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A FAMILY OF M^*-GROUPS

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1. Introduction. A compact bordered Klein surface of (algebraic) genus $g \geqq 2$ is said to have *maximal symmetry* [**5**] if its automorphism group is of order $12(g - 1)$, the largest possible. An *M^*-group* acts as the automorphism group of a bordered surface with maximal symmetry. M^*-groups were first studied in [**6**], and additional results about these groups are in [**5**, **7**, **8**].

Here we construct a new, interesting family of M^*-groups. Each group G in the family is an extension of a cyclic group by the automorphism group of a torus T with holes that has maximal symmetry. Furthermore, G acts on a bordered Klein surface X that is a fully wound covering [**7**] of T, that is, an especially nice covering in which X has the same number of boundary components as T. The construction we use for the new family of M^*-groups is a standard one that employs group automorphisms to define extensions of groups. We obtain a complete presentation for each new group. Because there is a correspondence between bordered Klein surfaces with maximal symmetry and regular triangulations [**5**], our construction also produces a new family of regular maps.

The new family of M^*-groups contains infinitely many groups, and we use these groups to establish two notable results. We show that for any positive integer n, there is a positive integer k such that there are at least n non-isomorphic M^*-groups of order k. Perhaps this result could have been anticipated, since Cohen has established the corresponding result for Hurwitz groups, the groups that act on Riemann surfaces with the maximum possible number of automorphisms [**1**]. Also, M^*-groups are generally more abundant than Hurwitz groups [**8**].

We also settle another related question. For each value of the positive integer g, there are several different topological types of bordered surfaces of genus g; each distinct topological type is called a *species* of the genus. Some genera have more than one species with maximal symmetry [**8**, Theorem 8]. Thus there is the natural problem of determining whether there is a bound (independent of the genus g) for the number of species within a single genus that can have maximal symmetry [**8**, Problem 6]. There is no such bound; we prove that for any positive integer n, there is a positive integer g such that there are at least n distinct species of bordered Klein surfaces of genus g that have maximal symmetry.

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2. M^*-groups. We assume that all surfaces are compact and of genus $g \geqq 2$.

A finite group G is called an *M^*-group* [**6**, **7**] if it is generated by three distinct non-trivial elements x, u, and z which satisfy the relations

(2.1) $u^2 = x^2 = (ux)^2 = (uz)^2 = (xz)^3 = 1.$

The order of z is called an *index* of G. The fundamental result about M^*-groups is the following.

THEOREM A [**6**, **5**]. *A finite group G is an M^*-group with index q if and only if G is the automorphism group of a bordered Klein surface X with maximal symmetry and k boundary components, where*

$$o(G) = 2qk.$$

In this situation we will frequently say that *G acts on X with index q*. The index of G determines the number of boundary components of X but not the orientability of the surface. The dihedral group $\langle u, z \rangle$ is the subgroup of G that fixes one of the boundary components of X, with z acting as a rotation of the boundary component and u as a reflection. For more on the action of G on X, see [**5**, pp. 267, 280-282] and [**7**, p. 24].

If the M^*-group G has index $q \leqq 5$, then the relations (2.1) and $z^q = 1$ imply G is finite. G is then a quotient of the well-known group that Coxeter and Moser denote $[3, q]$; see [**2**, pp. 35-37] and [**8**, p. 376]. This fact was used to classify the M^*-groups with index $q \leqq 5$ and the surfaces on which they act [**8**, Section 3 and Theorem 8].

Let $2 \leqq n \leqq 5$. Also in [**8**, Section 3] is the construction of a family of M^*-groups that are quotients of $[3, n]$. The largest group in the family is the group H_n with generators u, x, and z and defining relations

(2.2) $u^2 = x^2 = (ux)^2 = (uz)^2 = (xz)^3 = 1, \; xz^n = z^n x.$

The order and index of H_n are given in a table [**8**, p. 377]. Let α_n be the order of z^n. For any m that divides α_n, let $H_{n,m}$ be the group with generators u, x, and z and defining relations (2.2) and $z^{nm} = 1$. Then $H_{n,m}$ is an M^*-quotient of H_n, and

$$o(H_{n,m}) = m \cdot o(H_n)/\alpha_n$$

[**8**, p. 378]. It is easy to see that $H_{n,1} \cong [3, n]$.

These groups are interesting in part because they act on surfaces with maximal symmetry that are in families of fully wound coverings. A *fully wound covering* [**7**] is a full covering $f: X \to Y$ of the bordered Klein surface Y such that X has the same number of boundary components as Y; if the degree of the covering is d, each component of ∂X is wrapped around its image d times. Now for each value of n, the groups $H_{n,m}$ act on surfaces in a family of fully wound coverings; each surface on which a group $H_{n,m}$ acts is a fully wound covering of a surface on which the smallest group $H_{n,1}$ acts [**8**, p. 378].

3. Tori with holes. Our original goal was to obtain a similar construction for an M^*-group G with index 6, that is, a construction that would produce larger M^*-groups having G as a quotient with the corresponding surfaces in a family of fully wound coverings. The group defined by the relations (2.1) and $z^6 = 1$ is not finite [**2**, p. 37]. Our starting point then was the following easy result.

PROPOSITION 1. *Let G be an M^*-group. G has index* 6 *if and only if G acts on a torus with holes that has maximal symmetry.*

Proof. Let G act on X, a bordered surface with genus g, topological genus p, and k boundary components. Then

$$o(G) = 12(g - 1) = 2qk,$$

and $g = 2p + k - 1$ if X is orientable and $g = p + k - 1$ if X is non-orientable [**5**, p. 266].

First suppose that X is a torus with holes. Then immediately $g = k + 1$ and $q = 6$. Note that simply $o(G) = 12k$.

Now suppose $q = 6$ so that $g = k + 1$. If X were non-orientable, then $g = p + k - 1$ and $p = 2$, so that X would be a Klein bottle with holes. But no such surface has maximal symmetry [**5**, p. 270]. Hence X is orientable so that

$$g = 2p + k - 1 = k + 1$$

and the topological genus $p = 1$.

The tori with holes that have maximal symmetry have been classified.

THEOREM B [**5**]. *There is a torus with k holes with maximal symmetry if and only if k has the form n^2 or $3n^2$ for some integer n.*

Our next step was to obtain for each M^*-group with index $q = 6$ a complete presentation containing the relations (2.1). We felt that it was important that the extra relations be intimately connected to the geometry of the surface. These presentations help illuminate the later theoretical development, and we shall indicate briefly how they were obtained.

Let G be an M^*-group with index 6, and suppose G acts on the torus X with holes. Embed X in a torus X^* without boundary, so that every automorphism of X extends to an automorphism of X^* [**5**, Theorem D]. Then $X^* = C/\Lambda$, where Λ is the lattice subgroup of C generated by $\{1, \omega\}$ and $\omega = e^{\pi i/3}$. Let Q be the set of centers of the discs adjoined to X to make X^*, and let $\Omega = f^{-1}(Q)$, where $f:C \to C/\Lambda = X^*$ is the natural quotient map. For more details, see [**5**, pp. 269, 270].

First assume that the bordered surface X has n^2 holes. Then Ω is the lattice generated by $\{1/n, \omega/n\}$ [**5**, p. 270]. Certain automorphisms of C naturally induce automorphisms of the bordered surface X. Let z be the automorphism of X induced by rotation 60° clockwise about the origin and u that induced by reflection across the x-axis. Also, translation by $1/n$ and translation by ω/n induce automorphisms a and b, respectively. Then

$$L = \langle u, z \rangle \cong D_6$$

is the subgroup of G that fixes the boundary component centered about $f(0)$, and the translations a and b generate a subgroup $K \cong C_n \times C_n$. Since

$$L \cap K = \{1\} \quad \text{and} \quad o(G) = 12n^2,$$

$G = \langle u, z, a, b \rangle$.

Now let $x = az^3$. Geometrically x is the half-turn that interchanges the boundary components centered about $f(0)$ and $f(1/n)$. It is routine to verify that the relations (2.1) hold and to find other relations among the generators. For example, $b = zaz^{-1}$. It turns out that the following is a complete presentation for G.

$$(3.1) \quad \begin{gathered} u^2 = x^2 = z^6 = (ux)^2 = (uz)^2 = (xz)^3 = 1, \\ a^n = 1, x = az^3. \end{gathered}$$

This follows easily from the construction of Section 5, and no details will be given here. Obviously the generator a is not needed, but we shall use it to remind us of the geometry.

Now suppose that the bordered surface X has $3n^2$ holes. In this case the lattice Ω is generated by $\{1/n, (1 + \omega)/3n\}$ [**5**, p. 270]. As before, let z and u be the automorphisms of X induced by rotation 60° clockwise about the origin and reflection across the x-axis. Let a and d be the automorphisms induced by translation by $(1 + \omega)/3n$ and translation by $1/n$, respectively. Then G is generated by u, z, a, and d.

Let $x = az^3$ be the half-turn that interchanges the boundary components about $f(0)$ and $f((1 + \omega)/3n)$. Then a complete presentation for G is the following.

$$(3.2) \quad \begin{gathered} u^2 = x^2 = z^6 = (ux)^2 = (uz)^2 = (xz)^3 = 1, \\ a^{3n} = d^n = 1, x = az^3, d = zaz^{-1}a. \end{gathered}$$

Again this is an easy consequence of the construction of Section 5.

Note that in each case the translations generate an abelian normal subgroup K such that the quotient group $G/K \cong D_6$.

After obtaining the presentations (3.1) and (3.2), our next step was to change the relation $z^6 = 1$ to $xz^6 = z^6x$ in each presentation. The resulting groups were then studied for several values of n. It was quite helpful to have available a computer program that systematically enumerates the cosets of a subgroup of a group defined by generators and relations. Both the data collected for small n and the following observation were important in discovering the general construction. Suppose G^* is a larger group obtained by changing the relation $z^6 = 1$ to $xz^6 = z^6x$ in a presentation of the form (3.1) or (3.2). Then $M = \langle z^6 \rangle$ is normal in G^*, and G^*/M is the automorphism group of a torus with holes. Thus G^* also has D_6 as a quotient, that is, G^* has a normal subgroup K such that $G^*/K \cong D_6$.

4. The general construction. Here we present the construction needed to build the new family of M^*-groups. The construction forms larger groups from 2-generator groups that admit an action of the dihedral group $D_6 \cong C_2 \times S_3$, the smallest M^*-group. Of course many interesting and complicated groups have presentations with only two generators. The construction consists of the repeated application of a standard technique described in [**2**, p. 5]. Throughout this section let $K = \langle a, b | R \rangle$ be a group with generators a and b and defining relations R.

LEMMA 1. *Let $K = \langle a, b | R \rangle$, and let g and f be automorphisms of K such that*

$$f^2 = g^6 = (fg)^2 = 1.$$

If for some $c \in Z(K)$, the center of K,

$$g(c) = c \quad \text{and} \quad f(c) = c^{-1},$$

then there is a group G of order $12 \cdot o(K)$ with generators a, b, u, z and relations R involving a and b together with

$$u^2 = (uz)^2 = 1, \quad z^6 = c,$$

$$zaz^{-1} = g(a), \quad zbz^{-1} = g(b),$$

$$uau = f(a), \quad ubu = f(b).$$

Proof. First adjoin to K a new element z that transforms the elements of K according to the automorphism g. Identify z^6 with c, which is in the center of K and is fixed by g. Then the order of the new group K' is $6 \cdot o(K)$, and K' has generators a, b, and z and relations R involving a and b together with

$$(4.1) \quad z^6 = c, \; zaz^{-1} = g(a), \; zbz^{-1} = g(b).$$

Next extend the automorphism f to K' in the natural way, that is, define $f':K' \to K'$ by $f'(a) = f(a), f'(b) = f(b)$, and $f'(z) = z^{-1}$. It is easy to see that f' is an automorphism of K' of order two. Now adjoin to K' a new element u of order two that transforms the elements of K' according to f'. The order of the larger group G is $2 \cdot o(K') = 12 \cdot o(K)$, and G has generators a, b, z, and u and relations R, (4.1), $u^2 = 1$, $uzu = z^{-1}$, $uau = f(a)$, and $ubu = f(b)$.

By construction, K is normal in G and $G/K \cong D_6$.

Now we consider a rather special action on the group $K = \langle a, b|R\rangle$ (motivated by the groups of Section 3). The action does have an interesting range of applications. As usual, let $[a, b] = a^{-1}b^{-1}ab$ denote the commutator of a and b.

Let $c = [a, b]$. Suppose $c \in Z(K)$, and

$$(4.2) \quad \begin{aligned} g(a) &= b, & g(b) &= a^{-1}b, \\ f(a) &= ac, & f(b) &= ab^{-1}c \end{aligned}$$

define automorphisms of K. Then it is routine to check that

$$g(c) = c, f(c) = c^{-1}, \quad \text{and } f^2 = g^6 = (fg)^2 = 1.$$

Applying Lemma 1 yields our general construction.

THEOREM 1. *Let $K = \langle a, b|R\rangle$ and $c = [a, b]$. Suppose $c \in Z(K)$ and the functions g and f defined by* (4.2) *are automorphisms of K. Then there is a group G of order* $12 \cdot o(K)$ *with generators u, x, and z and defining relations*:

$$(4.3) \quad u^2 = x^2 = (ux)^2 = (uz)^2 = (xz)^3 = 1, xz^6 = z^6x,$$

and the relations R involving $a = xz^{-3}$ and $b = zaz^{-1}$. Furthermore G is an M^*-group.

Proof. By the lemma there is a group G of order $12 \cdot o(K)$ with generators a, b, u, z and relations R involving a and b together with

$$(4.4) \quad \begin{aligned} u^2 &= (uz)^2 = 1, & z^6 &= c = [a, b], \\ zaz^{-1} &= b, & zbz^{-1} &= a^{-1}b, \\ uau &= ac, & ubu &= ab^{-1}c. \end{aligned}$$

Let $x = az^3$. Then G is generated by u, x, and z, since $a = xz^{-3}$ and $b = zaz^{-1}$. We first check that the relations (4.3) hold. Note that

$$z^2az^{-2} = zbz^{-1} = a^{-1}b.$$

Conjugating by z, we have

$$z^3az^{-3} = za^{-1}bz^{-1} = b^{-1}a^{-1}b = c^{-1}a^{-1}$$

since $c = [a, b]$. But $c = z^6$ and c commutes with a. Thus

$$x^2 = az^3az^3 = 1.$$

From $z^2az^{-2} = a^{-1}b$, we have

$$az^2az^3 = bz^5 = zaz^4 \quad \text{or} \quad xz^{-1}x = zxz.$$

Thus $(xz)^3 = 1$. Next

$$uxu = uaz^3u = (ac)z^{-3} = az^3 = x \quad \text{and} \quad (ux)^2 = 1.$$

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Since $z^6 \in Z(K)$, obviously $xz^6 = z^6x$. Hence the generators u, x, and z satisfy the relations (4.3) and the relations R involving a and b.

The relations (4.3) include the relations (2.1), and u, x, and z are distinct and non-trivial by the construction of G. Therefore G is an M^*-group.

We still must show that the relations (4.4) can be obtained from (4.3). Consider the abstract group with generators u, x, and z and relations R and (4.3). Obviously u, z, a, and b form another set of generators. From $(xz)^3 = 1$, we have $(zx)^2 = xz^{-1}$ and

$$zbz^{-1} = z^2xz^{-5} = z^2(zx)^2z^{-4} = (z^3x)z(xz^{-3})z^{-1} = a^{-1}b.$$

Next

$$uau = uxz^{-3}u = xz^3 = az^6.$$

Now

$$ubu = uzxz^{-4}u = z^{-1}xz^4.$$

Also

$$ab^{-1}z^6 = xz^{-3}(zxz^{-4})^{-1}z^6 = xzxz^5 = z^{-1}xz^4$$

since $(xz)^3 = 1$. Therefore $ubu = ab^{-1}z^6$.

Finally from $x^2 = (az^3)^2 = 1$, we have

$$z^6a = z^3a^{-1}z^{-3}.$$

Also

$$z^2az^{-2} = zbz^{-1} = a^{-1}b,$$

so that

$$z^3az^{-3} = za^{-1}bz^{-1} = b^{-1}a^{-1}b = (b^{-1}ab)^{-1}.$$

Hence

$$b^{-1}ab = z^3a^{-1}z^{-3} = z^6a.$$

Obviously a commutes with z^6, since $xz^6 = z^6x$. Thus

$$c = [a, b] = a^{-1}b^{-1}ab = z^6,$$

and we have obtained the relations (4.4). Therefore the relations (4.3) and R are an abstract definition for G.

We mention the important special case in which the group K is abelian. Then the commutator $c = 1$, of course.

COROLLARY. *Let* $K = \langle a, b | R, [a, b] = 1 \rangle$ *be an abelian group, and suppose*

$$\begin{aligned} &g(a) = b, \quad g(b) = a^{-1}b, \\ &f(a) = a, \quad f(b) = ab^{-1} \end{aligned} \tag{4.5}$$

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define automorphisms of K. Then there is a group G of order $12 \cdot o(K)$ *with generators u, x, and z and defining relations*

(4.6) $u^2 = x^2 = z^6 = (ux)^2 = (uz)^2 = (xz)^3 = 1$

and the relations R involving $a = xz^{-3}$ *and* $b = zaz^{-1}$.

5. Applications. Here we apply Theorem 1 to several groups with two generators to obtain an interesting family of M^*-groups. We first construct the groups with presentations (3.1) and (3.2) and then form some larger groups having these as quotients.

Application 1. Let $K = C_n \times C_n$. Then K has presentation

$$a^n = b^n = 1, ab = ba.$$

The functions g and f defined by (4.5) are clearly automorphisms of K. Then there is a group G of order $12n^2$ with generators u, x, and z and defining relations (4.6) and

$$a^n = b^n = 1, a = xz^{-3}, b = zaz^{-1}.$$

Obviously G has the simpler presentation consisting of (4.6), $a^n = 1$, and $a = xz^{-3}$. Thus we have the presentation (3.1) of the automorphism group of a torus with n^2 holes with maximal symmetry. Henceforth let Q_n denote the group with presentation (3.1).

Application 2. Let $K = C_{3n} \times C_n$. To apply Theorem 1 we need a presentation for K in which the two generators have the same order. The simplest one is

$$a^{3n} = (ba)^n = 1, ab = ba.$$

It is easy to see that the functions g and f defined by (4.5) are automorphisms of K. Then there is a group G of order $36n^2$ with generators u, x, and z and defining relations (4.6) and

$$a^{3n} = (ba)^n = 1, a = xz^{-3}, b = zaz^{-1}.$$

But this is the presentation (3.2) (with $ba = d$), so that G acts on a torus with $3n^2$ holes that has maximal symmetry. Now let P_n be the group with presentation (3.2).

Application 3. Let K be the group with generators a, b, and c and defining relations

$$(5.1)\quad \begin{aligned} &a^n = b^n = 1, ab = bac, \\ &ac = ca, bc = cb. \end{aligned}$$

Of course $c = [a, b]$ and K is generated by a and b. K is a non-abelian group of order n^3 (if $n > 1$). In fact, let

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$$L = \langle a, c | a^n = c^n = [a, c] = 1 \rangle.$$

The function σ defined by $\sigma(a) = ac^{-1}$ and $\sigma(c) = c$ is an automorphism of order n. Adjoining to L a new element b of order n that transforms the elements of L according to σ produces the group K. The relation $c^n = 1$ is not needed in (5.1), since

$$bc = aba^{-1} \quad \text{and} \quad 1 = (aba^{-1})^n = (bc)^n = b^n c^n = c^n.$$

We need the following result about the group K.

LEMMA 2. *Let $t = a^{-1}b$, and let k be a positive integer. Then*

$$t^k = a^{-k} b^k c^{k(k-1)/2}.$$

Proof. Since $ba^{-1} = a^{-1}bc$, it is easy to see first that

$$ba^{-k} = a^{-k}bc^k.$$

Then a simple argument using induction yields the result. A detailed proof of a very similar result is in [**4**, p. 19].

In order to apply our construction, the functions g and f defined by (4.2) must be automorphisms of K, and $g(b) = a^{-1}b = t$. Now

$$t^n = c^{n(n-1)/2},$$

so that if n is odd, $t^n = 1$ (and in fact $o(t) = n$). However, if n is even, then $t^n = c^{n/2} \neq 1$, $o(t) = 2n$, and g is not an automorphism of K.

Therefore let n be odd. It is not difficult to see that g and f are automorphisms of K; Lemma 2 is helpful. By Theorem 1 there is a group G of order $12n^3$ with generators u, x, and z and relations (4.3), (5.1), $a = xz^{-3}$, and $b = zaz^{-1}$. But (4.3) implies the relations (4.4), in particular, $z^6 = [a, b] = c$. Then $ac = ca$ and $bc = cb$ follow from $xz^6 = z^6x$. Also $a^n = 1$ implies $b^n = 1$. Therefore G has defining relations (4.3), $a^n = 1$, and $a = xz^{-3}$.

Application 4. Let n be even, and let H be the group with generators a, b, and c and defining relations

$$(5.2) \qquad \begin{aligned} a^n = b^n = c^{n/2} = 1,\ ab = bac, \\ ac = ca,\ bc = cb. \end{aligned}$$

Let K be the group with presentation (5.1), and let N be the subgroup of K generated by $c^{n/2}$. Clearly N is normal in K, $o(N) = 2$, and $K/N = H$. Thus H is a group of order $n^3/2$. H is non-abelian if $n > 2$.

Any relation that holds in K also holds in H, and in addition, $c^{n/2} = 1$. Lemma 2 now shows that in H, $(a^{-1}b)^n = 1$. The functions g and f defined by (4.2) are automorphisms of H, and there is a larger group G of order $6n^3$ with generators u, x, and z and relations (4.3), (5.2), $a = xz^{-3}$, and $b = zaz^{-1}$. Again the presentation may be simplified. First note that the relation $c^{n/2} = 1$ may be obtained from the others. Since a and b satisfy the defining relations (5.1) of the group K, we have $(a^{-1}b)^n = c^{n/2}$ by Lemma 2. But (4.3) implies (4.4) so that $zbz^{-1} = a^{-1}b$ and consequently

$$c^{n/2} = (a^{-1}b)^n = (zbz^{-1})^n = 1.$$

As before, the relations $ac = ca$, $bc = cb$, and $b^n = 1$ are not needed. Thus G has defining relations (4.3), $a^n = 1$, and $a = xz^{-3}$, and the final presentation with n even is the same as that with n odd.

We combine these two applications to give our first main result. For a positive integer n, we define

$$s_n = \begin{cases} n & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

THEOREM 2. *Let n be a positive integer. Let K_n be the group with generators u, x, and z and defining relations*

$$u^2 = x^2 = (ux)^2 = (uz)^2 = (xz)^3 = 1,\ xz^6 = z^6x,$$
$$a^n = 1,\ a = xz^{-3}.$$

Then the order of K_n is $12n^2s_n$, and K_n is an M^*-group with index $6s_n$.

Proof. It follows immediately from Applications 3 and 4 that

$$o(K_n) = 12n^2s_n$$

and the index of K_n is

$$q = 6 \cdot o(z^6) = 6s_n.$$

Note that $M = \langle z^6 \rangle$ is normal in K_n and of course $K_n/M = Q_n$. But in general there are some intermediate quotient groups.

THEOREM 3. *Let n be a positive integer, and let m divide s_n. Let $K_{n,m}$ be the group with generators u, x, and z and defining relations*

$$\text{(5.3)}\quad \begin{aligned} u^2 = x^2 = z^{6m} = (ux)^2 = (uz)^2 = (xz)^3 = 1,\ xz^6 = z^6x, \\ a^n = 1,\ a = xz^{-3}. \end{aligned}$$

Then the order of $K_{n,m}$ is $12n^2m$, and $K_{n,m}$ is an M^*-group with index $6m$.

Proof. Fix the integer m. Let J be the subgroup of K_n generated by z^{6m}. Then J is normal in K_n, and clearly $K_{n,m} = K_n/J$. Since

$$o(J) = o(z^6)/m = s_n/m,$$

$$o(K_{n,m}) = 12n^2s_n/(s_n/m) = 12n^2m.$$

With a presentation of the form (2.1), $K_{n,m}$ is an M^*-group with index $q = o(z) = 6m$.

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For each n the groups $K_{n,m}$ act on bordered surfaces with maximal symmetry that are in a family of fully wound coverings. Each surface is a fully wound covering of a torus with holes and is therefore orientable [**8**, p. 376]. Consider $n = 16$, for example. Let X_8 be a surface with maximal symmetry on which the largest group $K_{16,8} = K_{16}$ acts. Let J_m be the subgroup of K_{16} generated by z^{6m} so that $K_{16,m} = K_{16}/J_m$, and let X_m be the surface X_8/J_m. Then we have the following families of fully wound coverings and automorphism groups.

$$\begin{array}{ccccccc} X_8 & \rightarrow & X_4 & \rightarrow & X_2 & \rightarrow & X_1 \\ K_{16,8} & \rightarrow & K_{16,4} & \rightarrow & K_{16,2} & \rightarrow & K_{16,1} \end{array}$$

Each surface X_m is an orientable surface with maximal symmetry that has $256 = (16)^2$ holes. X_1 is a torus with holes.

We return to applications of Theorem 1. The next two produce generalizations of the groups P_n in the same way that Applications 3 and 4 produce generalizations of Q_n. We will be rather brief.

Application 5. Let K be the group with generators a, b, and c and defining relations

$$(5.4) \qquad \begin{array}{c} a^{3n} = (ba)^n = 1, \; ab = bac, \\ ac = ca, \; bc = cb. \end{array}$$

K is a non-abelian group of order $3n^3$ (if $n > 1$).

Now let n be odd. Then the functions g and f defined by (4.2) are automorphisms of K. (If n is even, this is not the case.) Then there is a group G of order $36n^3$ with generators u, x, and z and relations (4.3), (5.4), $a = xz^{-3}$, and $b = zaz^{-1}$. Just as in Application 3, the relations $ac = ca$ and $bc = cb$ are unnecessary. Hence G has defining relations (4.3), $a^{3n} = (ba)^n = 1$, $a = xz^{-3}$, and $b = zaz^{-1}$.

Application 6. Let n be even, and H be the group with generators a, b, and c and defining relations

$$(5.5) \qquad \begin{array}{c} a^{3n} = (ba)^n = c^{n/2} = 1, \; ab = bac, \\ ac = ca, \; bc = cb. \end{array}$$

Then H is a quotient group of the group with presentation (5.4). H has order $3n^3/2$ and is non-abelian if $n > 2$. Applying the construction to H yields a larger group G of order $18n^3$ with generators u, x, and z and relations (4.3), (5.5), $a = xz^{-3}$, and $b = zaz^{-1}$. After some simplification, G has the same presentation as the group obtained in Application 5.

Combining the last two applications yields the following.

THEOREM 4. *Let n be a positive integer. Let L_n be the group with generators u, x, and z and defining relations*

$$u^2 = x^2 = (ux)^2 = (uz)^2 = (xz)^3 = 1,\ xz^6 = z^6x,$$
$$a^{3n} = (ba)^n = 1,\ a = xz^{-3},\ b = zaz^{-1}.$$

Then the order of L_n is $36n^2s_n$, and L_n is an M^*-group with index $6s_n$.

The proof of the following is like the proof of Theorem 3.

THEOREM 5. *Let n be a positive integer, and let m divide s_n. Let $L_{n,m}$ be the group with generators u, x, and z and defining relations*

$$u^2 = x^2 = z^{6m} = (ux)^2 = (uz)^2 = (xz)^3 = 1,\ xz^6 = z^6x,$$
$$a^{3n} = (ba)^n = 1,\ a = xz^{-3},\ b = zaz^{-1}.$$

Then the order of $L_{n,m}$ is $36n^2m$, and $L_{n,m}$ is an M^*-group with index $6m$.

For each n the surfaces with maximal symmetry on which the groups $L_{n,m}$ act are again in a family of fully wound coverings. Each surface is a covering of a torus with $3n^2$ holes.

The groups Q_n, P_n, K_n, $K_{n,m}$, L_n and $L_{n,m}$ are of course quite intimately related. We give some of the most important relationships.

$$Q_n = K_{n,1} \quad P_n = L_{n,1} \quad K_n = K_{n,s_n} \quad L_n = L_{n,s_n}$$
$$K_{n,m} = L_{n,m}/\langle a^n \rangle \quad L_{n,m} = K_{3n,m}/\langle (ba)^n \rangle.$$

Only the last two require any checking, and they are not hard to establish. These relationships justify our considering the groups as part of one large family.

6. Regular maps. There is an important correspondence between bordered Klein surfaces with maximal symmetry and regular maps. For the basic definitions on regular maps, see [**5**, pp. 278, 279] and [**2**, pp. 20, 101-103]. We use "regular" in the strong sense of [**5**] and [**11**]. A map is said to be of type $\{r, q\}$ if it is composed of r-gons, q meeting at each vertex.

Suppose the M^*-group G acts on the bordered surface X with index q. Then the surface X with maximal symmetry corresponds to a regular map $\mathscr{M}$ of type $\{3, q\}$ on the surface X^* obtained from X by attaching a disc to each boundary component. The map $\mathscr{M}$ is then a regular triangulation of X^*. Further G is isomorphic to the automorphism group of the map $\mathscr{M}$, and the number of boundary components of X is equal to the number of vertices of $\mathscr{M}$. For the details on this correspondence, see [**5**, pp. 278-282].

Thus each M^*-group constructed in Section 5 is the automorphism group of a regular map. In particular the groups Q_n and P_n act on maps on a torus. These maps are of course well-known [**2**, pp. 107-109]. Q_n is the automorphism group of the map $\{3, 6\}_{n,0}$, and P_n is the group of $\{3, 6\}_{n,n}$.

However the other groups produce a new family of regular maps. These maps are themselves quite interesting. Each is a covering of a map on a torus. For the basic definitions and facts about map coverings, see [**11**, pp. 765-768]. The new maps are similar in some respects to families of maps obtained by Sherk [**9**, pp. 458-460]. Garbe has also obtained families of maps that are coverings of toroidal maps [**3**]; Garbe's maps are not of type $\{3, q\}$ however.

We simply state the consequences of Theorems 3 and 5 for regular maps.

THEOREM 6. *Let n be a positive integer, and let m divide* s_n. *Then there is a regular map of type* $\{3, 6m\}$ *on an orientable surface of topological genus* $n^2(m - 1)/2 + 1$. *The map has automorphism group* $K_{n,m}$ *and is an m-fold covering of the torodial map* $\{3, 6\}_{n,0}$.

THEOREM 7. *Let n be a positive integer, and let m divide* s_n. *Then there is a regular map of type* $\{3, 6m\}$ *on an orientable surface of topological genus* $3n^2(m - 1)/2 + 1$. *The map has automorphism group* $L_{n,m}$ *and is an m-fold covering of the torodial map* $\{3, 6\}_{n,n}$.

7. Surfaces with maximal symmetry. In [**8**] there is the classification of all species of bordered Klein surfaces with maximal symmetry of genus $g \leqq 40$. The number is surprisingly high; there are 32 species in 18 different genera. Although there are several genera in that range with 2 species, there are none with 3. Thus there is the natural problem of determining whether there is a bound (independent of the genus g) for the number of species within a single genus that can have maximal symmetry.

Here we use the groups $K_{n,m}$ to show that there is no such bound, that is, that there are genera with an arbitrarily large number of species with maximal symmetry. The crucial observations are that it is easy to find groups of the same order and that the groups act on different species.

The group $K_{n,m}$ has order $12n^2m$ and index $q = 6m$ and acts on a surface with maximal symmetry with k holes, where

$$2qk = o(K_{n,m}).$$

Thus $k = n^2$. If it happens that $K_{n,m}$ and $K_{n',m'}$ have the same order but $n \neq n'$, then the two groups act on different species.

A convenient way to find numerous groups of the same order is to let $n = 2^i$ where $i \geqq 1$. Then $s_n = 2^{i-1}$. Let m divide s_n, so $m = 2^j$ for some j, $0 \leqq j \leqq i - 1$. Now

$$o(K_{n,m}) = 12n^2m = 12 \cdot 2^{2i+j}.$$

Fix the integer $t \geqq 2$. The number of different values of n (and m) for which $o(K_{n,m}) = 12 \cdot 2^t$ is just the number of solutions to the integer equation

$$2i + j = t$$

with $i \geqq 1$ and $0 \leqq j \leqq i - 1$. Let x be a real number, and let $\{x\}$ be the integer for which $\{x\} - 1 < x \leqq \{x\}$, that is, $\{x\}$ is the smallest integer greater than or equal to x. The following requires only a simple argument using induction.

LEMMA 3. *Let t be an even positive integer. Then there are $\{t/6\}$ solutions to the integer equation*

$$2i + j = t$$

with $i \geqq 1, 0 \leqq j \leqq i - 1$.

THEOREM 8. *Let r be a positive integer. Then there is a positive integer g such that there are at least r distinct species of bordered Klein surfaces of genus g that have maximal symmetry.*

Proof. Let $t = 6r$ and $g = 2^t + 1$. By Lemma 3 there are exactly r solutions to the equation $2i + j = t$ with $i \geqq 1, 0 \leqq j \leqq i - 1$. For each solution let $n = 2^i$, $m = 2^j$. Then the order of the group $K_{n,m}$ is $12 \cdot 2^t$, and it acts on an orientable surface with maximal symmetry of genus g, since

$$12(g - 1) = o(K_{n,m}).$$

The surface has n^2 boundary components. Now there are r different values of n (and m) for which

$$o(K_{n,m}) = 12 \cdot 2^t.$$

Thus there are at least r different species of genus g that have maximal symmetry.

There still remains the related question of whether or not there are arbitrarily many non-isomorphic M^*-groups of the same order, because many M^*-groups act on more than one species with maximal symmetry. The simplest example is $C_2 \times S_3$, the smallest M^*-group, which acts on a sphere with three holes and a torus with one [**5**, p. 270]. Singerman has given an interesting example of an M^*-group that acts on two different species with the same index [**10**]; in this case one of the species must be orientable and the other non-orientable. For more examples of one M^*-group acting on two species, see [**8**].

On the other hand it is possible for distinct M^*-groups to act on the same topological type of bordered surface. We use regular maps to present an example. There is a regular map of type $\{3, 7\}$ with 78 vertices on a non-orientable surface; the automorphism group of the map is $G = PSL(2, 13)$, the simple group of order 1092 [**2**, p. 140]. By the correspondence between surfaces with maximal symmetry and regular maps [**5**, Theorem 16, p. 280], G acts on a non-orientable surface X of genus 92 with $k = 78$ boundary components. Let X_0 be the orienting double of X. Then X_0 is an orientable surface with maximal symmetry of genus 183 with $2k = 156$ boundary components, topological genus 14, and automorphism group $C_2 \times G$. But there is a regular map of type $\{3, 7\}$ on an orientable surface of topological genus 14 that has automorphism group $PGL(2, 13)$; the map also has 156 vertices [**2**, p. 139]. The associated bordered surface has the same topological type as X_0. Thus $C_2 \times PSL(2, 13)$ and $PGL(2, 13)$ act on surfaces with maximal symmetry of the same species. It would be interesting to see a similar example in a lower genus.

To establish the result about M^*-groups that is a companion to Theorem 8, we show that the groups in the proof of that theorem are not isomorphic.

LEMMA 4. *Let n be an even positive integer such that $(3, n) = 1$, let m divide s_n, and let $G = K_{n,m}$ be the group with presentation* (5.3) *with $b = zaz^{-1}$. Then*

$$G'' = \langle a, b, z^6 \rangle \quad \text{and} \quad G''' = \langle z^6 \rangle \cong C_m.$$

Proof. Since G is an M^*-group, $[G:G']$ divides 4 and $[G':G'']$ divides 9 [**5**, p. 278]. Let

$$H = \langle a, b, z^6 \rangle.$$

H is the quotient of the group with presentation (5.2) (with $c = z^6$) by its subgroup $\langle z^{6m} \rangle$; see the proof of Theorem 3. Then H is normal in G, and $G/H = C_2 \times S_3$. Since $C_2 \times S_3$ has $C_2 \times C_2$ as a quotient, so does G. Hence $[G:G'] = 4$. Now $H \subset G'$ and $G'/H = C_3$. But $o(G) = 12n^2m$ and 3 does not divide n^2m. Therefore $H = G''$.

Let $M = \langle z^6 \rangle$. M is normal in H, and H/M is abelian. Thus $H' \subset M$. But

$$z^6 = c = [a, b]$$

so $z^6 \in H'$. Therefore $G''' = H' = M$.

THEOREM 9. *Let r be a positive integer. Then there is a positive integer N such that there are at least r non-isomorphic M^*-groups of order N.*

Proof. Let $t = 6r$ and $N = 12 \cdot 2^t$. Then as in the proof of Theorem 8, there are r different values of n (and m) for which $o(K_{n,m}) = N$. None of these groups could be isomorphic by Lemma 4.

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