to δ may be complicated expressions in terms of the negative solutions of Lundberg's fundamental equation.

It is also possible to use this approach to find an expression for the Laplace transform with respect to δ of the time of ruin for the modified process, and to then find the moments of the time of ruin for the modified process by differentiating this Laplace transform. However, this can be a very cumbersome process; interested readers can try to reproduce the results for $\psi_n^m(u)$ in Example 6.1 using this approach.

8. The Erlang(n) Risk Model

In this section, we consider the Erlang(n) risk model and take a different approach to finding the quantities for this model that are equivalent to H_j^n in section 5 and $H_j^{n,m}$ in section 6. We extend ideas from Dickson & Li (2012), and we start with some definitions from that paper. For $m = 0, 1, 2, \ldots$ let

$$p_m(t) = e^{-\beta t} \frac{(\beta t)^m}{m!} ,$$

where β is the scale parameter of the Erlang(n) distribution, and for $j = 0, 1, 2, \ldots, n-1$ define

$$g_{n,j}(x, t) = \sum_{r=1}^{\infty} p_{rn+j}(t) f^{r*}(x)$$

where f is still the density function for individual claims and f^{r*} is the r-fold convolution of f with itself. Dickson & Li (2012) show that when the factorisation (13) holds, for a modified Erlang(n) risk model under which the distribution of the time to the first claim is Erlang(j) with scale parameter β, then, with m_j denoting this modification, the joint density of the time of ruin and deficit at ruin when the initial surplus is zero is of the form

$$w^{m_j}(0, y, t) = \sum_{l=1}^{m} h_{j,l}(t)\, \tau_l(y) , \tag{19}$$

and this holds for $j = 1, 2, 3, \ldots, n-1$. They give an expression for $\tilde{h}_{j,l}(\delta)$ (the final formula in their section 3.1), but this expression does not appear helpful for our purpose. It is convenient to adopt the notation $w^{m_n}(0, y, t)$ for the joint density of T_0 and Y_0 for the ordinary Erlang(n) risk model.

Using the arguments in Dickson & Li (2012), for this modified Erlang(n) risk model claims occur at occurrences $j, n+j, 2n+j, \ldots$ of an underlying Poisson process, and we find that

$$\begin{aligned} w^{m_j}(0, y, t) &= \beta\, p_{j-1}(t) f(ct + y) + \beta \sum_{r=1}^{\infty} p_{rn+j-1}(t) \int_0^{ct} f^{r*}(x) f(ct - x + y)\, dx \\ &\quad - c \sum_{i=0}^{n-1} \int_0^t g_{n,j+i-n}(cs, s)\, w^{m_{n-i}}(0, y, t-s)\, ds . \end{aligned}$$

The first two terms in the above consider the situation when a claim occurs in the interval $(t, t + dt)$, while the third term adjusts for upcrossings of the surplus process through level 0 prior to time t. Suppose the last upcrossing through 0 occurs with $n - i$ Poisson events until the next claim, for $i = 0, 1, 2, \ldots, n-1$. Then, we have a contribution $w^{m_{n-i}}(0, y, t - s)$. This means there have been i Poisson events from the time of the previous claim to time s, so the number of Poisson events to time s is $rn+j+i$ for some r, $r=0,1,2,\ldots$. Using (19) and (13) we get

$$w^{m_j}(0,y,t)=\beta\,p_{j-1}(t)\sum_{l=1}^{m}\eta_l(ct)\,\tau_l(y)+\beta\sum_{r=1}^{\infty}p_{rn+j-1}(t)\int_0^{ct}f^{r*}(x)\sum_{l=1}^{m}\eta_l(ct-x)\,\tau_l(y)\,dx$$
$$-\,c\sum_{i=0}^{n-1}\int_0^t g_{n,j+i-n}(cs,s)\sum_{l=1}^{m}h_{n-i,l}(t-s)\,\tau_l(y)\,ds\,,$$

so that

$$h_{j,l}(t)=\beta\,p_{j-1}(t)\,\eta_l(ct)+\beta\sum_{r=1}^{\infty}p_{rn+j-1}(t)\int_0^{ct}f^{r*}(x)\,\eta_l(ct-x)\,dx$$
$$-\,c\sum_{i=0}^{n-1}\int_0^t g_{n,j+i-n}(cs,s)\,h_{n-i,l}(t-s)\,ds\,. \tag{20}$$

In order to find moments of the time of ruin for the modified processes, we require quantities we denote by $H_{j,l}^r$ given by

$$H_{j,l}^r=\int_0^\infty t^r\,h_{j,l}(t)\,dt$$

for $r=0,1,2,\ldots$. For brevity, we now write (20) as

$$h_{j,l}(t)=a_{j,l}(t)-\sum_{i=0}^{n-1}\int_0^t\gamma_{j,i}(s)\,h_{n-i,l}(t-s)\,ds \tag{21}$$

where

$$a_{j,l}(t)=\beta\,p_{j-1}(t)\,\eta_l(ct)+\beta\sum_{r=1}^{\infty}p_{rn+j-1}(t)\int_0^{ct}f^{r*}(x)\,\eta_l(ct-x)\,dx$$

and $\gamma_{j,i}(s)=cg_{n,j+i-n}(cs,s)$. Further, for $r=0,1,2,\ldots$ let

$$A_{j,l}^r=\int_0^\infty t^r\,a_{j,l}(t)\,dt\qquad\text{and}\qquad\Gamma_{j,i}^r=\int_0^\infty t^r\,\gamma_{j,i}(t)\,dt\,.$$

Then from (21), we get

$$\begin{aligned}H_{j,l}^r&=A_{j,l}^r-\sum_{i=0}^{n-1}\int_0^\infty t^r\int_0^t\gamma_{j,i}(s)\,h_{n-i,l}(t-s)\,ds\,dt\\&=A_{j,l}^r-\sum_{i=0}^{n-1}\int_0^\infty\gamma_{j,i}(s)\int_s^\infty t^r\,h_{n-i,l}(t-s)\,dt\,ds\\&=A_{j,i}^r-\sum_{i=0}^{n-1}\int_0^\infty\gamma_{j,i}(s)\int_0^\infty(y+s)^r\,h_{n-i,l}(y)\,dy\,ds\\&=A_{j,l}^r-\sum_{i=0}^{n-1}\sum_{k=0}^{r}\binom{r}{k}\Gamma_{j,i}^{r-k}\,H_{n-i,l}^k\,.\end{aligned} \tag{22}$$

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Table 1. Values of $H^r_{j,i}$.

j, i	$r=0$	$r=1$	$r=2$
1,1	0.3339	0.6153	14.1243
2,1	0.4497	1.3429	32.9785
3,1	0.4664	1.9877	52.0599
1,2	0.5956	0.2946	4.8506
2,2	0.3848	0.5449	11.2958
3,2	0.2697	0.7453	17.7962

Thus, we have a system of equations that we can use to find the quantities $H^r_{j,l}$. We note that for $n=0, 1, 2, \ldots$

$$\psi_n^{m_j}(0) = \int_0^\infty t^n \int_0^\infty w^{m_j}(0, y, t)\, dy\, dt = \sum_{l=1}^m H^n_{j,l}\,.$$

Example 8.1. We consider the situation when the claim inter-arrival time distribution is Erlang(3,β), and the individual claim amount distribution is Erlang(2,α), and we find the moments $\psi_n(u)$ and $\psi_n^e(u)$ for $n=1, 2$. As

$$k^e(t) = \frac{\beta}{3}\, e^{-\beta x} \sum_{j=0}^2 \frac{(\beta x)^j}{j!} = \frac{1}{3} \sum_{i=1}^3 \kappa_i(t)$$

where $\kappa_i(t)$ is the Erlang(i,β) density, we can find the moments $\psi_n^e(u)$ using Theorem 3.3 by finding $\psi_n^{m_j}(u), j=1, 2, 3$. The factorisation (13) gives

$$\eta_1(x) = \alpha x e^{-\alpha x}, \quad \eta_2(x) = e^{-\alpha x}, \quad \tau_1(y) = \alpha e^{-\alpha y}, \quad \tau_2(y) = \alpha^2 y e^{-\alpha y}\,,$$

so that for $j=1, 2, 3$

$$w^{m_j}(0, y, t) = h_{j,1}(t)\, \tau_1(y) + h_{j,2}(t)\, \tau_2(y)$$

and for the ordinary Erlang(3) risk model

$$g(0, y) = H^0_{3,1}\, \tau_1(y) + H^0_{3,2}\, \tau_2(y)\,. \tag{23}$$

For $j=1, 2$ we find that

$$A^r_{j,1} = \sum_{m=0}^\infty \frac{\beta^{3m+j}}{(3m+j-1)!} \frac{(\alpha c)^{2m+1}}{(2m+1)!} \frac{(r+5m+j)!}{(\beta+\alpha c)^{r+5m+j+1}}$$

$$A^r_{j,2} = \sum_{m=0}^\infty \frac{\beta^{3m+j}}{(3m+j-1)!} \frac{(\alpha c)^{2m}}{(2m)!} \frac{(r+5m+j-1)!}{(\beta+\alpha c)^{r+5m+j}}\,,$$

and for $i=0, 1, 2$

$$\Gamma^r_{j,i} = \sum_{m=1}^\infty \frac{\beta^{3m+j+i-3}}{(3m+j+i-3)!} \frac{(\alpha c)^{2m}}{(2m-1)!} \frac{(5m+j+i+r-4)!}{(\beta+\alpha c)^{5m+j+i+r-3}}\,.$$

We now set the parameter values as $\alpha=2$, $\beta=3$ and $c=1.2$. Solving the system of equations (22) for $r=0, 1, 2$ yields the values in Table 1.

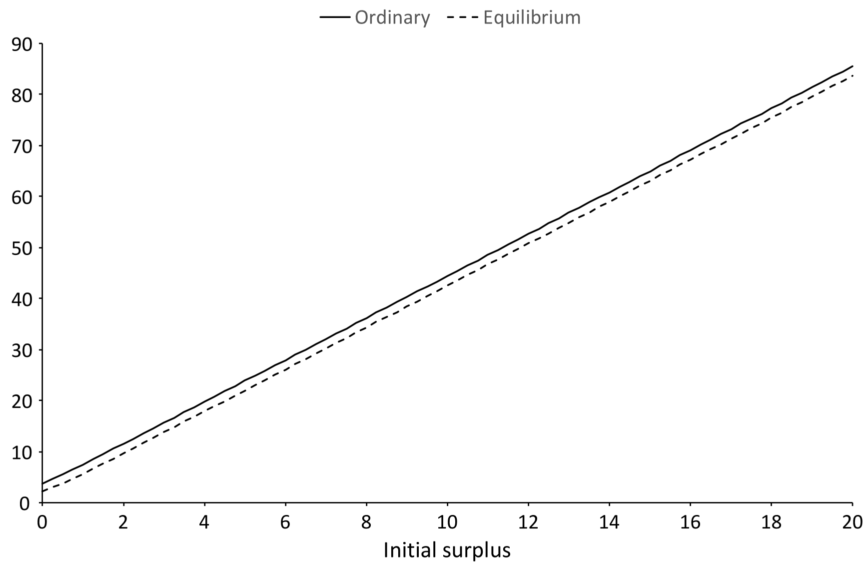

Figure 1. Mean time of ruin.

Noting that $\phi(0) = 1 - H^0_{3,1} - H^0_{3,2} = 0.2640$, we can use (1) and (23) to obtain $\phi(u)$, from which

$$\psi(u) = 0.7556\, e^{-R_1 u} - 0.0196\, e^{-R_2 u}$$

where $R_1 = 0.3952$ and $R_2 = 2.6721$. Applying the theorems from section 3, we obtain the first moments as

$$\psi_1(u) = (2.5005 + 3.1066u)\, e^{-R_1 u} + (0.2325 + 0.0027u)\, e^{-R_2 u}\,,$$

$$\psi_1^{m_2}(u) = (1.4012 + 3.5976u)\, e^{-R_1 u} + (0.4866 + 0.0055u)\, e^{-R_2 u}\,,$$

$$\psi_1^{m_1}(u) = (-0.1080 + 4.1663u)\, e^{-R_1 u} + (1.0179 + 0.0114u)\, e^{-R_2 u}\,,$$

$$\psi_1^{e}(u) = (1.2646 + 3.6235u)\, e^{-R_1 u} + (0.5790 + 0.0065u)\, e^{-R_2 u}\,,$$

and the second moments as

$$\psi_2(u) = (64.500 + 99.264u + 12.772u^2)\, e^{-R_1 u} + (5.3566 - 0.0625u - 0.0004u^2)\, e^{-R_2 u}\,,$$

$$\psi_2^{m_2}(u) = (33.322 + 102.67u + 14.791u^2)\, e^{-R_1 u} + (10.952 - 0.1307u - 0.0007u^2)\, e^{-R_2 u}\,,$$

$$\psi_2^{m_1}(u) = (-3.4108 + 104.66u + 17.129u^2)\, e^{-R_1 u} + (22.386 - 0.2735u - 0.0016u^2)\, e^{-R_2 u}\,,$$

$$\psi_2^{e}(u) = (31.470 + 102.20u + 14.897u^2)\, e^{-R_1 u} + (12.898 - 0.1556u - 0.0009u^2)\, e^{-R_2 u}\,.$$

Figures 1 and 2 respectively show the mean and standard deviation of the time of ruin, given that ruin occurs, for both the ordinary and the equilibrium models. It is not surprising that the plots are close in each figure, given that the only difference for these risk processes is the distribution of the time to the first claim.

https://doi.org/10.1017/S1748499522000124 Published online by Cambridge University Press

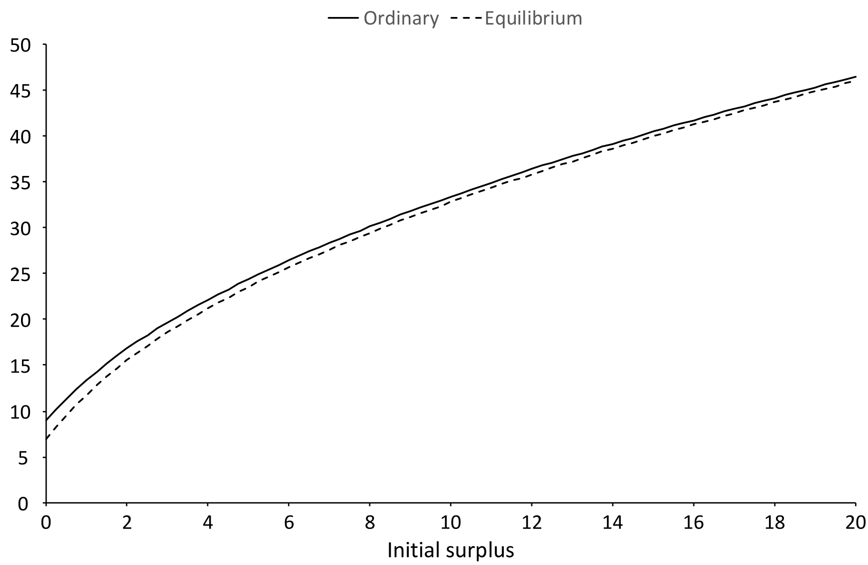

Figure 2. Standard deviation of time of ruin.

9. Concluding Remarks

We have presented general formulae for moments of the time of ruin in ordinary and modified Sparre Andersen risk models, and we have shown how they can be applied to find exact results. Our approach does not offer any advantage over the existing result by Lin & Willmot (2000) for the classical risk model. However, it does provide a fairly simple way of finding solutions for $\psi_n(u)$ in phase-type(2) risk models. In the case of Erlang risk models, we are able to find moments in the ordinary, equilibrium, and some modified cases.

Application of our approach in sections 5 and 6 hinges on some knowledge of the functions $w(0, y, t)$ and $w^m(0, y, t)$, as well as the factorisations we have imposed. In practice, these factorisations are not serious impositions as many individual claim amount distributions satisfy them. The approach in section 7 is more generally applicable, but is likely to require more tedious differentiation than in sections 5 and 6. To apply the approach in section 8, we must be able to compute convolutions of the individual claim amount distribution. This is not a great restriction as we can do this when the individual claim amount distribution is an infinite mixture of Erlang distributions with the same scale parameter, and Willmot & Woo (2007) show that a wide variety of distributions are of this type.

An open question is how to find the quantity $\nu_n(y)$ given by (5), either analytically or numerically (which is perhaps more likely), for forms of f for which the factorisation (13) does not apply. If we can do this, then it should be possible to use numerical approaches to find $\psi_n(u)$ for $u > 0$. An approach for the case $u = 0$ is presented in Dickson & Hipp (2001) where the inter-claim times are Erlang(2) distributed and the claim amount distribution is Pareto.

References

Abramowitz, M. & **Stegun, I.A.** (1965). *Handbook of Mathematical Functions*. Dover, New York.

Albrecher, H. & **Boxma, O.J.** (2005). On the discounted penalty function in a Markov-dependent risk model. *Insurance: Mathematics & Economics*, **37**, 650–672.

Borovkov, K.A. & **Dickson, D.C.M.** (2008). On the ruin time distribution for a Sparre Andersen process with exponential claim sizes. *Insurance: Mathematics & Economics*, **42**, 1104–1108.

https://doi.org/10.1017/S1748499522000124 Published online by Cambridge University Press

Dickson, **D.C.M.** & **Drekic**, **S.** (2006). Optimal dividends under a ruin probability constraint. *Annals of Actuarial Science*, **1**, 291–306.

Dickson, **D.C.M.** & **Hipp**, **C.** (2001). On the time to ruin for Erlang(2) risk processes. *Insurance: Mathematics & Economics*, **29**, 333–344.

Dickson, **D.C.M.** (2008). Some explicit solutions for the joint density of the time of ruin and the deficit at ruin. *ASTIN Bulletin*, **38**, 259–276.

Dickson, **D.C.M.** (2016). *Insurance Risk and Ruin*, 2nd edition. Cambridge University Press, Cambridge.

Dickson, **D.C.M.** & **Li**, **S.** (2010). Finite time ruin problems for the Erlang(2) risk model. *Insurance: Mathematics & Economics*, **46**, 12–18.

Dickson, **D.C.M.** & **Li**, **S.** (2012). Erlang risk models and finite time ruin problems. *Scandinavian Actuarial Journal*, **2012**(3), 183–202.

Drekic, **S.** & **Willmot**, **G.E.** (2003). On the density and moments of the time to ruin with exponential claims. *ASTIN Bulletin*, **33**, 11–21.

Gerber, **H.U.** (1979). *An Introduction to Mathematical Risk Theory*. S.S. Huebner Foundation, Philadelphia, PA.

Gerber, **H.U.** & **Shiu**, **E.S.W.** (1998). On the time value of ruin. *North American Actuarial Journal*, **2**(1), 48–78.

Gerber, **H.U.** & **Shiu**, **E.S.W.** (2005). The time value of ruin in a Sparre Andersen model. *North American Actuarial Journal*, **9**(2), 1–21.

Ji, **L.** & **Zhang**, **C.** (2012). Analysis of the multiple roots of the Lundberg fundamental equation in the PH(n) risk model. *Applied Stochastic Models in Business and Industry*, **28**, 73–90.

Kim, **S.-Y.** & **Willmot**, **G.E.** (2016). On the analysis of ruin-related quantities in the delayed renewal risk model. *Insurance: Mathematics & Economics*, **66**, 77–85.

Lee, **W.Y.** & **Willmot**, **G.E.** (2014). On the moments of the time to ruin in dependent Sparre Andersen models with emphasis on Coxian interclaim times. *Insurance: Mathematics & Economics*, **59**, 1–10.

Lee, **W.Y.** & **Willmot**, **G.E.** (2016). The moments of the time to ruin in dependent Sparre Andersen models with Coxian claim sizes. *Scandinavian Actuarial Journal*, **2016**(6), 550–564.

Li, **S.** & **Garrido**, **J.** (2005). On a general class of renewal risk process: analysis of the Gerber-Shiu function. *Advances in Applied Probability*, **37**, 836–856.

Lin, **X.** & **Willmot**, **G.E.** (2000). The moments of the time of ruin, the surplus before ruin, and the deficit at ruin. *Insurance: Mathematics & Economics*, **27**, 19–44.

Sparre Andersen, **E.** (1957). On the collective theory of risk in the case of contagion between the claims. In Transactions of the XV International Congress of Actuaries, Volume 2 (pp. 219–229).

Willmot, **G.E.** (2007). On the discounted penalty function in the renewal risk model with general interclaim times. *Insurance: Mathematics & Economics*, **41**, 17–31.

Willmot, **G.E.** & **Lin**, **X.S.** (2001). *Lundberg Approximations for Compound Distributions with Insurance Applications*. Springer, New York.

Willmot, **G.E.** & **Woo**, **J.K.** (2007). On the class of Erlang mixtures with risk theoretic applications. *North American Actuarial Journal*, **11**(2), 99–115.

Willmot, **G.E.** & **Woo**, **J.K.** (2017). *Surplus Analysis of Sparre Andersen Insurance Risk Processes*. Springer International Publishing AG, Switzerland.

Yu, **K.**, **Ren**, **J.** & **Stanford**, **D.A.** (2010). The moments of the time of ruin in Markovian risk models. *North American Actuarial Journal*, **14**(4), 464–471.

Cite this article: Dickson DCM (2023). The moments of the time of ruin in Sparre Andersen risk models, *Annals of Actuarial Science*, **17**, 63–82. https://doi.org/10.1017/S1748499522000124