6.   I am much indebted to Professor H. W. Turnbull for pointing out to me the existence of a set of reciprocal theorems which may be obtained from the "force diagrams" for the original nets.   They are:—

(i)   *If from a net of triangles with six edges per vertex, any interior set be removed and the vacated space be filled with triangles in any manner, then the average number of edges per vertex is precisely six.*   (Fig. 4)

(ii)   *If from a net of quadrilaterals with four edges per vertex any interior set be removed, and the vacated space be filled with quadrilaterals in any manner, then the average number of edges per vertex remains precisely four.*

(iii)   *The only (interior) modification in a net of hexagons, which replaces them by hexagons, is a mere deformation.*

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# A Note on Stirling's Theorem

By J. R. WILTON.

Let   $\Gamma(1+x) = \sqrt{(2\pi x)}\, x^x\, e^{-x}\, \phi(x)$;

then   $\lim\limits_{x \to \infty} \phi(x) = 1.$

This result is Stirling's theorem.   A simple proof is given in § 1.87 of Titchmarsh's *Theory of Functions* (Oxford Univ. Press, 1932).

Rather more than Stirling's theorem can be proved by a method which assumes nothing but the definition of the $\Gamma$-function, and $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$, from which it follows that

$$\left. \begin{aligned} &\text{if } x > 0,\ \int_0^\infty e^{-\frac{1}{2}x\tau^2}\, d\tau = \sqrt{\left(\frac{\pi}{2x}\right)}, \\ &\text{and} \qquad \int_0^\infty \tau^2 e^{-\frac{1}{2}x\tau^2}\, d\tau = \frac{\sqrt{\pi}}{x\sqrt{(2x)}}. \end{aligned} \right\} \tag{1}$$

The method is not new.   It was given, in essentials, in lectures by the late R. A. Herman, at the beginning of the century; and it may be much older.

If $x > 0$, the function $e^{-t} t^x$ steadily increases from 0 to $e^{-x} x^x$ as $t$ increases from 0 to $x$, and then steadily decreases, with limit 0 as $t \to \infty$. In

$$\Gamma (x + 1) = \int_0^\infty e^{-t} t^x \, dt,$$

then, put

$$e^{-t} t^x = e^{-x} x^x e^{-\frac{1}{2} x \tau^2},$$

i.e.,

$$\tfrac{1}{2} x \tau^2 = x \log x - x + t - x \log t. \tag{2}$$

For a given $\tau$, let the roots of (2) be

$$t_1 = x (1 - \xi) \leqq x$$

and

$$t_2 = x/(1 - \eta) \geqq x;$$

then $\quad 0 \leqq \xi < 1, \quad 0 \leqq \eta < 1, \quad$ and

$$\Gamma (x + 1) = \left( \int_0^x + \int_x^\infty \right) e^{-t} t^x \, dt$$

$$= e^{-x} x^{x+1} \int_0^\infty e^{-\frac{1}{2} x \tau^2} \left( \frac{1}{\xi} + \frac{1}{\eta} - 1 \right) \tau \, d\tau. \tag{3}$$

The equation (2) is equivalent to either of the following:

$$\frac{\tau^2}{\xi^2} = \frac{2}{\xi^2} \left( \log \frac{1}{1 - \xi} - \xi \right) = 1 + \frac{2}{3} \xi + \dots + \frac{2}{n + 2} \xi^n + \dots \tag{4.1}$$

$$\frac{\tau^2}{\eta^2} = \frac{2}{\eta^2} \left( \frac{\eta}{1 - \eta} - \log \frac{1}{1 - \eta} \right) = 1 + \frac{4}{3} \eta + \dots + \frac{2n + 2}{n + 2} \eta^n + \dots \tag{4.2}$$

From (4.1) we have

$$\left( \frac{\tau}{\xi} - \frac{1}{3} \tau \right)^2 = 1 + \frac{2}{9} \sum_{n=2}^\infty \frac{2n^2 + 1}{(n + 1)(n + 2)} \cdot \frac{2}{n} \xi^n < 1 + \frac{4}{9} \tau^2 < \left( 1 + \frac{2}{9} \tau^2 \right)^2,$$

so that

$$0 < \frac{\tau}{\xi} - \frac{1}{3} \tau - 1 < \frac{2}{9} \tau^2. \tag{5.1}$$

In the same way, from (4.2), we have

$$0 < \frac{\tau}{\eta} - \frac{2}{3} \tau - 1 < \frac{1}{12} \tau^2. \tag{5.2}$$

From (1), (3), (5.1) and (5.2) we have the theorem: *if* $x > 0$, *then* $1 < \phi (x) < 1 + 11/(72x)$.

With rather more elaborate analysis (using Bürmann's theorem to expand $\tau/\xi$ and $\tau/\eta$ in powers of $\tau$) there would be no difficulty in obtaining the initial terms in the asymptotic expansion of $\phi (x)$.