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DEGREE OF APPROXIMATION OF A FUNCTION BY NÖRLUND MEANS OF ITS FOURIER SERIES

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Two theorems of T.M. Flett [*Quart. J. Math. Oxford Ser.* (2) 7 (1956), 81-95] on the degree of approximation to a function by the Cesàro means of its Fourier series are extended to Nörlund means. Their conjugate analogues are also proved.

1.

Let $f(x)$ be Lebesgue integrable and periodic with period 2π, and let

$$(1.1) \qquad f(x) \sim \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} \left(a_k \cdot \cos kx + b_k \cdot \sin kx\right) = \sum_{k=0}^{\infty} A_k(x)$$

be its Fourier series.

The conjugate Fourier series of (1.1) is

$$(1.2) \qquad \sum_{k=1}^{\infty} \left(b_k \cdot \cos kx - a_k \cdot \sin kx\right) = \sum_{k=1}^{\infty} B_k(x) .$$

The Nörlund mean of an infinite series $\sum_{k=1}^{\infty} a_k$, with the sequence of partial sums $\{s_n\}$, is defined (Nörlund [4], Woronoi [5]) by the sequence-

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to-sequence transformation

$$t_n(p_n) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \cdot s_k \tag{1.3}$$

where $\{p_n\}$ is a sequence of non-negative strictly monotonic decreasing constants, and

$$P_n = \sum_{k=0}^{n} p_k \neq 0 \ , \quad P_{-1} = p_{-1} = 0 \ .$$

We use the following notation:

$$\phi(t) = \phi_x(t) = f(x+t) + f(x-t) - 2f(x) \ ,$$

$$\psi(t) = \psi_x(t) = f(x+t) - f(x-t) \ ,$$

$$N_n(p_n;\ t) = \frac{1}{2\pi P_n} \sum_{k=0}^{n} \frac{\sin(k+\frac{1}{2})t}{\sin t/2} \ ,$$

$$\bar{N}_n(p_n;\ t) = \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_{n-k} \frac{\cos(k+\frac{1}{2})t}{\sin t/2} \ ,$$

$$\bar{f}(x) = \frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot t/2 \, dt \ ,$$

$$\Phi_r(t) = \frac{1}{\Gamma(r)} \int_0^{t} (t-u)^{r-1} \phi(u) du \ , \quad r > 0 \ ,$$

$$\Phi_0(t) = \phi(t) \ , \quad \Phi_r(t) = \Phi'_{1+r}(t) \quad (-1 < r < 0) \ ,$$

$$\Psi_r(t) = \frac{1}{\Gamma(r)} \int_0^{t} (t-u)^{r-1} \psi(u) du \ , \quad r > 0 \ ,$$

$$\Psi_0(t) = \psi(t) \ , \quad \Psi_r(t) = \Psi'_{1+r}(t) \quad (-1 < r < 0) \ .$$

$[x]$ denotes the largest integer less than or equal to x .

2.

Flett [2] has proved the following theorems for the degree of approximation to a function by Cesàro means of its Fourier series.

THEOREM A. *Suppose that* f *is integrable in* $(-\pi, \pi)$ *and of class* Lip α *in the closed interval* (a, b) *where* $0 < \alpha < 1$ *and that* $a_n, b_n = O\left(n^{-\beta}\right)$. *If* $0 \le \beta < \alpha$ *and* $k \ge \alpha-\beta$, *then*

$$\sigma_n^k(x) - f(x) = O\left(n^{-\alpha}\right) ,$$

$\sigma_n^k(x)$ *being the* (C, k) *mean of series* (1.1).

THEOREM B. *Let* $0 < \alpha < 1$, $0 \le \beta < 1$, $-1 < r < 0$, $0 < \delta \le \pi$, $k \ge \alpha-\beta$, $k > \alpha + r$, *and let* x *be a point such that*

(i) $A_n(x) = O\left(n^{-\beta}\right)$,

(ii) $\Phi_{1+r}(+0) = 0$, *and* $\int_0^t u^{-r}\left|\Phi_{1+r}(u)\right| \le At^{1+\alpha}$ $(0 \le t \le \delta)$,

and

(iii) $\int_0^t \phi(u)du = O\left(t^{-1+\alpha}\right)$;

then

$$\sigma_n^k(x) - f(x) = O\left(n^{-\alpha}\right) .$$

In the present paper we generalise the above theorems for Nörlund means and also prove their conjugate analogues. Precisely we prove the following theorems.

THEOREM 1. *Suppose that* f *is integrable in* $(-\pi, \pi)$ *and of class* Lip α *in the closed interval* (a, b) *where* $0 < \alpha < 1$, *and that*

$$a_n, b_n = O\left\{\left(q_n/Q_n\right)^\beta\right\} . \tag{2.1}$$

If $0 < \beta < \alpha$ *and* $\{p_n\}$, $\{q_n\}$ *and* $\{r_n\}$ *be monotonic decreasing sequences of non-negative constants such that*

$$r_n/R_{n-1} \ge p_n/P_{n-1} - q_n/Q_{n-1} = P_n/P_{n-1} - Q_n/Q_{n-1} ,$$

Q_n *and* R_n *being defined similarly to* P_n , *then*

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$$t_n(r_n) - f(x) = O\left\{\left(R_{[P_n/p_n]}/R_n\right)\cdot\left(p_n/P_n\right)^{\alpha}\right\} \cong O\left\{\left(p_n/P_n\right)^{\alpha}\right\},$$

where $t_n(r_n)$ *is the Nörlund mean* (1.3) *of series* (1.1) *generated by the sequence* $\{r_n\}$.

THEOREM 2. *Suppose that* f *is integrable in* $(-\pi, \pi)$ *and of class* Lip α *in the closed interval* (a, b) , $0 < \alpha < 1$ *and that*

$$a_n, b_n = O\left\{\left(q_n/Q_n\right)^{\beta}\right\}. \tag{2.1}$$

If $0 < \beta < \alpha$ *and* $\{p_n\}$, $\{q_n\}$ *and* $\{r_n\}$ *are sequences as defined in Theorem* 1, *then*

$$\bar{t}_n(r_n) - \bar{f}(x) = O\left\{\left(R_{[P_n/p_n]}/R_n\right)\cdot\left(p_n/P_n\right)^{\alpha}\right\} \cong O\left\{\left(p_n/P_n\right)^{\alpha}\right\}$$

provided that the conjugate function exists, $\bar{t}_n(r_n)$ *being the Nörlund mean* (1.3) *of series* (1.2) *generated by sequence* $\{r_n\}$.

THEOREM 3. *Let* $0 < \alpha < 1$, $0 \le \beta < 1$, $-1 < r < 0$, $0 < \delta \le \pi$ *and let* x *be a point such that*

$$A_n(x) = O\left\{\left(q_n/Q_n\right)^{\beta}\right\}, \tag{2.2}$$

$$\Phi_{1+r}(+0) = 0 \quad \text{and} \quad \int_0^t u^{-r}|d\Phi_{1+r}(u)| \le At^{1+\alpha} \quad (0 \le t \le \delta), \tag{2.3}$$

$$\int_0^t \phi(u)du = O\left(t^{1+\alpha}\right); \tag{2.4}$$

then

$$t_n(r_n) - f(x) = O\left\{\left(R_{[P_n/p_n]}/R_n\right)\cdot\left(p_n/P_n\right)^{\alpha}\right\} \cong O\left\{\left(p_n/P_n\right)^{\alpha}\right\},$$

where $\{p_n\}$, $\{q_n\}$ *and* $\{r_n\}$ *are sequences as defined in Theorem* 1.

THEOREM 4. *Let* $0 < \alpha < 1$, $0 \le \beta < 1$, $-1 < r < 0$, $0 < \delta \le \pi$ *and let* x *be a point such that*

$$B_n(x) = O\left\{\left(q_n/Q_n\right)^{\beta}\right\}, \tag{2.5}$$

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(2.6) $\Psi_{1+r}(+0) = 0$ *and* $\int_0^t u^{-r}|d\Psi_{1+r}(u)| \le At^{1+\alpha}$ $(0 \le t \le \delta)$,

(2.7) $\int_0^t \psi(u)du = o\left(t^{1+\alpha}\right)$,

then

$$\bar{t}_n\left(r_n\right) - \bar{f}(x) = o\left\{\left(R_{\left[P_n/p_n\right]}/R_n\right)\cdot\left(p_n/P_n\right)^{\alpha}\right\} \cong o\left\{\left(p_n/P_n\right)^{\alpha}\right\} ,$$

provided that the conjugate function exists, $\{p_n\}$, $\{q_n\}$ *and* $\{r_n\}$ *being sequences as defined in Theorem* 1.

3.

We shall need the following lemmas in the proof of our theorems.

LEMMA 3.1. *For* $0 \le t \le p_n/P_n$,

$$N_n\left(p_n;\ t\right) = O(n) .$$

Proof.

$$\begin{aligned} N_n\left(p_n;\ t\right) &= \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_{n-k}\frac{\sin(k+\frac{1}{2})t}{\sin t/2} \\ &\le \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_{n-k}(2k+1) \\ &\le \frac{1}{2\pi P_n}(2n+1)\sum_{k=0}^{n} p_{n-k} \\ &= O(n) . \end{aligned}$$

LEMMA 3.2 [3]. *If the sequence* $\{p_n\}$ *is non-negative and non-increasing then for* $0 \le a \le b \le \infty$, $0 \le t \le \pi$ *and any* n , *we have*

$$\left|\sum_{k=a}^{b} p_k e^{i(n-k)t}\right| \le P_\tau$$

for any a , *where* $\tau = \left[t^{-1}\right]$.

LEMMA 3.3. *For* $0 < p_n/P_n \le t \le \delta \le \pi$,

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(i) $|N_n(p_n;\ t)| = O\{P_\tau/tP_n\}$,

(ii) $|\overline{N}_n(p_n;\ t)| = O\{P_\tau/tP_n\}$.

Proof. (i)

$$\begin{aligned}|N_n(p_n;\ t)| &= \frac{1}{2\pi P_n}\left|\sum_{k=0}^{n} p_k \frac{\sin(n-k+\frac{1}{2})t}{\sin t/2}\right| \\ &\le \frac{1}{2\pi P_n}\left\{\left|\sum_{k=0}^{n} p_k \sin(n-k)t \cot t/2\right| + \left|\sum_{k=0}^{n} p_k \cos(n-k)t\right|\right\} \\ &= O\{(P_\tau \cot t/2)/P_n\} + O\{P_\tau/P_n\} \quad \text{using Lemma 3.2} \\ &= O\{P_\tau/tP_n\} .\end{aligned}$$

(ii) The estimate for $\overline{N}_n(p_n;\ t)$ can be proved similarly.

LEMMA 3.4. *For* $0 \le u \le p_n/P_n$,

(i)

$$\begin{aligned}J(p_n;\ u) &= \int_{p_n/P_n}^{\delta} (t-u)^{-r-1} N_n(r_n;\ t)\,dt \\ &= O\left\{\left(R_{[P_n/p_n]}/R_n\right)\cdot\left(P_n/p_n\right)^{r+1}\right\} ;\end{aligned}$$

(ii)

$$\overline{J}(p_n;\ u) = \int_{p_n/P_n}^{\delta} (t-u)^{-r-1}\overline{N}_n(r_n;\ t)\,dt = O\left\{\left(R_{[P_n/p_n]}/R_n\right)\cdot\left(P_n/p_n\right)^{r+1}\right\} .$$

Proof. (i)

$$\begin{aligned}J(p_n;\ u) &= \int_{p_n/P_n}^{\delta} (t-u)^{-r-1} N_n(r_n;\ t)\,dt \\ &= \int_{p_n/P_n}^{\delta} (t-u)^{-r-1} O\{R_\tau/tR_n\}\,dt \quad \text{using Lemma 3.3 (i)} \\ &\le O\{(R_{[P_n/p_n]}/R_n)\cdot(P_n/p_n)\} \int_{p_n/P_n}^{\delta} (t-u)^{-r-1}\,dt\end{aligned}$$

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$$= O\{(R_{[P_n/p_n]}/R_n)\cdot(P_n/p_n)\} \cdot \left\{\frac{(t-u)^{-r}}{-r}\right\}_{p_n/P_n}^{\delta}$$

$$= O\{(R_{[P_n/p_n]}/R_n)\cdot(P_n/p_n)\} \cdot \left\{\frac{(\delta-u)^{-r}}{-r} + \frac{1}{r}\left(\frac{p_n}{P_n} - u\right)^{-r}\right\}$$

$$\le O\left\{(R_{[P_n/p_n]}/R_n)\cdot(P_n/p_n)^{r+1}\right\} \quad \text{for} \quad 0 \le u \le p_n/P_n .$$

(ii) The estimate for $\bar{J}(p_n; u)$ can be proved similarly.

LEMMA 3.5. *(i)*

$$K(u) = \int_u^{\delta} (t-u)^{-r-1} N_n(r_n; t) = O\left\{u^{-1-r}\,\frac{R_{[1/u]}}{R_n}\right\} .$$

(ii)

$$\bar{K}(u) = \int_u^{\delta} (t-u)^{-r-1} \bar{N}_n(r_n; t) = O\left\{u^{-1-r}\,\frac{R_{[1/u]}}{R_n}\right\} .$$

Proof. *(i)*

$$K(u) = \int_u^{\delta} (t-u)^{-r-1} N_n(r_n; t)\,dt$$

$$= \int_u^{\delta} (t-u)^{-r-1} O\{R_{[\tau]}/tR_n\}dt \quad \text{using Lemma 3.3 } (i)$$

$$= O\left\{\frac{R_{[1/u]}}{uR_n}\right\} \cdot \int_u^{\delta} (t-u)^{-r-1}dt$$

$$= O\left\{\frac{R_{[1/u]}}{uR_n}\right\} \cdot \left\{\frac{(t-u)^{-r}}{-r}\right\}_u^{\delta}$$

$$= O\left\{\frac{R_{[1/u]}}{uR_n}\right\} \cdot \left\{\frac{(\delta-u)^{-r}}{-r}\right\} \quad (-1 < r < 0)$$

$$= O\left\{\frac{R_{[1/u]}}{uR_n} \cdot u^{-1-r}\right\} .$$

(ii) The estimate for $\bar{K}(u)$ can be obtained in a similar manner.

4.

Proof of Theorem 1. Let us write (Zygmund [6])

$$S_n(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n} A_k(x) ,$$

then we have

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi(t)\,\frac{\sin(n+\frac{1}{2})t}{\sin t/2}\,dt ,$$

using (1.3) for $\sum A_k(x)$, we have

$$\begin{aligned}
(4.1) \qquad t_n(r_n) - f(x) &= \frac{1}{R_n}\sum_{k=0}^{n} r_{n-k}S_k(x) - f(x) \\
&= \int_0^\pi \phi(t)\,\frac{1}{2\pi R_n}\sum_{k=0}^{n} r_{n-k}\,\frac{\sin(k+\frac{1}{2})t}{\sin t/2}\,dt \\
&= \int_0^\pi \phi(t)N_n(r_n;\ t)dt \\
&= \left\{\int_0^{p_n/P_n} + \int_{p_n/P_n}^{\delta} + \int_\delta^\pi\right\}\phi(t)N_n(r_n;\ t)dt \\
&= I_1 + I_2 + I_3 \text{ , say.}
\end{aligned}$$

Now

$$\begin{aligned}
(4.2) \qquad |I_1| &\le O(n)\int_0^{p_n/P_n} |\phi(t)|dt \quad \text{using Lemma 3.1} \\
&= O(n)\cdot o\left(p_n/P_n\right)^{\alpha+1} \\
&= o\left\{\left(p_n/P_n\right)^{\alpha}\right\} \quad \text{as} \quad np_n \le P_n .
\end{aligned}$$

Further

$$\begin{aligned}
(4.3) \qquad |I_2| &\le \int_{p_n/P_n}^{\delta} |\phi(t)|\cdot|N_n(r_n;\ t)|dt \\
&= \int_{p_n/P_n}^{\delta} o\left(t^{\alpha}\right)\cdot O\,\frac{R_\tau}{tR_n}\,dt \quad \text{using Lemma 3.3 } (i) \\
&= \frac{1}{R_n}\int_{p_n/P_n}^{\delta} o\left(t^{\alpha-1}R_\tau\right)dt
\end{aligned}$$

$$= O\left\{\left(R_{[P_n/p_n]}/R_n\right)\cdot\left(p_n/P_n\right)^{\alpha}\right\}$$

$$\cong O\left\{\left(p_n/P_n\right)^{\alpha}\right\} .$$

Now we have

$$\phi_x(t) \sim A_0 + 2\sum_{k=1}^{\infty} A_k(x)\cos kt$$

where $A_0 = a_0 - 2f(x)$. Hence we have

$$(4.4)\quad |I_3| \le \int_\delta^\pi |A_0 N_n(r_n;\ t)|dt + 2\int_\delta^\pi \left|\sum_{k=1}^{\infty} A_k(x)\cos kt\right| \cdot |N_n(r_n;\ t)|dt$$

$$\le \int_\delta^\pi |A_0 N_n(r_n;\ t)|dt + 2\int_\delta^\pi \left|\sum_{k=1}^{\infty} a_k \cos kx \cos kt\right|$$

$$\cdot\ |N_n(r_n;\ t)|dt + 2\int_\delta^\pi \left|\sum_{k=1}^{\infty} b_k \sin kx \cos kt\right| \cdot |N_n(r_n;\ t)|dt$$

$$= I_{3.1} + I_{3.2} + I_{3.3}\ ,\ \text{say.}$$

Now

$$(4.5)\qquad I_{3.1} = \int_\delta^\pi \left|A_0 \frac{R_\tau}{tR_n}\right| dt \quad \text{by using Lemma 3.3 } (i)$$

$$= O\left(1/R_n\right)$$

$$= o(1)\ ,\ \text{as}\quad n \to \infty .$$

Further

$$2\sum_{k=1}^{\infty} a_k(x)\cos kx\cos kt = \sum_{k=1}^{\infty} O\left\{\left(q_k/Q_k\right)^{\beta}\right\}\cdot\{\cos k(x+t) + \cos k(x-t)\}$$

using (2.1)

$$= \sum_{k=1}^{\infty} O(1)\cdot\{\cos k(x+t) + \cos k(x-t)\}$$

$$= O(1) .$$

Thus

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$$(4.6) \qquad I_{3.2} = O(1) \cdot \int_\delta^\pi |N_n(r_n;\ t)|dt$$

$$= O(1) \int_\delta^\pi |R_\tau/tR_n|dt \quad \text{by Lemma 3.3 } (i)$$

$$= O(1/R_n)$$

$$= o(1) \text{ , as } n \to \infty \text{ .}$$

Similarly,

$$(4.7) \qquad I_{3.3} = o(1) \text{ .}$$

In view of (4.4), (4.5), (4.6) and (4.7),

$$(4.8) \qquad |I_3| = O(1) \text{ .}$$

Finally considering (4.1), (4.2), (4.3) and (4.8), the proof of the theorem is complete.

5.

Proof of Theorem 2. We have

$$\bar{S}_n(x) = \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos t/2 - \cos(n+\frac{1}{2})t}{\sin t/2} dt \text{ .}$$

Using (1.3) for $\sum B_n(x)$, we have

$$(5.1) \quad \bar{t}_n(r_n) - \bar{f}(x) = \frac{1}{R_n} \sum_{k=0}^{n} r_k \bar{S}_{n-k}(x) - \frac{1}{2\pi} \int_0^\pi \psi(t) \cot t/2\, dt$$

$$= \frac{1}{R_n} \sum_{k=0}^{n} r_k \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos t/2 - \cos(n-k+\frac{1}{2})t}{\sin t/2} dt$$

$$- \frac{1}{2\pi} \int_0^\pi \psi(t) \cot t/2\, dt$$

$$= - \int_0^\pi \frac{1}{2\pi R_n} \psi(t) \sum_{k=0}^{n} r_k \frac{\cos(n-k+\frac{1}{2})t}{\sin t/2} dt$$

$$= - \int_0^\pi \psi(t) \bar{N}_n(r_n;\ t) dt$$

$$= -\left\{\int_0^{p_n/P_n} + \int_{p_n/P_n}^{\delta} + \int_{\delta}^{\pi}\right\}\psi(t)\bar{N}_n\left(r_n;\ t\right)dt$$

$$= \bar{I}_1 + \bar{I}_2 + \bar{I}_3\ , \text{ say.}$$

Since the conjugate function exists, we have

$$\frac{1}{2\pi}\int_0^{p_n/P_n} \psi(t)\cot t/2\, dt = o(1)\ .$$

Hence

$$\frac{1}{2\pi}\int_0^{p_n/P_n} \psi(t)\cot t/2\, dt + \bar{I}_1$$

$$= \frac{1}{2\pi}\int_0^{p_n/P_n} \psi(t)\left\{\cot t/2 - \frac{1}{R_n}\sum_{k=0}^{n} \frac{r_k\cos(n-k+\frac{1}{2})t}{\sin t/2}\right\}dt$$

$$= \frac{1}{2\pi R_n}\int_0^{p_n/P_n} \psi(t)\sum_{k=0}^{n} r_k\left\{\sum_{\nu=0}^{n-k} 2\sin \nu t\right\}dt$$

$$\le \frac{1}{2\pi R_n}\int_0^{p_n/P_n} |\psi(t)|\sum_{k=0}^{n} nr_k dt$$

$$= O(n)\int_0^{p_n/P_n} o\left(t^{\alpha}\right)dt$$

$$\cong o\left\{\left(p_n/P_n\right)^{\alpha}\right\}\ .$$

Thus

$$|\bar{I}_1| = o\left\{\left(p_n/P_n\right)^{\alpha}\right\} + o(1)\ . \tag{5.2}$$

Further

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$$(5.3)\qquad |\bar{I}_2| = \int_{p_n/P_n}^{\delta} |\psi(t)| \cdot |\bar{N}_n(r_n;\ t)|dt$$

$$= \int_{p_n/P_n}^{\delta} |\psi(t)| \cdot O(R_\tau/tR_n)dt \quad \text{using Lemma 3.3 } (ii)$$

$$= O(1/R_n) \int_{p_n/P_n}^{\delta} o(t^{\alpha-1}) \cdot O(R_\tau)dt$$

$$\le O(R_{[P_n/p_n]}/R_n) \cdot o\{(p_n/P_n)^{\alpha}\}$$

$$= o\{(R_{[P_n/p_n]}/R_n)\cdot(p_n/P_n)^{\alpha}\}$$

$$\cong o\{(p_n/P_n)^{\alpha}\} .$$

Finally, since

$$\psi(t) \sim 2 \sum_{k=1}^{\infty} B_k(x) \sin kt ,$$

we have

$$(5.4)\quad |\bar{I}_3| \le 2 \int_{\delta}^{\pi} \left|\sum_{k=1}^{\infty} B_k(x) \sin kt\right| |\bar{N}_n(r_n;\ t)|dt$$

$$\le 2 \int_{\delta}^{\pi} \left|\sum_{k=1}^{\infty} b_k \cos kx \sin kt\right| |\bar{N}_n(r_n;\ t)|dt$$

$$+ 2 \int_{\delta}^{\pi} \left|\sum_{k=1}^{\infty} a_k \sin kx \sin kt\right| |\bar{N}_n(r_n;\ t)|dt$$

$$= \bar{I}_{3.1} + \bar{I}_{3.2} , \text{ say.}$$

Now

$$2 \sum_{k=1}^{\infty} b_k \cos kx \sin kt = \sum_{k=1}^{\infty} o\{(q_k/Q_k)^{\beta}\} \cdot \{\sin k(x+t) - \sin k(x-t)\}$$

$$\le \sum_{k=1}^{\infty} O(1)\{\sin k(x+t) - \sin k(x-t)\}$$

$$= O(1) .$$

Therefore, we have

$$\bar{I}_{3.1} = O(1) \int_{\delta}^{\pi} |\bar{N}_n(r_n;\ t)|dt \tag{5.5}$$

$$= O(1) \int_{\delta}^{\pi} O\{R_{\tau}/tR_n\}dt \quad \text{by using Lemma 3.3 } (ii)$$

$$= O(1/R_n)$$

$$= o(1) \ , \text{ as } \ n \to \infty \ .$$

Similarly

$$\bar{I}_{3.2} = o(1) \ . \tag{5.6}$$

Considering (5.4), (5.5) and (5.6), we get

$$|\bar{I}_3| = o(1) \ . \tag{5.7}$$

Consequently in view of (5.1), (5.2), (5.3) and (5.7) the proof of Theorem 2 is complete.

6.

Proof of Theorem 3. We have, as in Theorem 1,

$$t_n(r_n) - f(x) = \left\{\int_0^{p_n/P_n} + \int_{p_n/P_n}^{\delta} + \int_{\delta}^{\pi}\right\}\phi(t) \cdot N_n(r_n;\ t)dt \tag{6.1}$$

$$= J_1 + J_2 + J_3 \ , \text{ say.}$$

By using Lemma 3.1, we get

$$J_1 \le \int_0^{p_n/P_n} \phi(t) \cdot O(n)dt \tag{6.2}$$

$$= O(n) \cdot O\left\{(p_n/P_n)^{1+\alpha}\right\} \quad \text{by using (2.4)}$$

$$= O\left\{(p_n/P_n)^{\alpha}\right\} \quad \text{since } \ np_n \le P_n \ .$$

Further, since

$$\phi(t) \sim A_0 + 2 \sum_{k=1}^{\infty} A_k(x) \cos kt \ ,$$

where $A_0 = a_0 - 2f(x)$, we have

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$$(6.3)\quad |J_3| \le \int_\delta^\pi |A_0N_n(r_n;\ t)|dt + 2\int_\delta^\pi \left|\sum_{k=1}^\infty A_k(x)\cos kt\right| \cdot |N_n(r_n;\ t)|dt$$

$$= J_{3.1} + J_{3.2}\ ,\ \text{say.}$$

Now

$$(6.4)\qquad J_{3.1} = o(1)$$

as in Theorem 1. Further

$$\sum_{k=1}^\infty A_k(x)\cdot\cos kt\cdot N_n(r_n;\ t) = N_n(r_n;\ t)\sum_{k=1}^\infty O\left\{(q_k/Q_k)^\beta\right\}\cdot\cos kt$$

using (2.2)

$$\le O\{N_n(r_n;\ t)\}\cdot\sum_{k=1}^\infty\{\cos kt\}$$

$$= O\{N_n(r_n;\ t)\}\ .$$

Thus

$$(6.5)\qquad J_{3.2} = \int_\delta^\pi O\{N_n(r_n;\ t)\}dt$$

$$= \int_\delta^\pi O(R_\tau/tR_n)dt \quad \text{by using Lemma 3.3 } (ii)$$

$$= O(1/R_n)$$

$$= o(1)\ ,\ \text{as}\ n\to\infty\ .$$

Considering (6.3), (6.4) and (6.5), we have

$$(6.6)\qquad |J_3| = o(1)\ .$$

Finally, following Bosanquet [1], we get

$$(6.7)\quad J_2 = \frac{1}{\Gamma(-r)}\int_{p_n/P_n}^\delta N_n(r_n;\ t)dt\int_0^t (t-u)^{-r-1}d\Phi_{r+1}(u)$$

$$= \frac{1}{\Gamma(-r)}\left\{\int_0^{p_n/P_n} d\Phi_{r+1}(u)\int_{p_n/P_n}^\delta (t-u)^{-r-1}N_n(r_n;\ t)dt\right.$$

$$\left.+\int_{p_n/P_n}^\delta d\Phi_{r+1}(u)\int_u^\delta (t-u)^{-r-1}N_n(r_n;\ t)dt\right\}$$

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by changing the order of integration

$$= \frac{1}{\Gamma(-r)} \left\{ \int_0^{p_n/P_n} J(p_n, u)\, d\Phi_{r+1}(u) + \int_{p_n/P_n}^{\delta} K(u)\, d\Phi_{r+1}(u) \right\}$$

$$= J_{2.1} + J_{2.2}\text{, say.}$$

Using Lemma 3.4 *(i)*, we get

$$(6.8)\quad J_{2.1} \le \frac{1}{\Gamma(-r)} \int_0^{p_n/P_n} O\left\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(P_n/p_n\right)^{r+1} \right\} |d\Phi_{r+1}(u)|$$

$$= O\left\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(P_n/p_n\right)^{r+1} \right\} \int_0^{p_n/P_n} u^r \left\{ u^{-r} |d\Phi_{r+1}(u)| \right\}$$

$$= O\left\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(P_n/p_n\right)^{r+1} \right\}$$

$$\cdot \left[\left\{ u^r \cdot o\left(u^{1+\alpha}\right) \right\}_0^{p_n/P_n} - \int_0^{p_n/P_n} r \cdot u^{r-1} \cdot o\left(u^{1+\alpha}\right) du \right]$$

by partial integration and using (2.3)

$$= O\left\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(P_n/p_n\right)^{r+1} \right\}$$

$$\cdot \left[o\left\{ \left(p_n/P_n\right)^{\alpha+r+1} \right\} - r\left\{ o\left(p_n/P_n\right)^{\alpha+r+1} \right\} \right]$$

$$= o\left\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(p_n/P_n\right)^{\alpha} \right\} .$$

Similarly,, using Lemma 3.5 *(i)* we get

$$(6.9)\qquad J_{2.2} \le O\left(1/R_n\right) \int_{p_n/P_n}^{\delta} u^{-1-r} R_{[1/u]} |d\Phi_{r+1}(u)|$$

$$\le O\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(P_n/p_n\right) \} \int_{p_n/P_n}^{\delta} u^{-r} |d\Phi_{r+1}(u)|$$

$$= o\left\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(p_n/P_n\right)^{\alpha} \right\} \text{ by using (2.3).}$$

From (6.7), (6.8) and (6.9), we get

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$$J_2 = o\left\{\left(R_{|p_n/p_n|}/R_n\right) \cdot \left(p_n/P_n\right)^{\alpha}\right\} \tag{6.10}$$
$$\cong o\left\{\left(p_n/P_n\right)^{\alpha}\right\} .$$

By virtue of (6.1), (6.2), (6.6) and (6.10), the proof of Theorem 3 is complete.

7.

Proof of Theorem 4. We have, as in Theorem 2,

$$\bar{t}_n\left(r_n\right) - \bar{f}(x) = -\left\{\int_0^{p_n/P_n} + \int_{p_n/P_n}^{\delta} + \int_{\delta}^{\pi}\right\}\psi(t)\bar{N}_n\left(r_n;\ t\right)dt \tag{7.1}$$
$$= \bar{J}_1 + \bar{J}_2 + \bar{J}_3 \text{ , say.}$$

Proceeding as in Theorem 2, we get

$$\bar{J}_1 = o\left\{\left(p_n/P_n\right)^{\alpha}\right\} + o(1) . \tag{7.2}$$

Further, we write as in Theorem 2,

$$\psi(t) \sim 2 \sum_{k=1}^{\infty} B_k(x) \sin kt$$

whence we have

$$|\bar{J}_3| \le 2 \int_{\delta}^{\pi} \left|\sum_{k=1}^{\infty} B_k(x) \sin kt\right| |\bar{N}_n\left(r_n;\ t\right)| dt .$$

Now

$$\sum_{k=1}^{\infty} B_k(x) \sin kt = \sum_{k=1}^{\infty} O\left(q_k/Q_k\right)^{B} \sin kt \quad \text{by using (2.5)}$$
$$= O(1) \sum_{k=1}^{\infty} \sin kt$$
$$= O(1) .$$

Thus

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(7.3) $$|\bar{J}_3| \le O(1) \int_\delta^\pi |\bar{N}_n(r_n;\ t)|dt$$
$$= O(1/R_n) \quad \text{by using Lemma 3.3 } (ii)$$
$$= o(1)\ ,\ \text{as}\quad n \to \infty\ .$$

Now, using the fractional integration for $\psi(t)$, we get

(7.4) $$\bar{J}_2 = \frac{1}{\Gamma(-r)} \int_{p_n/P_n}^{\delta} \bar{N}_n(r_n;\ t) \int_0^t (t-u)^{-r-1} d\Psi_{r+1}(u)$$
$$= \frac{1}{\Gamma(-r)} \left\{ \int_0^{p_n/P_n} d\Psi_{r+1}(u) \int_{p_n/P_n}^{\delta} (t-u)^{-r-1} \bar{N}_n(r_n;\ t)dt \right.$$
$$\left. + \int_{p_n/P_n}^{\delta} d\Psi_{r+1}(u) \int_u^\delta (t-u)^{-r-1} \bar{N}_n(r_n;\ t)dt \right\}$$

by changing the order of integration

$$= \frac{1}{\Gamma(-r)} \left\{ \int_0^{p_n/P_n} \bar{J}(p_n,\ u)d\Psi_{r+1}(u) + \int_{p_n/P_n}^{\delta} \bar{K}(u)d\Psi_{r+1}(u) \right\}$$
$$= \bar{J}_{2.1} + \bar{J}_{2.2}\ ,\ \text{say}.$$

Using Lemma 3.4 *(ii)* and 3.5 *(ii)* and (2.6) and proceeding similarly as in Theorem 3, we get

(7.5) $$\bar{J}_{2.1} = O\left\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(p_n/P_n\right)^\alpha \right\}$$

and

(7.6) $$\bar{J}_{2.2} = O\left\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(p_n/P_n\right)^\alpha \right\} .$$

From (7.4), (7.5) and (7.6) we get

(7.7) $$\bar{J}_2 = O\left\{ \left(R_{[P_n/p_n]}/R_n\right) \cdot \left(p_n/P_n\right)^\alpha \right\} .$$

Consequently, in view of (7.1), (7.2), (7.3) and (7.7), the proof of Theorem 4 is complete.

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