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ON A TYPE OF MATRIX RING

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In this note we discuss a type of matrix ring that has nice properties concerning the injectivity and quasi-injectivity of one-sided ideals.

1. PRELIMINARIES

We consider associative rings with identity, and all modules are unitary. A module is said to be *uniserial* if its submodules are linearly ordered by inclusion. The uniserial modules encountered here will also be both Artinian and Noetherian and so have a (composition) length. A ring R is called *serial* if both R_R and ${}_RR$ are direct sums of finitely many uniserial modules.

Recall that a ring R is called a *right* V*-ring* if all simple right R-modules are injective. If all right ideals of the ring R are actually two-sided then R is called a *right duo* ring.

For other undefined terminology, we refer the reader to the text [2] by Dung, Huynh, Smith and Wisbauer.

2. THE MATRIX RING

Here we define a type of matrix ring which was introduced first by Ivanov [4] in his study of the structure of non-local rings whose right ideals are quasi-injective. Since then these rings have proven to be very useful for finding examples or counter-examples of certain classes of rings (see, for example, Beidar, Fong, Ke and Jain[1], Huynh and Rizvi [3]).

Let T be a ring having a two-sided ideal M such that $D = T/M$ is a division ring. Let

$$V = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \subset \left\{ \begin{pmatrix} d & x \\ 0 & d \end{pmatrix} : d,\ x \in D \right\}.$$

Then V is a D-bimodule with $\dim({}_DV) = \dim(V_D) = 1$ and $V \cdot V = 0$. Moreover, $VT = TV = V$.

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Let $n \in \mathbb{N}$, with $n \geqslant 3$. We consider the $n \times n$ matrix ring R of the form:

$$R = \begin{pmatrix} D & V & 0 & \cdots & 0 & 0 \\ 0 & D & V & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \ddots & \cdot \\ 0 & 0 & 0 & \cdots & D & V \\ 0 & 0 & 0 & \cdots & 0 & T \end{pmatrix}.$$

We let R_i (respectively C_i) denote the right (respectively left) ideal of R which has the same ith row (respively ith column) as R, but all other rows (respectively columns) are zero. Then each R_i is a uniserial right R-module of length 2 for $1 \leqslant i \leqslant n-1$, each C_i is a uniserial left R-module of length 2 for $1 < i \leqslant n-1$, while C_1 is a simple left R-module.

Part (1) of Theorem 1 below was obtained also in [**1**], where this type of ring was used to describe rings in which all right ideals are quasi-injective. However the proofs in [**1**] involve complicated computation using elements. One might hope that the simple arguments we use here can streamline parts of [**1**] as well as provide a better understanding of this type of matrix ring.

Theorem 1. *Let T and R be as above. Then*

(1) *R is never left self-injective.*

(2) *$Q := R_1 \oplus \cdots \oplus R_{n-1}$ is a quasi-injective right R-module and $P := C_2 \oplus \cdots \oplus C_{n-1}$ is a quasi-injective left R-module.*

(3) *If T is a right nonsingular, right self-injective, right V-ring, and M is essential in T_T, then R is a right self-injective ring.*

(4) *T is indecomposable as a ring if and only if R is indecomposable as a ring.*

(5) *The ring R is never indecomposable if T is a von Neumann regular right duo ring which is not a division ring.*

Proof: (1) Write ${}_RR = C_1 \oplus C_2 \oplus \cdots \oplus C_n$. If ${}_RR$ is injective, then ${}_RC_1$ is injective. However C_2 is a local left R-module of length 2 and $\mathrm{Soc}(C_2) \cong C_1$. Hence $\mathrm{Soc}(C_2)$ splits in C_2, a contradiction. This proves (1).

(2) Let M^* be the subset of C_n consisting of those matrices where the (n,n)th entry is in M. Since $VM = 0$, M^* is a two-sided ideal of R. Using the decomposition ${}_RR = C_1 \oplus \cdots \oplus C_n$, we get $R/M^* \cong C_1 \oplus \cdots \oplus C_{n-1} \oplus (C_n/M^*)$, a direct sum of uniserial left R/M^*-modules of length at most 2. (Clearly C_n/M^* is uniserial of length 2.) Similarly we have $R/M^* \cong R_1 \oplus \cdots \oplus R_{n-1} \oplus (R_n/M^*)$, a direct sum of uniserial right (R/M^*)-modules of length at most 2. (Note that R_n/M^* is a simple right R-module.) Hence R/M^* is an Artinian serial ring with Jacobson radical square zero. This implies

that every uniform right (or left) (R/M^*)-module of length 2 is injective (see [2, 13.5]). Thus Q_R and $_RP$ are quasi-injective, proving (2).

(3) Assume that T is a right nonsingular, right self-injective, and right V-ring, such that M is essential in T_T. Then, by (2), we have $R_R = Q \oplus R_n$, a direct sum of two quasi-injective right R-modules. Thus to prove the right self-injectivity of R we need only to show that Q is R_n-injective and R_n is Q-injective.

First note that the right R-module $\mathrm{Soc}(Q) \oplus M^*$ is essential in R_R and moreover $\mathrm{Soc}(Q) \cdot (\mathrm{Soc}(Q) \oplus M^*) = 0$. This shows that $\mathrm{Soc}(Q)$ is a singular right R-module. Then, since R_n is a nonsingular right R-module, there are no nonzero homomorphisms $Q_R \to R_n$ and so, trivially, R_n is Q-injective.

Next note that, as a right R-module, R_n is a V-module, that is, every R_n-subgenerated simple module is R_n-injective. Now let U be a submodule of R_n and let φ be a homomorphism of U to Q. Since $U/\operatorname{Ker}\varphi$ is isomorphic to a submodule of Q, $\mathrm{Soc}(U/\operatorname{Ker}\varphi)$ is finitely generated, and hence $(R_n/\operatorname{Ker}\varphi)$-injective. Therefore $\mathrm{Soc}(U/\operatorname{Ker}\varphi)$ splits in $R_n/\operatorname{Ker}\varphi$. Since $\mathrm{Soc}(U/\operatorname{Ker}\varphi)$ is essential in $U/\operatorname{Ker}\varphi$, it follows that $U/\operatorname{Ker}\varphi = \mathrm{Soc}(U/\operatorname{Ker}\varphi)$. Thus there is a submodule $W \subseteq R_n$ containing $\operatorname{Ker}\varphi$ such that $R_n/\operatorname{Ker}\varphi = (U/\operatorname{Ker}\varphi) \oplus (W/\operatorname{Ker}\varphi)$. This implies that we can extend φ to a homomorphism of R_n to Q, proving that Q is R_n-injective. This establishes the right self-injectivity of R.

(4) Assume that T is indecomposable as a ring. Since $D = T/M$, we have

$$R/M^* \cong \begin{pmatrix} D & V & 0 & \cdots & 0 & 0 \\ 0 & D & V & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & D & V \\ 0 & 0 & 0 & \cdots & 0 & D \end{pmatrix}.$$

It is easy to see that this matrix ring is right and left Artinian and indecomposable as a ring. Hence the ring R/M^* is also indecomposable.

Now, if $R = A \oplus B$ is a ring decomposition with A and B both nonzero, then, since $M^* = (A \cap M^*) \oplus (B \cap M^*)$, we have $R/M^* \cong [A/(A \cap M^*)] \oplus [B/(B \cap M^*)]$. Hence one of these latter direct summands of R/M^* must be zero, say $A/(A \cap M^*) = 0$. Then $A = A \cap M^*$, and so $A \subseteq M^* \subset R_n$. Thus $R_n = A \oplus (B \cap R_n)$, a ring direct decomposition of R_n with $B \cap R_n \neq 0$. This is a contradiction, because $T \cong R_n$ and, by our assumption, T is an indecomposable ring.

Conversely, suppose that R is ring-indecomposable. Let's assume that $T = U \oplus W$, a ring-direct sum with U and W both nonzero. Then, adapting our definition of M^*, we can define corresponding right ideals U^* and W^* in R to give a ring-direct sum

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$R_n = U^* \oplus W^*$ for R_n. It follows that $R_n/M^* \cong [U^*/(U^* \cap M^*)] \oplus [W^*/(W^* \cap M^*)]$. Since $R_n/M^* \cong T/M$, a division ring, one of the two summands of R_n/M^* must be zero, say $U^*/(U^* \cap M^*) = 0$. Then $U^* \subseteq M^*$,and so $U \subseteq M$. It follows that $VU = 0$. Hence $V = VT = V(U \oplus W) = VU \oplus VW = VW$. Consequently, we get the following ring-direct decomposition of R in which the first summand is U^*:

$$R = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & U \end{pmatrix} \oplus \begin{pmatrix} D & V & 0 & \cdots & 0 & 0 \\ 0 & D & V & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \vdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & D & V \\ 0 & 0 & 0 & \cdots & 0 & W \end{pmatrix}.$$

This contradiction establishes (3).

(4) This is clear from (3), because every von Neumann regular right duo ring is decomposable if and only if it is not a division ring. ∎

We remark that in Theorem 1, statement (3) is not true if M is not essential in T_T. Moreover, if M is not essential in T_T, then the ring R is right nonsingular if T is right nonsingular.

From the proof of (3) we see that if M is essential in T_T, then R contains an indecomposable ring-direct summand if and only if T has an indecomposable ring-direct summand.

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