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PARTITIONS OF NATURAL NUMBERS AND THEIR WEIGHTED REPRESENTATION FUNCTIONS

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Abstract

For any positive integers k_1, k_2 and any set $A \subseteq \mathbb{N}$, let $R_{k_1,k_2}(A, n)$ be the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. Let g be a fixed integer. We prove that if k_1 and k_2 are two integers with $2 \le k_1 < k_2$ and $(k_1, k_2) = 1$, then there does not exist any set $A \subseteq \mathbb{N}$ such that $R_{k_1,k_2}(A, n) - R_{k_1,k_2}(\mathbb{N} \setminus A, n) = g$ for all sufficiently large integers n, and if $1 = k_1 < k_2$, then there exists a set A such that $R_{k_1,k_2}(A, n) - R_{k_1,k_2}(\mathbb{N} \setminus A, n) = 1$ for all positive integers n.

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1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For a set $A \subseteq \mathbb{N}$, let $R_1(A, n)$, $R_2(A, n)$ and $R_3(A, n)$ denote the number of solutions of $a_1 + a_2 = n, a_1, a_2 \in A$; $a_1 + a_2 = n, a_1, a_2 \in A, a_1 < a_2$ and $a_1 + a_2 = n, a_1, a_2 \in A, a_1 \le a_2$, respectively. For $i = 1, 2, 3$, Sárközy asked whether there exist two sets A and B with $|(A \cup B) \setminus (A \cap B)| = +\infty$ such that $R_i(A, n) = R_i(B, n)$ for all sufficiently large integers n. We call this problem the Sárközy problem. In 2002, Dombi [2] proved that the answer is negative for $i = 1$ and positive for $i = 2$. For $i = 3$, Chen and Wang [1] proved that the answer is also positive. In 2004, Lev [3] provided a new proof by using generating functions. Later, Sándor [5] determined the partitions of $\mathbb{N}$ into two sets with the same representation functions by using generating functions. In 2008, Tang [6] provided a simple proof by using the characteristic function.

In 2012, Yang and Chen [7] first considered the Sárközy problem with weighted representation functions. For any positive integers $k_1, \dots, k_t$ and any set $A \subseteq \mathbb{N}$, let $R_{k_1,\dots,k_t}(A, n)$ be the number of solutions of the equation $n = k_1a_1 + \cdots + k_ta_t$ with $a_1, \dots, a_t \in A$. They posed the following question.

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PROBLEM 1.1 [7, Problem 1]. Does there exist a set $A \subseteq \mathbb{N}$ such that $R_{k_1,\dots,k_t}(A,n) = R_{k_1,\dots,k_t}(\mathbb{N} \setminus A,n)$ for all $n \ge n_0$?

They answered this question for $t = 2$ and proved the following results.

THEOREM 1.2 [7, Theorem 1]. *If k_1 and k_2 are two integers with $k_2 > k_1 \ge 2$ and $(k_1,k_2) = 1$, then there does not exist any set $A \subseteq \mathbb{N}$ such that $R_{k_1,k_2}(A,n) = R_{k_1,k_2}(\mathbb{N} \setminus A,n)$ for all sufficiently large integers n.*

THEOREM 1.3 [7, Theorem 2]. *If k is an integer with $k > 1$, then there exists a set $A \subseteq \mathbb{N}$ such that*

$$R_{1,k}(A,n) = R_{1,k}(\mathbb{N} \setminus A,n) \tag{1.1}$$

for all integers $n \ge 1$.

Furthermore, if $0 \in A$, then (1.1) holds for all integers $n \ge 1$ if and only if

$$A = \{0\} \bigcup \Big(\bigcup_{i=0}^{\infty} [(k+1)k^{2i}, (k+1)k^{2i+1} - 1] \Big),$$

where $[x,y] = \{n : n \in \mathbb{Z}, x \le n \le y\}$.

Later, Li and Ma [4] proved the same results by using generating functions.

Let g be a fixed integer. In this paper, we consider whether there exists a set $A \subseteq \mathbb{N}$ such that $R_{k_1,k_2}(A,n) - R_{k_1,k_2}(\mathbb{N} \setminus A,n) = g$ for all $n \ge n_0$. First, we answer this problem in the negative if k_1 and k_2 are two integers with $2 \le k_1 < k_2$ and $(k_1,k_2) = 1$.

THEOREM 1.4. *Let g be a fixed integer. If k_1 and k_2 are two integers with $2 \le k_1 < k_2$ and $(k_1,k_2) = 1$, then there does not exist any set $A \subseteq \mathbb{N}$ such that*

$$R_{k_1,k_2}(A,n) - R_{k_1,k_2}(\mathbb{N} \setminus A,n) = g$$

for all sufficiently large integers n.

Similar to Theorem 1.3, we seek a set $A \subseteq \mathbb{N}$ such that $R_{1,k}(A,n) - R_{1,k}(\mathbb{N} \setminus A,n) = g$ for all integers $n \ge 1$. In fact, if $|g| > 1$, then such a set A does not exist by the simple observation that $0 \le R_{1,k}(A,n) \le 1$ and $0 \le R_{1,k}(\mathbb{N} \setminus A,n) \le 1$ for all positive integers $n < k$. So we only need to consider the case $g = 1$.

THEOREM 1.5. *If k is an integer with $k > 1$, then there exists a set $A \subseteq \mathbb{N}$ such that*

$$R_{1,k}(A,n) - R_{1,k}(\mathbb{N} \setminus A,n) = 1 \tag{1.2}$$

for all integers $n \ge 1$.

Furthermore, (1.2) holds for all integers $n \ge 1$ if and only if

$$A = \{0\} \bigcup \Big(\bigcup_{i=0}^{\infty} [k^{2i}, k^{2i+1} - 1] \Big).$$

2. Proofs

LEMMA 2.1. *Let $k_1 < k_2$ be two positive integers, $\{a(n)\}_{n=-\infty}^{+\infty}$ be a sequence of integers with $a(n) = 0$ for $n < 0$ and $A \subseteq \mathbb{N}$. Then the equality*

$$R_{k_1,k_2}(A, n) - R_{k_1,k_2}(\mathbb{N} \setminus A, n) = a(n) \tag{2.1}$$

holds for all nonnegative integers n if and only if

$$\chi_A\Big(\Big[\frac{n}{k_1}\Big]\Big) + \chi_A\Big(\Big[\frac{n}{k_2}\Big]\Big) = 1 + \sum_{j=0}^{k_1-1}(a(n-j) - a(n-k_2-j))$$

holds for all nonnegative integers n, where $\chi_A(i)$ is the characteristic function of A, that is, $\chi_A(i) = 1$ if $i \in A$ and $\chi_A(i) = 0$ if $i \notin A$.

PROOF. Let $f(x)$ be the generating function associated with A, that is,

$$f(x) = \sum_{a\in A} x^a = \sum_{i=0}^{\infty} \chi_A(i)x^i.$$

Then,

$$\begin{aligned}
&\sum_{n=0}^{\infty}(R_{k_1,k_2}(A, n) - R_{k_1,k_2}(\mathbb{N} \setminus A, n))x^n \\
&\quad = f(x^{k_1})f(x^{k_2}) - \Big(\frac{1}{1-x^{k_1}} - f(x^{k_1})\Big)\Big(\frac{1}{1-x^{k_2}} - f(x^{k_2})\Big) \\
&\quad = \frac{f(x^{k_1})}{1-x^{k_2}} + \frac{f(x^{k_2})}{1-x^{k_1}} - \frac{1}{(1-x^{k_1})(1-x^{k_2})}.
\end{aligned}$$

Let

$$p(x) = \sum_{n=0}^{\infty} a(n)x^n.$$

It follows that (2.1) holds for all nonnegative integers n if and only if

$$\frac{f(x^{k_1})}{1-x^{k_2}} + \frac{f(x^{k_2})}{1-x^{k_1}} - \frac{1}{(1-x^{k_1})(1-x^{k_2})} = p(x),$$

that is,

$$f(x^{k_1})\frac{1-x^{k_1}}{1-x} + f(x^{k_2})\frac{1-x^{k_2}}{1-x} = \frac{1}{1-x} + (1-x^{k_2})\frac{1-x^{k_1}}{1-x}p(x). \tag{2.2}$$

Note that

$$f(x^{k_1})\frac{1-x^{k_1}}{1-x} = (1 + x + \cdots + x^{k_1-1})\sum_{n=0}^{\infty}\chi_A(n)x^{k_1n} = \sum_{n=0}^{\infty}\chi_A\Big(\Big[\frac{n}{k_1}\Big]\Big)x^n,$$

$$f(x^{k_2})\frac{1-x^{k_2}}{1-x} = (1+x+\cdots+x^{k_2-1})\sum_{n=0}^{\infty}\chi_A(n)x^{k_2 n} = \sum_{n=0}^{\infty}\chi_A\Big(\Big[\frac{n}{k_2}\Big]\Big)x^n,$$
$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$$

and

$$(1-x^{k_2})\frac{1-x^{k_1}}{1-x}p(x) = (1-x^{k_2})(1+x+\cdots+x^{k_1-1})\sum_{n=0}^{\infty}a(n)x^n$$
$$= \sum_{n=0}^{\infty}\Big(\sum_{j=0}^{k_1-1}(a(n-j)-a(n-k_2-j))\Big)x^n.$$

It follows from (2.2) that for all nonnegative integers n,

$$\chi_A\Big(\Big[\frac{n}{k_1}\Big]\Big)+\chi_A\Big(\Big[\frac{n}{k_2}\Big]\Big) = 1+\sum_{j=0}^{k_1-1}(a(n-j)-a(n-k_2-j)).$$

This completes the proof of Lemma 2.1. □

LEMMA 2.2. *Let n_0 be a positive integer and $k_1 < k_2$ be two positive integers with $(k_1, k_2) = 1$ and $A \subseteq \mathbb{N}$ be a set with*

$$\chi_A\Big(\Big[\frac{i}{k_1}\Big]\Big)+\chi_A\Big(\Big[\frac{i}{k_2}\Big]\Big) = 1 \quad \text{for all } i \geq k_1+k_2+n_0. \tag{2.3}$$

If $n \geq k_1+k_2+n_0$ and $\chi_A(n)+\chi_A(n+1) = 1$, then $k_2 \mid n+1$.

PROOF. Since $\chi_A(n)+\chi_A(n+1) = 1$, it follows that

$$\chi_A\Big(\Big[\frac{(n+1)k_1-1}{k_1}\Big]\Big)+\chi_A\Big(\Big[\frac{(n+1)k_1}{k_1}\Big]\Big) = \chi_A(n)+\chi_A(n+1) = 1. \tag{2.4}$$

By (2.3),

$$\chi_A\Big(\Big[\frac{(n+1)k_1-1}{k_1}\Big]\Big)+\chi_A\Big(\Big[\frac{(n+1)k_1-1}{k_2}\Big]\Big) = 1$$

and

$$\chi_A\Big(\Big[\frac{(n+1)k_1}{k_1}\Big]\Big)+\chi_A\Big(\Big[\frac{(n+1)k_1}{k_2}\Big]\Big) = 1.$$

It follows from (2.4) that

$$\chi_A\Big(\Big[\frac{(n+1)k_1-1}{k_2}\Big]\Big)+\chi_A\Big(\Big[\frac{(n+1)k_1}{k_2}\Big]\Big) = 1.$$

Let t and r be integers with

$$(n+1)k_1 = tk_2+r, \quad 0 \leq r \leq k_2-1.$$

If $r \geq 1$, then

$$1 = \chi_A\Big(\Big[\frac{(n+1)k_1 - 1}{k_2}\Big]\Big) + \chi_A\Big(\Big[\frac{(n+1)k_1}{k_2}\Big]\Big) = 2\chi_A(t),$$

which is a contradiction. Hence, $r = 0$ and $(n+1)k_1 = tk_2$. Noting that $(k_1, k_2) = 1$, we have $k_2 \mid n+1$. This completes the proof of Lemma 2.2. □

Proof of Theorem 1.4. Let g be an integer and let k_1, k_2 be integers with $2 \leq k_1 < k_2$ and $(k_1, k_2) = 1$. Suppose that

$$R_{k_1,k_2}(A, n) - R_{k_1,k_2}(\mathbb{N} \setminus A, n) = g \tag{2.5}$$

for all integers $n \geq n_0$. Let $\{a(n)\}_{n=-\infty}^{+\infty}$ be a sequence of integers with $a(n) = 0$ for $n < 0$ and $a(n) = g$ for all integers $n \geq n_0$. It follows from Lemma 2.1 that for all integers $i \geq k_1 + k_2 + n_0$,

$$\chi_A\Big(\Big[\frac{i}{k_1}\Big]\Big) + \chi_A\Big(\Big[\frac{i}{k_2}\Big]\Big) = 1. \tag{2.6}$$

If A is a finite set, then $R_{k_1,k_2}(A, n) = 0$ for all sufficiently large integers n, and $R_{k_1,k_2}(\mathbb{N} \setminus A, n)$ cannot be a fixed constant as $n \to +\infty$, which implies that (2.5) cannot hold. So A is an infinite set. Similarly, $\mathbb{N} \setminus A$ is also an infinite set.

Since $2 \leq k_1 < k_2$, it follows that there exists an integer $t > 1$ such that $k_2 < k_1^t$. Note that both A and $\mathbb{N} \setminus A$ are infinite sets. So there exists an integer $n = k_1^{\alpha} k_2^{\beta} h - 1 > (k_1 + k_2 + n_0)^{t+1}$ such that $n \in A$ and $n + 1 \notin A$, where α and β are nonnegative integers and h is a positive integer with $(h, k_1 k_2) = 1$. It follows from (2.6) and Lemma 2.2 that $k_2 \mid n + 1$ and $\beta \geq 1$. Since

$$(k_1 + k_2 + n_0)^{t+1} < n < k_1^{\alpha} k_2^{\beta} h < k_1^{t(\alpha+\beta)} h,$$

it follows that $k_1^{\alpha+\beta} > k_1 + k_2 + n_0$ or $h > k_1 + k_2 + n_0$. Hence, for any $0 \leq i \leq \beta$,

$$k_1^{\alpha+i} k_2^{\beta-i} h \geq k_1^{\alpha+\beta} h > k_1 + k_2 + n_0. \tag{2.7}$$

By (2.6),

$$\chi_A\Big(\Big[\frac{k_1^{\alpha+1} k_2^{\beta} h}{k_1}\Big]\Big) + \chi_A\Big(\Big[\frac{k_1^{\alpha+1} k_2^{\beta} h}{k_2}\Big]\Big) = 1 \tag{2.8}$$

and

$$\chi_A\Big(\Big[\frac{k_1^{\alpha+1} k_2^{\beta} h - k_1}{k_1}\Big]\Big) + \chi_A\Big(\Big[\frac{k_1^{\alpha+1} k_2^{\beta} h - k_1}{k_2}\Big]\Big) = 1. \tag{2.9}$$

Since $k_1^{\alpha} k_2^{\beta} h = n + 1 \notin A$ and $k_1^{\alpha} k_2^{\beta} h - 1 = n \in A$, it follows from (2.8) and (2.9) that

$$\chi_A(k_1^{\alpha+1} k_2^{\beta-1} h - 1) + \chi_A(k_1^{\alpha+1} k_2^{\beta-1} h) = 1.$$

By Lemma 2.2, $k_2 \mid k_1^{\alpha+1}k_2^{\beta-1}h$ and so $\beta \geq 2$. Continuing this procedure yields

$$\chi_A(k_1^{\alpha+\beta}h - 1) + \chi_A(k_1^{\alpha+\beta}h) = 1.$$

By (2.7) and Lemma 2.2, we also have $k_2 \mid k_1^{\alpha+\beta}h$, which is impossible. Hence, there does not exist any set $A \subseteq \mathbb{N}$ such that (2.5) holds for all sufficiently large integers n. This completes the proof of Theorem 1.4. □

PROOF OF THEOREM 1.5. Suppose that there is a set A such that

$$R_{1,k}(A, n) - R_{1,k}(\mathbb{N} \setminus A, n) = 1 \tag{2.10}$$

for all integers $n \geq 1$. Then $0 \in A$ and (2.10) holds for all integers $n \geq 0$. Let $\{a(n)\}_{n=-\infty}^{+\infty}$ be a sequence of integers with $a(n) = 0$ for $n < 0$ and $a(n) = 1$ for $n \geq 0$. By Lemma 2.1,

$$R_{1,k}(A, n) - R_{1,k}(\mathbb{N} \setminus A, n) = a(n)$$

for all nonnegative integers n if and only if

$$\chi_A(n) + \chi_A\Big(\Big[\frac{n}{k}\Big]\Big) = 1 + a(n) - a(n-k)$$

for all nonnegative integers n, that is,

$$\begin{aligned} \chi_A(n) + \chi_A(0) &= 2 \quad \text{for } 0 \leq n \leq k-1, \\ \chi_A(n) + \chi_A\Big(\Big[\frac{n}{k}\Big]\Big) &= 1 \quad \text{for } n \geq k. \end{aligned}$$

Thus,

$$A = \{0\} \bigcup \Big(\bigcup_{i=0}^{\infty} [k^{2i}, k^{2i+1} - 1]\Big). \qquad \square$$

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