Canad. J. Math. Vol. **44 (6)** 1992, pp. 1289–1302.

BEST POLYNOMIAL APPROXIMATION WITH LINEAR CONSTRAINTS

K. PAN AND E. B. SAFF

ABSTRACT. Let A be a $(k+1)\times(k+1)$ nonzero matrix. For polynomials $p \in \mathcal{P}_n$, set $\underline{p} := \big(p(0), p'(0), \ldots, p^{(k)}(0)\big)^T$ and $B_n(A) := \{p \in \mathcal{P}_n : A\underline{p} = \underline{0}\}$. Let $E \subset \mathbf{C}$ be a compact set that does not separate the plane and f be a function continuous on E and analytic in the interior of E. Set $E_n(A,f) := \inf\{\|f-p\|_E : p \in B_n(A)\}$ and $E_n(f) := \inf\{\|f-p\|_E : p \in \mathcal{P}_n\}$. Our goal is to study approximation to f on E by polynomials from $B_n(A)$. We obtain necessary and sufficient conditions on the matrix A for the convergence $E_n(A,f) \longrightarrow 0$ to take place. These results depend on whether zero lies inside, on the boundary or outside E and yield generalizations of theorems of Clunie, Hasson and Saff for approximation by polynomials that omit a power of z. Let $p^*_{n,A} \in B_n(A)$ be such that $E_n(A,f) = \|f - p^*_{n,A}\|_E$. We also study the asymptotic behavior of the zeros of $p^*_{n,A}$ and the asymptotic relation between $E_n(f)$ and $E_n(A,f)$.

1. Introduction and notation. Let E be a compact set in the complex plane $\mathbf{C}$ containing infinitely many points and let $\|\cdot\|$ denote the uniform norm on E. For a function f, if the derivatives $f^{(i)}(0)$, $i = 0, \ldots, k$, exist, define:

$$\underline{f} := \big(f(0), f'(0), \ldots, f^{(k)}(0)\big)^T.$$

Let $A := (a_{i,j})_{i,j=0}^k \neq 0$ be a given $(k+1)\times(k+1)$ matrix with complex constant entries. With $\mathcal{P}_n$ denoting the collection of all algebraic polynomials of degree at most n, we set

$$\alpha_{n,A}(f) := \inf\{\|p\| : p \in \mathcal{P}_n \text{ and } A\underline{p} = A\underline{f}\}, \quad n \geq k.$$

We also define

$$B_n(A) := \{p \in \mathcal{P}_n : A\underline{p} = \underline{0}\},$$

$$C(E) := \{f : f \text{ continuous on } E\},$$

$$\mathcal{A}(E) := \{f \in C(E) : f \text{ analytic in the interior of } E\},$$

$$E_n(A,f) := \inf\{\|f-p\| : p \in B_n(A)\},$$

$$E_n(f) := \inf\{\|f-p\| : p \in \mathcal{P}_n\},$$

$$\mathcal{B}_n(f) := \{p \in B_n(A) : \|f-p\| = E_n(A,f)\}.$$

The research of the first author was done in partial fulfillment of the Ph.D. degree at the University of South Florida under the direction of Profs. E. B. Saff and B. Shekhtman.

Research of the second author supported in part by the U.S. National Science Foundation under grant DMS-881-4026.

Received by the editors October 25, 1990.

AMS subject classification: 41A29.

© Canadian Mathematical Society, 1992.

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

Throughout we let $p^*_{n,A} := p^*_{n,A}(f)$ denote an arbitrary but fixed element of $\mathcal{B}_n(f)$, and we let $p^*_n := p^*_n(f)$ denote the unique polynomial in $\mathcal{P}_n$ satisfying $\|f - p^*_n(f)\| = E_n(f)$. As we shall show, the behavior of $E_n(A,f)$ depends on whether zero lies inside E, on the boundary of E or outside E. Our results generalize theorems of Clunie, Hasson and Saff [CHS] for approximation by polynomials that omit a single power of z. One important aspect of our investigation is the relation between $E_n(f)$ and $E_n(A,f)$. We also study the asymptotic behavior of the zeros of $p^*_{n,A}$.

It is natural to consider the more general problem of approximation from the set $B_n(A,\underline{a}) := \{p \in \mathcal{P}_n : A\underline{p} = \underline{a}\}$. If $\underline{a} \neq \underline{0}$, then one can replace the function $f(z)$ by the new function $g(z) := f(z) - \sum_{i=0}^k (w_i/i!)z^i$, where $\underline{w} := (w_0, \ldots, w_k)^T$ is a solution of $A\underline{x} = \underline{a}$ (the existence of $\underline{w}$ is assumed; otherwise $B_n(A,\underline{a}) = \emptyset$). Then for each $n \geq k$, the polynomial $p^*_{n,A}(g) + \sum_{i=0}^k (w_i/i!)z^i$ is a best approximation to f from $B_n(A,\underline{a})$. Thus, without loss of generality, we only need consider approximation from $B_n(A)$.

2. **Asymptotic behavior of $E_n(A,f)$.** For k a fixed nonnegative integer and $f \in \mathcal{A}(E)$, we shall examine the asymptotic behavior of $E_n(A,f)$ as $n \longrightarrow \infty$. We begin with some basic lemmas.

LEMMA 2.1. *If $f \in C(E)$ and $n \geq k$, then*

$$\alpha_{n,A}(p^*_n) - E_n(f) \leq E_n(A,f) \leq \alpha_{n,A}(p^*_n) + E_n(f). \tag{2.1}$$

PROOF. Note that

$$E_n(A,f) = \|f - p^*_{n,A}\| \geq \|p^*_n - p^*_{n,A}\| - \|f - p^*_n\|. \tag{2.2}$$

Since $A(\underline{p}^*_n - \underline{p}^*_{n,A}) = A\underline{p}^*_n$, we have $\|p^*_n - p^*_{n,A}\| \geq \alpha_{n,A}(p^*_n)$ and so the lower estimate in (2.1) follows from (2.2).

Now let $q \in \mathcal{P}_n$ be such that $A\underline{q} = A\underline{p}^*_n$ and $\|q\| = \alpha_{n,A}(p^*_n)$. Then we have

$$E_n(A,f) \leq \|f - p^*_n + q\| \leq E_n(f) + \|q\| = E_n(f) + \alpha_{n,A}(p^*_n).$$

■

LEMMA 2.2. *If $f \in C(E)$, then*

$$\lim_{n\to\infty} E_n(A,f) = 0$$

if and only if

$$\lim_{n\to\infty} E_n(f) = 0 \text{ and } \lim_{n\to\infty} \alpha_{n,A}(p^*_n) = 0.$$

PROOF. If $E_n(A,f) \longrightarrow 0$, then clearly $E_n(f) \longrightarrow 0$ as $n \longrightarrow \infty$. From the lower estimate in Lemma 2.1 we then deduce that $\lim_{n\to\infty} \alpha_{n,A}(p^*_n) = 0$.

The sufficiency of the conditions follows immediately from the upper estimate in Lemma 2.1. ■

Multiplying the inequalities in (2.1) by $\alpha^{-1}_{n,A}(p^*_n)$ we immediately get

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

LEMMA 2.3. *Let $f \in C(E)$. Suppose $\alpha_{n,A}(p_n^*) \neq 0$ for all n large and*

$$\lim_{n\to\infty} [\alpha_{n,A}(p_n^*)]^{-1} E_n(f) = 0. \tag{2.3}$$

Then

$$E_n(A,f) \cong \alpha_{n,A}(p_n^*) \text{ as } n \to \infty. \tag{2.4}$$

Here we use the notation $a_n \cong b_n$ to mean $a_n / b_n \to 1$ as $n \to \infty$.

To give conditions under which (2.3) is satisfied we need some further notation. We denote by K the unbounded component of $\overline{\mathbf{C}} \setminus E$ and by $g_K(z,\infty)$ the Green function with pole at infinity for K. We say that K is regular if for each point $z_0 \in \partial K$, the boundary of K, we have

$$\lim_{z\to z_0} g_K(z,\infty) = 0, \quad z \in K.$$

The following result is known as the Bernstein-Walsh lemma.

LEMMA 2.4 ([W, §4.6]). *Let E be a compact set whose complement K is connected and regular. If the polynomial $p \in \mathcal{P}_n$ satisfies the inequality $|p(z)| \leq L$ for z on E, then*

$$|p(z)| \leq L \exp\big(n g_K(z,\infty)\big), \quad z \in K.$$

We can now establish

THEOREM 2.5. *Suppose E is a compact set whose complement $\overline{\mathbf{C}} \setminus E$ is connected and regular. Assume that $f(z)$ is analytic on E and $0 \in E$. If $A\underline{f} \neq \underline{0}$, then the asymptotic formula* (2.4) *holds.*

PROOF. It is well-known (*cf.* [W, §4.7]) that since f is analytic on E,

$$\limsup_{n\to\infty} E_n^{1/n}(f) < 1 \tag{2.5}$$

and $\{p_n^*\}_0^\infty$ converges uniformly to f on some open set containing E. The latter property implies that

$$\lim_{n\to\infty} p_n^{*(j)}(0) = f^{(j)}(0), \quad j = 0,\dots,k. \tag{2.6}$$

Next, define

$$\beta_{n,A} := \sup\Big\{ \max_{0\leq i\leq k} \Big|\sum_{j=0}^{k} a_{i,j} p^{(j)}(0)\Big|, \|p\| \leq 1 \text{ and } p \in \mathcal{P}_n \Big\}. \tag{2.7}$$

We claim that

$$\limsup_{n\to\infty} \beta_{n,A}^{1/n} \leq 1. \tag{2.8}$$

In fact, if we define for each $\delta > 1$, the level curve

$$\Gamma_\delta := \{z : g_K(z,\infty) = \log \delta\},$$

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

then 0 is surrounded by Γ_δ since $0 \in E$. Now by the Cauchy integral formula, we have for all $p \in \mathcal{P}_n$,

$$p^{(j)}(0) = \frac{j!}{2\pi i} \int_{\Gamma_\delta} \frac{p(z)}{z^{j+1}}\, dz, \quad j = 0, 1, \ldots \text{ and } \delta > 1.$$

So for $p \in \mathcal{P}_n$ with $\|p\| \leq 1$, we obtain from Lemma 2.4 that

$$|p^{(j)}(0)| \leq \frac{j!}{2\pi} \delta^n \frac{\text{length}(\Gamma_\delta)}{\text{dist}(0, \Gamma_\delta)^{j+1}}, \quad j = 0, 1, \ldots.$$

According to the definition of $\beta_{n,A}$ in (2.7), we therefore get

$$\limsup_{n \to \infty} \beta_{n,A}^{1/n} \leq \delta,$$

and by letting $\delta \longrightarrow 1^+$ we have verified the claim (2.8).

Since $A\underline{f} \neq \underline{0}$, there is an i_0, $0 \leq i_0 \leq k$, such that

$$\sum_{j=0}^{k} a_{i_0,j} f^{(j)}(0) \neq 0. \tag{2.9}$$

For $n \geq k$, let $q_n \in \mathcal{P}_n$ satisfy $\|q_n\| = \alpha_{n,A}(p_n^*)$ and $A\underline{q}_n = A\underline{p}_n^*$. Then from (2.6) and (2.9) it follows that, for n large, $\|q_n\| \neq 0$ and so

$$\left| \sum_{j=0}^{k} a_{i_0,j} q_n^{(j)}(0) \right| \Big/ \|q_n\| \leq \beta_{n,A}.$$

Thus, for n large,

$$\alpha_{n,A}^{-1}(p_n^*) \leq \beta_{n,A} \left\{ \left| \sum_{j=0}^{k} a_{i_0,j} p_n^{*(j)}(0) \right| \right\}^{-1}. \tag{2.10}$$

Furthermore, from (2.6) we have, for n large,

$$\left| \sum_{j=0}^{k} a_{i_0,j} p_n^{*(j)}(0) \right| \geq \frac{1}{2} \left| \sum_{j=0}^{k} a_{i_0,j} f^{(j)}(0) \right|, \tag{2.11}$$

and so from (2.8), (2.10) and (2.11) we get

$$\limsup_{n \to \infty} [\alpha_{n,A}^{-1}(p_n^*)]^{1/n} \leq 1. \tag{2.12}$$

Combining (2.5) and (2.12) yields

$$\lim_{n \to \infty} \alpha_{n,A}^{-1}(p_n^*) E_n(f) = 0,$$

and so the theorem follows from Lemma 2.3. ∎

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

3. **Approximation with linear constraints.** It is well-known that, by Mergelyan's theorem, $E_n(f) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $f \in \mathcal{A}(E)$ if and only if the compact set E does not separate the plane; that is, $\overline{\mathbf{C}} \setminus E$ is connected. In this section, we shall study the conditions on the matrix A that imply $E_n(A,f) \longrightarrow 0$ as $n \longrightarrow \infty$.

THEOREM 3.1. *Let $f \in \mathcal{A}(E)$ and assume $\overline{\mathbf{C}} \setminus E$ is connected and $0 \notin E$. Then*

$$\lim_{n \to \infty} E_n(A,f) = 0. \tag{3.1}$$

PROOF. Since $0 \notin E$, the function $f(z)z^{-(k+1)} \in \mathcal{A}(E)$. Using Mergelyan's theorem, we have that for any $\varepsilon > 0$ there is a $p_{n-k-1} \in \mathcal{P}_{n-k-1}$ such that

$$\|f(z)z^{-(k+1)} - p_{n-k-1}(z)\| \leq \varepsilon, \quad \text{for } n \text{ large.}$$

Hence

$$\|f(z) - z^{k+1}p_{n-k-1}(z)\| = \left\|z^{k+1}\left(f(z)z^{-(k+1)} - p_{n-k-1}(z)\right)\right\| \leq \|z^{k+1}\|\varepsilon.$$

But $z^{k+1}p_{n-k-1} \in B_n(A)$; thus (3.1) follows. ∎

The case when zero lies interior to E is also easy to handle.

THEOREM 3.2. *Assume $0 \in E^o$, the interior of E, and $\overline{\mathbf{C}} \setminus E$ is connected. If $f \in \mathcal{A}(E)$, then*

$$\lim_{n \to \infty} E_n(A,f) = 0 \text{ if and only if } A\underline{f} = \underline{0}.$$

PROOF. First assume that $\lim_{n\to\infty} E_n(A,f) = 0$ and $0 \in E^o$. Then $\lim_{n\to\infty} p_{n,A}^{*(j)}(0) = f^{(j)}(0), j = 0, \ldots, k$. Also note that $A\underline{p}_{n,A}^* = \underline{0}$ and so letting $n \longrightarrow \infty$ we get $A\underline{f} = \underline{0}$.

Next assume that $A\underline{f} = \underline{0}$ and set

$$v_n(z) := \sum_{i=0}^{k} \frac{\left(p_n^{*(i)}(0) - f^{(i)}(0)\right)}{i!} z^i.$$

Since $A\underline{f} = \underline{0}$, we have $A\underline{v}_n = A(\underline{p}_n^* - \underline{f}) = A\underline{p}_n^*$. Thus

$$\alpha_{n,A}(p_n^*) \leq \|v_n\| \leq \sum_{i=0}^{k} \frac{|p_n^{*(i)}(0) - f^{(i)}(0)|}{i!} \|z^i\|, \quad n \geq k. \tag{3.2}$$

Now, by Mergelyan's theorem, $\lim_{n\to\infty} E_n(f) = 0$ and since $0 \in E^o$, we have $\lim_{n\to\infty} p_n^{*(j)}(0) = f^{(j)}(0), j = 0, \ldots, k$. Hence with (3.2) we get

$$\lim_{n \to \infty} \alpha_{n,A}(p_n^*) = 0$$

and the theorem follows from Lemma 2.2. ∎

It remains to consider the more interesting case when $0 \in \partial E$, the boundary of E. It can be seen from the results of Nersesyan [N] that the essential condition needed for convergence is that the constraint $A\underline{p} = \underline{0}$ does not imply that $p(0) = 0$. Here we provide a simple direct proof that utilizes the following result of [CHS, p. 68] stated in a slightly more general form.

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

LEMMA 3.3. *Let $0 \in \partial E$. For any $\varepsilon > 0$ and positive integer m there is a polynomial $q_0(z)$ such that*

$$\|z - z^{2m+1} q_0(z)^{2m}\| < \varepsilon.$$

Now we can state

THEOREM 3.4. *Assume $\overline{\mathbf{C}} \setminus E$ is connected and $0 \in \partial E$. Then the following conditions are equivalent:*

(i) $\lim_{n\to\infty} E_n(A,f) = 0$ *for all* $f \in \mathcal{A}(E)$;

(ii) $B_k(A) \setminus \mathcal{P}_{k,0} \neq \emptyset$, *where* $\mathcal{P}_{k,0} := z\mathcal{P}_{k-1}$ *and* $\mathcal{P}_{0,0} := \{0\}$; *that is, there exists a polynomial* $p \in B_k(A)$ *such that* $p(0) \neq 0$;

(iii) A *has* 0 *as an eigenvalue (i.e.* $\det A = 0$*) and has an associated eigenvector with first component equal to* 1.

PROOF. First observe that (ii) $\Leftrightarrow$ (iii) is trivial.

We now show that (iii) $\Rightarrow$ (i). For the linear system $A\underline{x} = \underline{0}$, where $\underline{x} := (x_0, \ldots, x_k)^T$, assertion (iii) states that there is a solution with first component not equal to zero. So it is easy to see that there is a submatrix $A_{i_1,\ldots,i_l;1,j_1,\ldots,j_{l-1}}$ whose determinant is nonzero, where $l := k + 1 - \operatorname{rank}(A)$, and $A_{i_1,\ldots,i_l;1,j_1,\ldots,j_{l-1}}$ denotes the submatrix obtained by deleting the i_1th, $\ldots, i_l$th rows and 1st, j_1th, $\ldots, j_{l-1}$th columns from A. (We remark that $l \leq k$ since $A \neq 0$.) Without loss of generality we can assume

$$\det \begin{pmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,k+1-l} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,k+1-l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-l,1} & a_{k-l,2} & \cdots & a_{k-l,k+1-l} \end{pmatrix} \neq 0.$$

Hence there exist constants $b_{i,j}$ and c_i such that for $n \geq k$

$$B_n(A) = \Big\{ p \in \mathcal{P}_n : p^{(i)}(0) = c_i p(0) + \sum_{j=k-l+2}^{k} b_{i,j} p^{(j)}(0),\ i = 1, \ldots, k + 1 - l \Big\}.$$

For any $0 < \varepsilon < 1$, choose a polynomial p_0, using Mergelyan's theorem, such that $\|f - p_0\| < \varepsilon$. Assuming $p_0 \in \mathcal{P}_n$ with $n \geq k$, set

$$d_i := p_0^{(i)}(0) - c_i p_0(0) - \sum_{j=k-l+2}^{k} b_{i,j} p_0^{(j)}(0), \quad i = 1, \ldots, k + 1 - l$$

and $d := \max_{1 \leq i \leq k+1-l} \{|d_i|\}$. Also define

$$\varepsilon_1 := \begin{cases} \varepsilon & \text{if } d \leq \varepsilon \\ \varepsilon / d & \text{otherwise.} \end{cases}$$

Let m be a fixed positive integer with $2m + 2 > k$. From Lemma 3.3, we know that there exists a polynomial q_0 such that

$$\|z - z^{2m+1} q_0(z)^{2m}\| < \varepsilon_1.$$

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

Then we have

$$[z - z^{2m+1}q_0(z)^{2m}]^i = z^i - z^{2m+2}p_i(z) =: Q_i(z), \quad i = 1, \ldots, k+1-l,$$

where the p_i's are polynomials. Also note that $\varepsilon_1 < 1$ so that

$$\|Q_i\| < \varepsilon_1, \quad i = 1, \ldots, k+1-l. \tag{3.3}$$

Consider

$$r(z) := p_0(z) - \sum_{j=1}^{k+1-l} d_j Q_j(z)/j!.$$

Then

$$\begin{aligned} r(0) &= p_0(0), \\ r^{(i)}(0) &= p_0^{(i)}(0) - d_i, \quad i = 1, \ldots, k+1-l, \\ r^{(i)}(0) &= p_0^{(i)}(0), \quad i = k+2-l, \ldots, k. \end{aligned}$$

Thus we have for $i = 1, \ldots, k+1-l$:

$$\begin{aligned} r^{(i)}(0) &= p_0^{(i)}(0) - p_0^{(i)}(0) + c_i p_0(0) + \sum_{j=k-l+2}^{k} b_{i,j} p_0^{(j)}(0) \\ &= c_i p_0(0) + \sum_{j=k-l+2}^{k} b_{i,j} p_0^{(j)}(0) \\ &= c_i r(0) + \sum_{j=k-l+2}^{k} b_{i,j} r^{(j)}(0), \end{aligned}$$

and so $r(z) \in B_t(A)$ for some positive integer t. From (3.3) and the definition of ε_1, we have $\|d_i Q_i\| \leq \varepsilon, i = 1, \ldots, k+1-l$, and so

$$\begin{aligned} \|f - r\| &= \left\| f - p_0 + \sum_{j=1}^{k+1-l} d_j Q_j / j! \right\| \\ &\leq \|f - p_0\| + \sum_{j=1}^{k+1-l} \|d_j Q_j / j!\| \\ &\leq \varepsilon + (k+1-l)\varepsilon. \end{aligned}$$

Thus $\lim_{n\to\infty} E_n(A, f) = 0$.

Finally, to show that (i) implies (iii), assume that (iii) is not true. Then $p(0) = 0$ for all $p \in B_k(A)$ and hence $p(0) = 0$ for all $p \in B_n(A)$, $n = 0, 1, 2, \ldots$. Thus for $f \equiv 1$, $E_n(A, f)$ does not tend to zero, which contradicts (i). ∎

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

4. **Distribution of zeros of $p^*_{n,A}$.** To state our results, we need to introduce some terminology from potential theory. We denote the logarithmic capacity (transfinite diameter) of the set E by cap(E) (*cf.* [T]). If cap(E) > 0, let μ_E be the unique positive unit measure with supp(μ_E) $\subset E$ that minimizes the energy integral

$$I[\mu] := \iint_E \log|z-t|^{-1}\,d\mu(t)\,d\mu(z)$$

over all unit measures supported on E. The extremal measure μ_E is called the *equilibrium distribution* for E and

$$U(\mu_E;z) := \int \log|z-t|^{-1}\,d\mu_E(t)$$

is the *conductor potential* of E. The minimum energy $I[\mu_E]$ is related to the capacity of E via

$$\mathrm{cap}(E) = \exp(-I[\mu_E]).$$

The Green function $g_K(z,\infty)$ with pole at infinity for K, the unbounded component of $\overline{\mathbf{C}}\setminus E$, is given by (*cf.* [T, p. 82])

$$g_K(z,\infty) = -\{\log[\mathrm{cap}(E)] + U(\mu_E;z)\}, \tag{4.1}$$

and is positive and harmonic in $K\setminus\{\infty\}$. We define for each $\sigma > 1$, the closed region

$$E_\sigma := E\bigcup\{z\in K : 0 < g_K(z,\infty) \le \log\sigma\},$$

which has boundary

$$\Gamma_\sigma := \{z\in K : g_K(z,\infty) = \log\sigma\}.$$

Note that if we define $K_\sigma := \overline{\mathbf{C}}\setminus E_\sigma$, then

$$g_{K_\sigma}(z,\infty) = g_K(z,\infty) - \log\sigma$$

and from (4.1) it is easy to see that

$$\mathrm{cap}(E_\sigma) = \mathrm{cap}(E)\sigma. \tag{4.2}$$

In this section, we will examine the geometric rate of convergence of $E_n(A,f)$ and the limiting distribution of the zeros of the polynomials $p^*_{n,A}$. For a polynomial p_n of precise degree n, we denote by $\nu_n = \nu(p_n)$ the discrete unit measure (defined on the Borel sets in $\mathbf{C}$) having mass $1/n$ at each zero of p_n, with the obvious modification in this definition for the case when p_n has multiple zeros. We say that ν_n converges in the weak-star topology to the measure μ as $n\longrightarrow\infty$ and write $\nu_n \xrightarrow{*} \mu$ if

$$\lim_{n\to\infty}\int\phi\,d\nu_n = \int\phi\,d\mu,$$

for every continuous function ϕ on $\mathbf{C}$ having compact support.

Before we state our main results, we need the following lemma of Blatt, Saff and Simkani.

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

LEMMA 4.1 ([BSS]). *Let E be a compact set with* $\text{cap}(E) > 0$ *and set* $E^* := \text{supp}(\mu_E)$. *Let Λ be an infinite subset of positive integers and $\{p_n\}_{n\in\Lambda}$ be a sequence of monic polynomials of respective degrees precisely n. Then $\nu_n = \nu(p_n)$ converges in the weak-star topology to μ_E as $n \longrightarrow \infty$, $n \in \Lambda$, if conditions* (i) *and* (ii) *below are satisfied.*

(*i*) $\limsup_{n\to\infty} \|p_n\|_{E^*}^{1/n} \le \text{cap}(E)$, $n \in \Lambda$;

(*ii*) $\lim_{n\to\infty} \nu_n(B) = 0$, $n \in \Lambda$, *for every closed set B contained in the union of the bounded (open) components of* $\overline{\mathbf{C}} \setminus E^*$.

We first consider the case when $0 \in E^o$.

THEOREM 4.2. *Suppose $\overline{\mathbf{C}} \setminus E$ is connected and regular,* $0 \in E^o$ *and* $\text{cap}(E) > 0$. *Assume $f \in \mathcal{A}(E)$, but f is not analytic on E and f does not vanish identically on any component of E^o. If $A\underline{f} = \underline{0}$, then*

(*i*) $\limsup_{n\to\infty} E_n^{1/n}(A,f) = 1$;

(*ii*) $\nu(p^*_{n,A}) \xrightarrow{*} \mu_E$ *as $n \longrightarrow \infty$, $n \in \Lambda$, where $\Lambda \subseteq \mathbf{N}$ is a sequence that depends on f.*

PROOF. Clearly $E_n(A,f) \le \|f\|$ and so

$$\limsup_{n\to\infty} E_n^{1/n}(A,f) \le 1.$$

Since f is not analytic on E, we also have $\limsup_{n\to\infty} E_n^{1/n}(f) = 1$ (*cf.* [W, §4.7]). Hence

$$1 = \limsup_{n\to\infty} E_n^{1/n}(f) \le \limsup_{n\to\infty} E_n^{1/n}(A,f) \le 1,$$

which yields (i).

Write $p^*_{n,A}(z) = a^*_{n,A}z^n + \cdots$ and for $n > k$ choose

$$T_{n,A}(z) \in B_n(A), \quad T_{n,A}(z) = z^n + \cdots,$$

such that

$$\|T_{n,A}\| = \inf\{\|p\| : p \in B_n(A) \text{ and } p = z^n + \cdots\}.$$

Then, for $n > k$,

$$E_{n-1}(A,f) \le \|f - p^*_{n,A} + a^*_{n,A}T_{n,A}\| \le E_n(A,f) + |a^*_{n,A}|\|T_{n,A}\|. \tag{4.3}$$

Let $T_n(z) = z^n + \cdots$ denote the (unconstrained) Chebyshev polynomials for E; that is

$$\|T_n\| = \inf\{\|p\| : p \in \mathcal{P}_n \text{ and } p(z) = z^n + \cdots\}.$$

It is well-known (*cf.* [T]) that $\lim_{n\to\infty} \|T_n\|^{1/n} = \text{cap}(E)$. Note that

$$\|T_n\| \le \|T_{n,A}\| \le \|z^{k+1}T_{n-k-1}\| \le \|z^{k+1}\|\|T_{n-k-1}\|,$$

so we have

$$\lim_{n\to\infty} \|T_{n,A}\|^{1/n} = \text{cap}(E). \tag{4.4}$$

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

From (4.3) it follows that

$$E_{n-1}(A,f) - E_n(A,f) \le |a^*_{n,A}| \|T_{n,A}\|. \tag{4.5}$$

Next observe from Theorem 3.2 that $E_n(A,f) \longrightarrow 0$ as $n \longrightarrow \infty$, and hence from (i) it follows that

$$\limsup_{n\to\infty} [E_{n-1}(A,f) - E_n(A,f)]^{1/n} = 1. \tag{4.6}$$

From (4.4), (4.5) and (4.6), it is easy to see that there is a subsequence $\Lambda \subseteq \mathbf{N}$ such that

$$\liminf_{n\to\infty} |a^*_{n,A}|^{1/n} \ge 1/\operatorname{cap}(E), \quad n \in \Lambda.$$

Since the $p^*_{n,A}$ are uniformly bounded on E, the monic polynomials $p_n(z) := p^*_{n,A}(z)/a^*_{n,A}$, $n \in \Lambda$, satisfy condition (i) of Lemma 4.1. Finally the assumption that f does not identically vanish in any component of E^o together with Hurwitz's theorem imply that condition (ii) of Lemma 4.1 also holds for the sequence $\{p_n\}_{n\in\Lambda}$. Hence $\nu(p^*_{n,A}) = \nu(p_n) \xrightarrow{*} \mu_E$, as $n \longrightarrow \infty$, $n \in \Lambda$, by Lemma 4.1. ∎

REMARK. As can be seen from the proof, conclusion (ii) of Theorem 4.2 holds for any sequence $\Lambda \subseteq \mathbf{N}$ such that

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} [E_{n-1}(A,f) - E_n(A,f)]^{1/n} = 1.$$

THEOREM 4.3. *Assume E is compact, $0 \in \partial E$, $\operatorname{cap}(E) > 0$, and $K = \overline{\mathbf{C}} \setminus E$ is connected and regular. Suppose f is analytic on E and f does not vanish identically on any component of E^o. Furthermore, assume $B_k(A) \setminus \mathcal{P}_{k,0} \ne \emptyset$ and $A\underline{f} \ne \underline{0}$. Then*
(i) $\limsup_{n\to\infty} E_n^{1/n}(A,f) = 1$;
(ii) $\nu(p^*_{n,A}) \xrightarrow{*} \mu_E$, *as $n \longrightarrow \infty$, $n \in \Lambda$, where $\Lambda \subseteq \mathbf{N}$ is a sequence that depends on f.*

PROOF. From Lemma 2.1 we know that

$$\alpha_{n,A}(p^*_n) \le E_n(f) + E_n(A,f) \le 2E_n(A,f). \tag{4.7}$$

Together with (2.6), (2.10) and (2.11), for n large we have (with the same i_0 as in (2.9))

$$2E_n(A,f) \ge \alpha_{n,A}(p^*_n) \ge \frac{1}{2}\left|\sum_{j=0}^{k} a_{i_0,j} f^{(j)}(0)\right| \beta^{-1}_{n,A}. \tag{4.8}$$

Thus (2.8) and (4.8) imply that

$$\limsup_{n\to\infty} E_n^{1/n}(A,f) \ge \limsup_{n\to\infty} 1/\beta^{1/n}_{n,A} \ge 1.$$

Since $1 \ge \limsup_{n\to\infty} E_n^{1/n}(A,f)$, we see that (i) holds.
The proof of (ii) is now the same as that of (ii) in Theorem 4.2. ∎

We next consider the case when 0 is outside E.

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

THEOREM 4.4. *Suppose E is compact, $K = \overline{\mathbf{C}} \setminus E$ is connected and regular, $0 \notin E$ and $g_K(0, \infty) = \log \sigma (\sigma > 1)$. Assume $f(z)$ is analytic on E_σ and does not vanish identically on any component of E_σ^o. If $A\underline{f} \neq \underline{0}$, then*

(i) $\limsup_{n\to\infty} E_n^{1/n}(A, f) = 1/\sigma$;

(ii) $\nu(p_{n,k}^*) \xrightarrow{*} \mu_{E_\sigma}$, as $n \to \infty$, $n \in \Lambda$, where $\Lambda \subseteq \mathbf{N}$ is a sequence that depends on f.

REMARK. If $f \in \mathcal{A}(E_\sigma)$, but f is not analytic on E_σ, then (i) holds because $\limsup_{n\to\infty} E_n^{1/n}(f) = 1/\sigma$. If f is analytic on E_σ, then $\limsup_{n\to\infty} E_n^{1/n}(f) < 1/\sigma$; however Theorem 4.4 asserts that (i) holds provided $A\underline{f} \neq \underline{0}$.

PROOF OF THEOREM 4.4. We know that (*cf.* [W, §4.7]) since f is analytic on E_σ

$$\limsup_{n\to\infty} E_n^{1/n}(f) < 1/\sigma \tag{4.9}$$

and $\{p_n^*\}_{n=0}^\infty$ converges uniformly to f on some open set containing E_σ. Consequently,

$$\lim_{n\to\infty} p_n^{*(j)}(0) = f^{(j)}(0), \quad j = 0, \ldots, k. \tag{4.10}$$

Let $\delta \in (\sigma, \infty)$. As in the proof of Theorem 2.5, for $p \in \mathcal{P}_n$, we have

$$|p^{(j)}(0)| \leq \frac{j!}{2\pi} \delta^n \frac{\operatorname{length}(\Gamma_\delta)}{\operatorname{dist}(0, \Gamma_\delta)^{j+1}}.$$

According to the definition of $\beta_{n,A}$ in (2.7) we get $\limsup_{n\to\infty} \beta_{n,A}^{1/n} \leq \delta$ and letting $\delta \to \sigma^+$ yields

$$\limsup_{n\to\infty} \beta_{n,A}^{1/n} \leq \sigma. \tag{4.11}$$

From (4.7), (2.10) and (4.10), we again deduce (4.8). Combining this with (4.11) we obtain

$$\limsup_{n\to\infty} E_n^{1/n}(A, f) \geq 1/\sigma. \tag{4.12}$$

Note that $f(z)/z^{k+1}$ has a pole at $z = 0$ (since $A\underline{f} \neq \underline{0}$) and so

$$\limsup_{n\to\infty} E_n^{1/n}\big(f(z)/z^{k+1}\big) = 1/\sigma.$$

Hence for $\varepsilon > 0$ there is a polynomial $q_{n-k-1} \in \mathcal{P}_{n-k-1}$ such that, for n large,

$$\|f(z)/z^{k+1} - q_{n-k-1}(z)\| \leq [(1 + \varepsilon)/\sigma]^{n-k-1},$$

and so

$$\|f(z) - z^{k+1} q_{n-k-1}(z)\| \leq \|z^{k+1}\| [(1 + \varepsilon)/\sigma]^{n-k-1}.$$

Note that $z^{k+1} q_{n-k-1}(z) \in B_n(A)$ so we have

$$\limsup_{n\to\infty} E_n^{1/n}(A, f) \leq \limsup_{n\to\infty} \|f(z) - z^{k+1} q_{n-k-1}(z)\|^{1/n} \leq (1 + \varepsilon)/\sigma.$$

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

As $\varepsilon > 0$ is arbitrary, we get $\limsup_{n\to\infty} E_n^{1/n}(A,f) \leq 1/\sigma$. Together with (4.12), this yields (i).

Now we prove (ii). It suffices to check that the conditions in Lemma 4.1 are satisfied for $p_n(z) := p^*_{n,A}(z)/a^*_{n,A}$ and E replaced by E_σ. Since $\{p^*_{n,A}\}_{n=0}^\infty$ converges uniformly to f on every closed set $D \subset E^o_\sigma$, the condition (ii) in Lemma 4.1 is satisfied for the sequence $\{p_n\}_{n\in\mathbf{N}}$.

From (4.3) we have

$$E_{n-1}(A,f) - E_n(A,f) \leq |a^*_{n,A}|\,||T_{n,A}||.$$

Also from (i) and the fact that $\lim_{n\to\infty} ||T_{n,A}||^{1/n} = \operatorname{cap}(E)$, we have for a suitable subsequence Λ

$$\liminf_{n\to\infty} |a^*_{n,A}|^{1/n} \geq \frac{1}{\operatorname{cap}(E)\sigma}, \quad n \in \Lambda \subseteq \mathbf{N}. \tag{4.13}$$

Note that by (i) for any $\rho < \sigma$ we have (*cf.* [W, §4.7])

$$||p^*_{n,A} - f||_{E_\rho} \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

where $||\cdot||_{E_\rho}$ denotes the uniform norm on E_ρ. Thus the sequence $\{||p^*_{n,A}||_{E_\rho}\}_{n=0}^\infty$ is bounded and using Lemma 2.4 we have

$$\limsup_{n\to\infty} ||p^*_{n,A}||_{E_\sigma}^{1/n} \leq \sigma/\rho.$$

Letting $\rho \longrightarrow \sigma^-$ we obtain

$$\limsup_{n\to\infty} ||p^*_{n,A}||_{E_\sigma}^{1/n} \leq 1. \tag{4.14}$$

For the monic polynomials p_n, by (4.13) and (4.14) we therefore have

$$\limsup_{n\to\infty} ||p_n||_{E_\sigma}^{1/n} \leq \operatorname{cap}(E)\sigma = \operatorname{cap}(E_\sigma), \quad n \in \Lambda.$$

This yields condition (i) in Lemma 4.1 and completes the proof. ∎

5. **Comparison of rates of convergence.** In this section we will prove that when $0 \notin E^o$ there are " relatively few " functions $f \in \mathcal{A}(E)$ (in the sense of category) with rate of the convergence of $E_n(f)$ faster than that of $E_n(A,f)$.

For the case when $0 \in E^o$ the following result is straightforward to establish (*cf.* the proof of Theorem 3.2).

THEOREM 5.1. *Let E be a compact set, $\overline{\mathbf{C}}\setminus E$ be connected, and $0 \in E^o$. If $f \in \mathcal{A}(E)$ and $A\underline{f} = \underline{0}$, then*

$$E_n(A,f) = O\big(E_n(f)\big).$$

In the proof of the main result of this section we follow an argument of Saff and Totik which utilizes the following.

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

LEMMA 5.2 ([ST, PROOF OF THEOREM 1]). *For any integer $n_0 > k$, there is an $f \in \mathcal{A}(E)$ such that $||f|| = 1$ and $p^*_{n_0}(f) \equiv 0$. In particular, $E_{n_0}(f) = E_{n_0}(A, f) = 1$.*

We can now state our main result.

THEOREM 5.3. *Let E be compact with $K = \overline{\mathbf{C}} \setminus E$ connected and $0 \notin E^o$. If $B_k(A) \setminus \mathcal{P}_{k,0} \neq \emptyset$, then the set S of functions $f \in \mathcal{A}(E)$ for which*

$$\limsup_{n \to \infty} \frac{E_n(f)}{E_n(A, f)} < 1 \tag{5.1}$$

is of the first category in the complete metric space $\mathcal{A}(E)$.

So that (5.1) is meaningful for all $f \in \mathcal{A}(E)$ we set $E_n(f)/E_n(A, f) = 0$ whenever $f \in \mathcal{P}_n$.

PROOF. Let

$$S_{m,l} := \left\{ f \in \mathcal{A}(E) : \frac{E_n(f)}{E_n(A, f)} \leq 1 - 1/m \text{ for all } n \geq l \right\}.$$

Then

$$S = \bigcup_{m=1}^{\infty} \bigcup_{l=1}^{\infty} S_{m,l}.$$

Assume to the contrary that S is not of the first category. Then for some m and l the set $S_{m,l}$ is not nowhere dense in $\mathcal{A}(E)$. We claim that $S_{m,l}$ is closed. In fact, if $\{f_\nu\}_{\nu=1}^{\infty} \subseteq S_{m,l}$ and f_ν converges to f uniformly on E, then $E_n(f_\nu) \longrightarrow E_n(f)$ and $E_n(A, f_\nu) \longrightarrow E_n(A, f)$ as $\nu \longrightarrow \infty$ for fixed $n \geq l$, and so $E_n(f)/E_n(A, f) \leq 1 - 1/m$; that is, $f \in S_{m,l}$.

Since $S_{m,l}$ is closed and not nowhere dense in $\mathcal{A}(E)$, there is an $f_0 \in \mathcal{A}(E)$ and a $\delta_0 > 0$ such that the δ_0-neighborhood of f_0 in $\mathcal{A}(E)$ is contained in $S_{m,l}$. Choose a polynomial $p_0 \in B_{\deg p_0}(A)$ with $||f_0 - p_0|| < \delta_0/2$ (this can be done by Theorems 3.1 and 3.4) and set $n_0 := \max\{l, \deg p_0\}$. If $f (||f|| \neq 0)$ is any function in $\mathcal{A}(E)$, then the function

$$f^*(z) := p_0(z) + \frac{1}{2}\delta_0 ||f||^{-1} f(z)$$

belongs to the δ_0-neighborhood of f_0. Hence

$$\frac{E_{n_0}(f^*)}{E_{n_0}(A, f^*)} \leq 1 - 1/m.$$

But note that since $p_0 \in B_{\deg p_0}(A)$ we have

$$E_{n_0}(f^*) = \delta_0 ||f||^{-1} E_n(f)/2,$$

and

$$E_{n_0}(A, f^*) = \delta_0 ||f||^{-1} E_n(A, f)/2.$$

Thus we can conclude that for every function $f \in \mathcal{A}(E) \setminus B_{n_0}(A)$,

$$\frac{E_{n_0}(f)}{E_{n_0}(A, f)} \leq 1 - 1/m,$$

which is impossible by Lemma 5.2. ∎

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press

REFERENCES

[BSS] H.-P. Blatt, E. B. Saff and M. Šimkani, *Jentzsch -Szegö type theorems for zeros of best approximants*, J. London Math. Soc . (2) **38**(1980), 307–316.

[CHS] J. Clunie, M. Hasson and E. B. Saff, *Approximation by polynomials that omit a power of z*, Approx. Theory & its Appl. **3**(1987), 63–73.

[N] A. A. Nersesyan, *Uniform approximation with simultaneous interpolation by analytic functions*, Izv. Akad. Nauk Armyan. SSR. Mat. **15**(1980), 249–257 (in Russian). See also Soviet J. Contemporary Math. Anal. (4) **15**(1980), 1–9.

[ST] E. B. Saff and V. Totik, *Behavior of polynomials of best uniform approximation*, Trans. Amer. Math. Soc. **316**(1988), 567–593.

[T] M. Tsuji, *Potential Theory in Modern Function Theory*, 2nd ed., Chelsea Publishing Company, New York, NY, (1958).

[W] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, 5th ed., Amer. Math. Soc. Colloquium Publ. **20**(1969), Providence, Rhode Island.

Department of Mathematics
University of California
Riverside, California 92521
USA

Institute for Constructive Mathematics
Department of Mathematics
University of South Florida
Tampa, Florida 33620
USA

https://doi.org/10.4153/CJM-1992-077-5 Published online by Cambridge University Press