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BIFURCATION OF POSITIVE SOLUTIONS FOR A NEUMANN BOUNDARY VALUE PROBLEM

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Abstract

Analytical, approximate and numerical methods are used to study the Neumann boundary value problem

$$\begin{aligned} &-u_{xx} + q^2 u = u^2(1+\sin x), \qquad \text{for } 0 < x < \pi, \\ &\text{subject to} \quad u_x(0) = 0, \quad u_x(\pi) = 0, \end{aligned} \tag{1}$$

for $q^2 \in (0, \infty)$. Asymptotic approximations to (1) are found for q^2 small and q^2 large. In the case where q^2 is large $u(x) \approx 3q\delta(x - \pi/2)$. When $q^2 = 0$ we show that the only possible solution is $u \equiv 0$. However, there exist non-zero solutions for $q^2 > 0$ as well as the trivial solution $u \equiv 0$. To $O(q^4)$ in the q^2 small case $u(x) = q^2\pi(\pi+2)^{-1}$, so that bifurcation occurs about the trivial solution branch $u \equiv 0$ at the first eigenvalue $\lambda_0 = 0$ and in the direction of the first eigenfunction $\xi_0 =$ constant.

We obtain a bifurcation diagram for (1), which confirms that there exists a positive solution for $q^2 \in (0, 10)$. Symmetry-breaking bifurcations and blow-up behaviour occur on certain regions of the diagram. We show that all non-trival solutions to the problem must be positive.

The formal outer solution $u = q^2\hat{u}$ appears to satisfy $\hat{u} = \hat{u}^2(1+\sin x)$, so that $\hat{u} \equiv 0$ and $\hat{u} = (1+\sin x)^{-1}$ are possible limit solutions. However, in the non-trivial case $\hat{u}_x(0) = -1$ and $\hat{u}_x(\pi) = 1$; this means that $\hat{u}$ does not satisfy the boundary conditions required for a solution of (1). This behaviour usually implies that for q^2 large a boundary layer exists near $x = 0$ (and one near $x = \pi$), which corrects the slope. However, we find no evidence for such a solution structure, and only find perturbations in the direction of a delta function about $u \equiv 0$. We show using the monotone convergence theorem for quadratic forms that the inverse of the operator on the left-hand side of (1) is strongly convergent as $q^2 \to \infty$. We show that strong convergence of the operator is sufficient to stop outer-layer behaviour occurring.

1. Introduction

In Mays and Norbury [7] we developed an existence theory for a general class of operator equations. Using this theory we proved in Mays and Norbury [6] that the

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Neumann boundary value problem,

$$\begin{aligned} &Lu \equiv -u_{xx} + q^2 u = u^2(1+\sin x), \qquad \text{for } 0 < x < \pi, \\ &\text{subject to} \qquad u_x(0) = 0, \quad u_x(\pi) = 0, \end{aligned} \tag{2}$$

has a non-zero solution u^* for all $q^2 \in (0, 1.226)$. We studied this problem for two reasons. First, it behaves in a similar way to a fluid dynamics equation, which we discuss more fully in the next paragraph. Secondly, it acts as an evaluation of our existence method, giving an indication of how conservative the method is.

A new pseudo-differential equation was introduced by Benjamin [1], describing the uni-directional propagation of nonlinear dispersive waves. The equation is obtained as an approximate model for long waves in a two-fluid system. The waves occur in a two-fluid system where a thin layer of incompressible fluid with density ρ_1 bounded above by a rigid horizontal plane, lies on a very deep incompressible fluid with density $\rho_2 > \rho_1$. Benjamin [1] is concerned with the case where the interfacial surface tension T satisfies $T \gg g(\rho_2 - \rho_1)h^2$, where h is the undisturbed thickness of the upper layer. Take, for example, the case of benzene on water. Then $\rho_2 - \rho_1 = 0.3\,\mathrm{g\,cm^{-3}}$ and $T = 35\,\mathrm{dyn\,cm^{-1}}$ (Kaye and Laby [5, p. 42]). It follows that for $h = 2\,\mathrm{mm}$, say, the model considered by Benjamin should be relevant. It was further shown by Benjamin [2] that there exist solitary wave solutions.

$$\text{Let } \alpha = \frac{\rho_2}{2\rho_1}; \quad \text{let } \beta = \frac{T}{2g(\rho_2-\rho_1)h^2}; \quad \text{let } \gamma = \frac{\alpha}{2(\beta - \beta c)^{1/2}},$$

where c is the phase velocity and is related to k, the wave number, by $c = 1 - \alpha|k| + \beta k^2$. Benjamin [2] begins by introducing a tidier version of the equation in question, given in Benjamin [1], and then takes Fourier transforms. No generality is lost by this simplification. The equation Benjamin then considers is, for $*$ the convolution operator,

$$u(x) = W^{-1}(x)\cdot(u*u)(x) \equiv \mathscr{A}u(x), \qquad \text{say}, \tag{3}$$

where

$$W(x) = 1 - 2\gamma|x| + x^2, \text{ for } x \in \Re,$$

and the physical parameter $\gamma \in (0,1)$. Because (3) cannot be readily recast as a compact-operator equation, Benjamin proceeds by considering the "regularized" version of (3):

$$Wu - \epsilon^2 u_{xx} = (u*u). \tag{4}$$

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Due to the inherent complexity of (4) considered by Benjamin, it is difficult to obtain a clear understanding of its behaviour. For this reason we study in (2) a problem which is more tractable than Benjamin's, but which has similar properties. The right-hand side in both cases is a symmetric, positive, non-autonomous, quadratic term. In Benjamin's paper cnoidal waves occur on the real interval $(-\infty, \infty)$. For this to be compatible with our problem on a finite interval, we require that Neumann boundary conditions be applied at each end of the interval. For solitary waves we require physically that the tails converge to a limit, which again means that Neumann conditions must be applied.

We tackle the problem using both analytical and numerical methods. Although the problem appears at a first glance to be extremely simple, the analytical results given in Section 2 show that in fact there is some unexpected behaviour. This underlines the importance of obtaining an understanding of a problem using analytical methods: because of the somewhat unusual behaviour found within this problem, conventional numerical methods may produce unreliable results. We show in § 2.1 that when q^2 is small the Neumann boundary conditions become important and prohibit the existence of any solution other than the trivial zero solution in the case $q^2 = 0$, and we also show that all solutions must be positive. We obtain in § 2.2 an approximation in the q^2-small case and show that the solutions converge in every sense to the zero function as $q^2 \to 0$. The convergence of solutions as $q^2 \to \infty$ is more complex. By making the substitution $\hat{u} = q^2 u$, the problem becomes

$$-q^{-2}\hat{u}_{xx} + \hat{u} = \hat{u}^2(1 + \sin x),$$

subject to $\hat{u}_x(0) = \hat{u}_x(\pi) = 0$; and the obvious non-trivial outer solution is given by $\hat{u} = (1 + \sin x)^{-1}$. However, it is apparent that this solution does not satisfy the boundary conditions at either $x = 0$ or $x = \pi$. This behaviour usually implies that for q^2 large a boundary layer exists near $x = 0$ (and near $x = \pi$) which corrects the slope. We have found no numerical evidence for this outer-layer behaviour.

The symmetric solution is shown in § 2.3 by means of a regular approximation to be converging to a delta function centred on the $\pi/2$-axis as $q^2 \to \infty$ — what might be described as inner-layer behaviour. Using a novel proof based on the monotone convergence theorem for quadratic forms, we are able to show that solutions can only converge to either the zero solution or to a non-$L_2(0, \pi)$ function. Because the delta function is not in L_2, convergence of the solutions to it is not prohibited; and we can observe numerically that by $q^2 = 100$ the solution is accurately approximated by our regular approximation, which itself tends to a delta function. Because of the strongly exponential behaviour of the solutions we rely in § 3.1 on the crude but robust method of shooting for a solution. Our proof for the absence of outer-layer behaviour is given in § 2.4. The key condition is that the Green's operator is strongly convergent as $q^2 \to \infty$. This is a sufficient condition to stop boundary-layer behaviour occurring.

Once strong convergence has been established, the proof requires little more than an application of the triangle inequality and argument by contradiction. To establish strong convergence we rely on the theorem known as the monotone convergence theorem for quadratic forms. This theorem seems not to have been used in the literature to account for the absence in a PDE problem of boundary-layer behaviour. In particular, our proof of strong convergence using this theorem could be adapted to a wide class of other ODE or PDE problems.

From our existence results in Mays and Norbury [6] we would expect that a solution exists in the range $q^2 \in (0, 1.226)$. This is indeed confirmed in § 3.2, and in fact the solution is both unique and symmetric in this range. However, we find that outside this range symmetry-breaking bifurcations occur, and we obtain a bifurcation diagram in the range $q^2 \in [0, 10]$. A plot of the symmetric solutions of (2) for certain values of q^2 is given in Figure 1. To check our bifurcation results we also study in § 3.3 the linearised problem, which we solve using a finite centralised second-difference method. Finally, we discuss the blow-up behaviour of the solutions in § 3.4.

2. Analytic results

2.1. Positivity of solutions Denote by B the inverse of the operator L. Then we may write B as a Green's operator

$$Bu = \int_0^\pi k(x, y)u(y)\,dy,$$

where the Green's function k is given by

$$k(x, y) = \begin{cases} K_1(y)\cosh qx, & x < y, \\ K_2(y)\cosh q(\pi - x), & x > y, \end{cases}$$

with

$$K_1(x) = \frac{e^{qx}(e^{2\pi q} - 1)}{e^{2\pi q} + e^{2qx}}$$

and

$$K_2(x) = \frac{e^{q(\pi+x)}(e^{2qx} + 1)(e^{2\pi q} - 1)}{e^{4\pi q} + 2e^{2q(\pi+x)} + e^{4qx}}.$$

Therefore, $k(x, y) > 0$ for all $x, y \in [0, \pi]$ and $q^2 > 0$. Let u^* be a non-zero solution of (2); then $u^* = BDu^*$. Now, by definition, $Du \equiv u^2(1 + \sin x) \geq 0$, for all $x \in [0, \pi]$. It follows that

$$u^*(x) = BDu^* = \int_0^\pi k(x, y)Du^*(y)\,dy \geq 0,$$

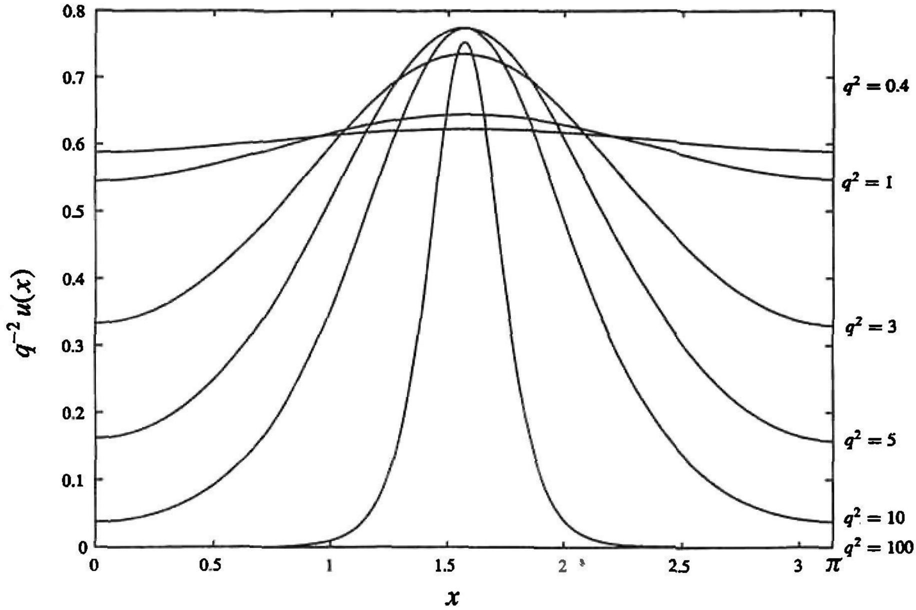

FIGURE 1. Symmetric non-trivial solutions of (2) scaled by q^{-2}.

for all $x \in [0, \pi]$ and $q^2 > 0$. Thus all non-zero solutions to (2) are positive.

When $q^2 = 0$ the solution to (2) must satisfy

$$-\int_0^\pi u_{xx}\,dx = \int_0^\pi u^2(1+\sin x)\,dx.$$

But from the Neumann boundary conditions $\int_0^\pi u_{xx}dx = 0$, and as both u^2 and $(1+\sin x)$ are positive, the only possible solution when $q^2 = 0$ is $u \equiv 0$. However, there exist non-zero solutions for $q^2 > 0$ as well as the trivial solution $u \equiv 0$. The linearised problem about $u \equiv 0$ has a null space at $q^2 = 0$, and bifurcation occurs from the trivial solution.

2.2. Behaviour for q^2 small We now study the behaviour for q^2 small. Write $u(x) = q^2\overline{u} + w(x)q^4$, where $\overline{u}$ and $w(x)$, a constant and a function, respectively, are to be determined. Substituting this last expression into (2) gives

$$\begin{aligned} &-q^4 w_{xx} + q^4\overline{u} + q^6 w = (q^2\overline{u} + wq^4)^2(1+\sin x), \quad 0 < x < \pi, \\ &\text{subject to} \qquad w_x(0) = w_x(\pi) = 0. \end{aligned} \tag{5}$$

Dividing (5) by q^4 and then neglecting the 2nd- and 4th-order terms we obtain

$$-w_{xx} + \overline{u} = \overline{u}^2(1+\sin x), \qquad \text{subject to } w_x(0) = w_x(\pi) = 0. \tag{6}$$

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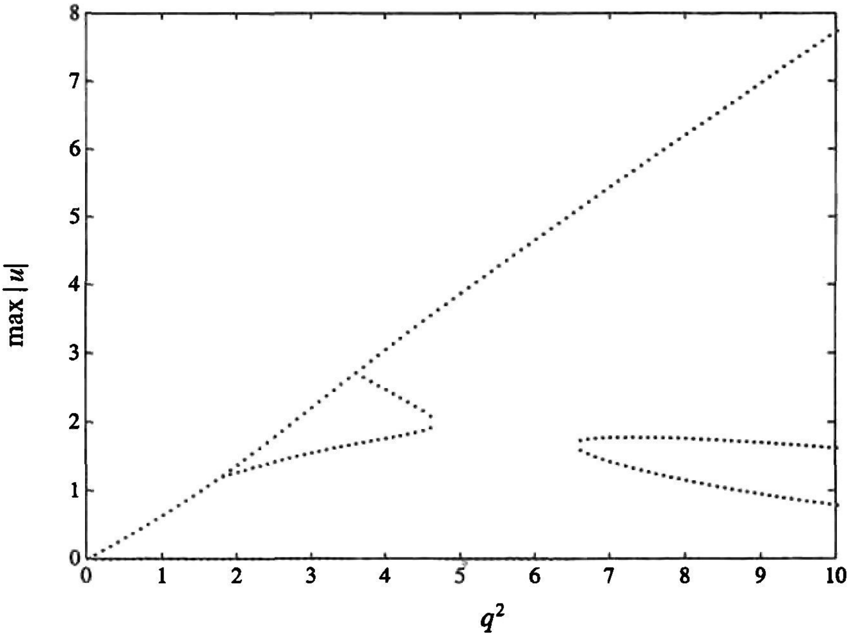

FIGURE 2. Bifurcation diagram for $\max_{x\in[0,\pi]} |u^*(x)|$ against $q^2 \in [0, 10]$.

By integrating (6) once and using the boundary conditions, we find $\overline{u} = \pi/(\pi+2)$ is a necessary choice of constant. Also, by integrating (6) twice and introducing the boundary conditions, we obtain from (6) that $w(x) = \overline{w}(x) + O(q^6)$, where $\overline{w}(x) = 2^{-1}x^2(\overline{u}-\overline{u}^2) - x\overline{u}^2 + \overline{u}^2\sin x + D$, and D is a constant of integration. Thus $u(x) = q^2\overline{u} + q^4\overline{w} + O(q^6)$.

Now integrating (2) with respect to x, and introducing the boundary conditions, we have that, for $\overline{u} = \pi/(\pi+2)$,

$$\int_0^\pi q^2 u = \int_0^\pi u^2(1+\sin x)\,dx. \tag{7}$$

By making the substitution $u(x) = q^2\overline{u} + q^4\overline{w} + O(q^6)$ in (7), we have that

$$\int_0^\pi \{q^2\overline{u} + q^4\overline{w} - (q^2\overline{u}^2 + 2q^4\overline{u}\,\overline{w})(1+\sin x)\}\,dx + O(q^6) = 0. \tag{8}$$

Comparing the q^4 coefficients we find that for (8) to hold we require

$$D = \frac{\pi}{6}\left(\frac{\pi^3 - 8\pi^2 - 12\pi + 72}{\pi^3 + 6\pi^2 + 12\pi + 8}\right) \approx -0.053.$$

For q^2 small, we may thus approximate $u(x)$ to $O(q^6)$ by

$$u(x) = q^2\overline{u} + q^4\left[\frac{x^2}{2}(\overline{u}-\overline{u}^2) - x\overline{u}^2 + \overline{u}^2\sin x + \frac{\pi}{6}\left(\frac{\pi^3-8\pi^2-12\pi+72}{\pi^3+6\pi^2+12\pi+8}\right)\right]. \tag{9}$$

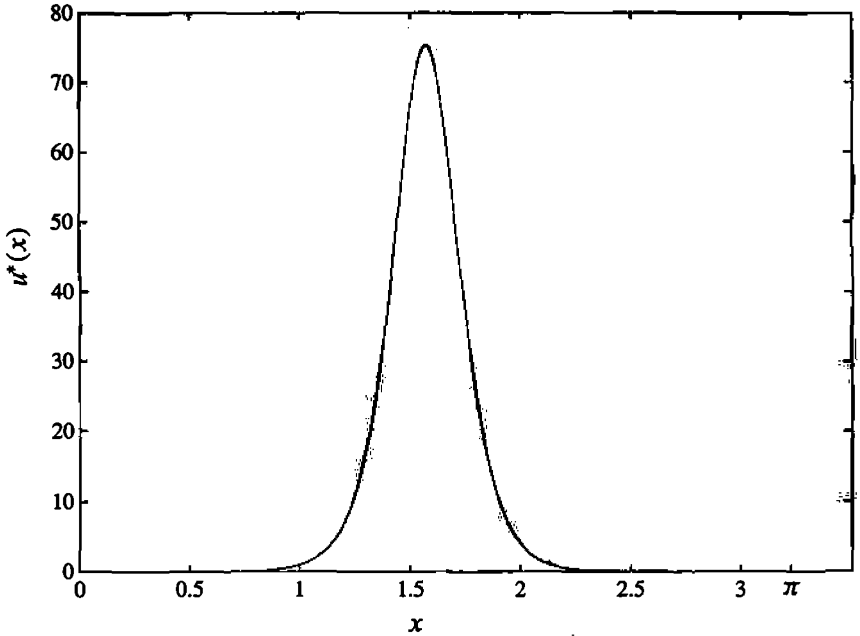

FIGURE 3. Non-trivial symmetric solution $u^*(x)$ of (2) for $q^2 = 100$ with $u(0) \equiv \varrho = 8.9150869 \times 10^{-5}$.

An alternative argument to determine the constant of integration D would have been to pass to the Fredholm alternative.

We observe that $u(x)$ converges to the zero solution in every sense as $q^2 \to 0$.

2.3. Behaviour for q^2 large We begin by studying the behaviour of (2) for q^2 large. We obtain in Section 3 a numerical solution of (2) for $q^2 = 100$. This symmetric solution is given in Figure 3. It appears to be an approximation to the delta-function centred on $\pi/2$.

Let $\psi = q^{-2}u$. Then under this substitution (2) becomes

$$\begin{aligned} &-q^{-2}\psi_{xx} + \psi = \psi^2(1 + \sin x), \qquad \text{for } 0 < x < \pi, \\ &\text{subject to} \qquad \psi_x(0) = 0, \quad \psi_x(\pi) = 0. \end{aligned} \tag{10}$$

The formal outer approximation of (10) is $\psi_0 = \psi_0^2(1+\sin x)$. That is, the non-trivial, formal outer solution is given by $\psi_0 = (1 + \sin x)^{-1}$. However, $(\psi_0)_x(0) = -1$ and $(\psi_0)_x(\pi) = 1$; which means that ψ_0 does not satisfy the boundary conditions required for a solution of (10). This behaviour usually implies that a boundary layer exists near $x = 0$ and near $x = \pi$. However, a glance at the computed solution given in Figure 3 shows that we should not expect such a solution structure. The behaviour in question is not the effect of the Neumann boundary conditions. Replacing the condition $u_x(0) = u_x(\pi) = 0$ in (2) with the condition $u(0) = u(\pi) = 0$, gives the corresponding Dirichlet problem. The formal outer solution of the Dirichlet version of (2) is again $\psi_0 \equiv (1+\sin x)^{-1}$, which does not satisfy the conditions $\psi_0(0) = \psi_0(\pi) = 0$. When $q^2 = 100$ there is a positive, non-zero, symmetric solution in the Dirichlet case of (2), with $u_x(0) = 8.9150869 \times 10^{-4}$. When q^2 is large there is a close similarity between the Neumann solution of (2) and the Dirichlet solution of (2), indicating that boundary-layer behaviour does not occur in either case.

We note that $\psi_0 \equiv 0$ is also a formal outer solution, and that it is the only solution which satisfies the boundary conditions.

Let $x = q^{-1}\eta + \pi/2$. Noting the identities $u_{xx} = q^2\psi_{xx}$ and $\psi_{xx} = q^2\psi_{\eta\eta}$, we may re-write (2) as

$$\begin{aligned} &-\psi_{\eta\eta} + \psi = \psi^2\left(1+\sin\left(q^{-1}\eta + \frac{\pi}{2}\right)\right), \quad -\frac{q\pi}{2} < \eta < \frac{q\pi}{2}, \\ &\text{subject to} \qquad \psi_\eta\left(-\frac{q\pi}{2}\right) = \psi_\eta\left(\frac{q\pi}{2}\right) = 0. \end{aligned} \tag{11}$$

Now $1 + \sin(q^{-1}\eta + \pi/2) = 1 + \cos q^{-1}\eta = 2 + O(q^{-2}\eta^2)$. We make the assumption that for q^2 large the right-hand side of the differential equation in (11) may be approximated by $2\psi^2$, and we study

$$-\psi_{\eta\eta} + \psi = 2\psi^2, \quad \text{subject to } \psi_\eta\left(-\frac{q\pi}{2}\right) = \psi_\eta\left(\frac{q\pi}{2}\right) = 0, \tag{12}$$

with $\eta \in (-q\pi/2, q\pi/2)$. Integrating both sides of (12) with respect to ψ, gives

$$-\int \psi_{\eta\eta}\, d\psi + \frac{\psi^2}{2} = \frac{2\psi^3}{3} + C, \tag{13}$$

where C is a constant of integration. Now integrating by parts, we may re-write (13) as

$$-\frac{\psi_\eta^2}{2} + \frac{\psi^2}{2} = \frac{2\psi^3}{3} + C. \tag{14}$$

Rearranging and integrating (14) with respect to η, we obtain

$$\int_{\psi(0)}^{\psi(\eta)} \frac{d\psi}{\sqrt{-4\psi^3/3 + \psi^2 - 2C}} = \int_0^\eta d\eta \equiv \eta. \tag{15}$$

Since $\eta \to \infty$ as $q \to \infty$, we must have that the integral on the left-hand side of (15) becomes unbounded as $q \to \infty$. For this to happen we need the integral on the left-hand side to have a singularity for $\psi \in [\psi(0), \psi(\eta)]$. That is, $-4\psi^3/3 + \psi^2 - 2C$ must have a double zero for $\psi \in [\psi(0), \psi(\eta)]$. This happens when either $C = 0$ or

$C = 1/24$. When $C = 0$ the double zero occurs at $\psi = 0$, and when $C = 1/24$ the double zero occurs at $\psi = 1/2$. However, in the case $C = 1/24$ we have that

$$\frac{1}{\sqrt{-4\psi/3 + \psi^2 - 2C}} \tag{16}$$

is real only for $\psi \leq -1/4$. Therefore, the domain of integration does not contain the pole when $C = 1/24$. Because (16) is real for all $\psi \leq 3/4$, and as the pole occurs at $\psi = 0$, it follows that $C \to 0$ as $q \to \infty$. Thus for q^2 large we may approximate (14) as

$$-\frac{\psi_\eta^2}{2} + \frac{\psi^2}{2} = \frac{2\psi^3}{3}. \tag{17}$$

Let $\psi_{\max}$ be the maximum value of ψ, with $\psi_\eta = 0$ at the maximum. Then from (17) we have $3\psi_{\max}^2 = 4\psi_{\max}^3$, that is, $\psi_{\max} = 3/4$ or $\psi_{\max} = 0$. Hence from the definition of ψ, $u_{\max}$, the maximum value of u, is $u_{\max} = 3q^2/4$. This implies that when for example $q^2 = 100$, $u_{\max} = 75$. This is indeed borne out by Figure 3.

Now by rearranging (17) and integrating, we have that

$$\int \frac{d\psi}{\psi\sqrt{1 - 4\psi/3}} = \int d\eta. \tag{18}$$

Since $\psi \leq 3/4$, we let $\phi = \sqrt{1 - 4\psi/3}$. We may now evaluate the integral on the left-hand side of (18) to obtain $\eta = -\hat{C}\log[(1 + \phi)/(1 - \phi)]$, for $\psi > 0$, where $\hat{C}$ is a constant of integration. Because $\psi = 3/4$ at $\eta = 0$, we have that $\hat{C} = 1$. Thus $\psi = 3e^{-\eta}(1 + e^{-\eta})^{-2}$. Hence we have by substitution

$$u^*(x) = 3q^2 \frac{e^{-q(x-\pi/2)}}{\left[1 + e^{-q(x-\pi/2)}\right]^2}.$$

We find that the approximation is indeed a good one. For instance, at $q^2 = 10$ the asymptotic approximation is $\max_{x\in[0,\pi]} |u^*(x)| = 7.5$ and $\max_{x\in[0,\pi]} |u^*(x)| = 7.742$ for the numerical solution of (2). At $q^2 = 100$ the asymptotic approximation is $\max_{x\in[0,\pi]} |u^*(x)| = 75$ and $\max_{x\in[0,\pi]} |u^*(x)| = 75.27$ for the numerical solution of (2), giving errors at $q^2 = 10$ and $q^2 = 100$ of 3.2% and 0.36%, respectively.

It is reasonable to ask whether the asymptotic approximations for q^2 small and q^2 large "match up" well. In Section 3 we study numerically the behaviour of (2), and obtain in Figure 2 a bifurcation diagram. The bifurcation diagram takes the form of a graph of q^2 against $\max_{x\in[0,\pi]} |u^*(x)|$. Now for q^2 small we have that $\max_{x\in[0,\pi]} |u^*(x)| = u(\pi/2) = q^2\overline{u} + q^4[8^{-1}\pi^2(\overline{u} - \overline{u}^2) + (1 - 2^{-1}\pi)\overline{u}^2 + D] + O(q^6)$; and for q^2 large we have that $\max_{x\in[0,\pi]} |u^*(x)| = u(\pi/2) \approx 3q^2/4$. We may therefore use the bifurcation diagram to obtain a comparison between the small q^2 approximations, the large q^2 approximations and the numerical solution. The combined plot is given in Figure 4.

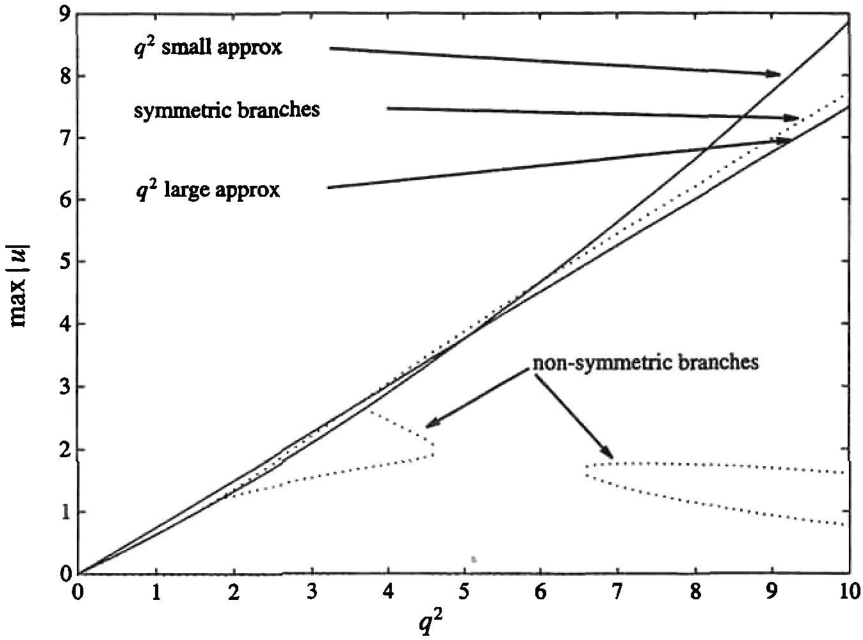

FIGURE 4. A plot of the numerically calculated $\max_{x\in[0,\pi]}|u^*(x)|$ for $q^2 \in [0, 10]$ as compared with the asymptotic approximations of $\max_{x\in[0,\pi]}|u^*(x)|$ for q^2 small and q^2 large.

2.4. Absence of outer-layer behaviour We now show why boundary-layer behaviour about the solution $(1+\sin x)^{-1}$ of the differential equation does not occur. To do this we re-scale the problem by taking $\psi = q^{-2}u$, $\epsilon = q^{-1}$, to give

$$-\epsilon^2\psi_{xx} + \psi = \psi^2(1+\sin x), \tag{19}$$

subject to $\psi_x(0) = \psi_x(\pi) = 0$, with $0 < x < \pi$. Let $u = (1+\sin x)^{-1}$. Then u satisfies the ODE in (19) but not the Neumann boundary conditions. Let $\{\epsilon_n\}$ be any sequence for which $\epsilon_n \to 0$ as $n \to \infty$. Let $\{u_n\}$ be a sequence of functions in $W^{1,2}(0,\pi)$ such that each u_n satisfies the corresponding ODE problem (19) and boundary conditions, with $\epsilon^2 = \epsilon_n^2$. Further suppose $u_n \to u$. These conditions correspond to u_n exhibiting boundary-layer behaviour. We now argue that this assumption leads to a contradiction.

Let $v_n = Du_n$ and $v = Du$. Then $v_n \to v$. Denote by B_n the inverse of the formal differential operator $-\epsilon_n^2\psi_{xx} + \psi$, subject to Neumann boundary conditions. Denote by B the inverse of the operator ψ, subject to Neumann boundary conditions. It follows by an application of the triangle inequality that

$$\|B_nv_n - Bv\| \le \|B_nv - Bv\| + \|B_nv_n - B_nv\| = \|(B_n - B)v\| + \|B_n(v_n - v)\|. \tag{20}$$

Now $\|B_n\| \to 1$ and $\|B_n(v_n - v)\| \le \|B_n\| \cdot \|v_n - v\|$; hence $\|B_n(v_n - v)\| \to 0$. Since B_n converges strongly to B (for proof see Appendix A), and $\|v\| < 2$, we have that $\|(B_n - B)v\| \to 0$. Thus from (20) $\|B_n v_n - Bv\| \to 0$. Now by definition $u_n = B_n v_n$; therefore $\|u_n - Bv\| \to 0$. Since $\|u_n - u\| \to 0$, $Bv = u$, that is, $BDu = u$. This means that u is a solution of the operator problem. But this cannot happen, as $u \equiv (1 + \sin x)^{-1}$ is not a solution (it does not satisfy the Neumann boundary conditions). Therefore we have a contradiction, and so this type of boundary-layer behaviour cannot occur.

We note that in the above proof, it was required that $\|v\|$ be finite. However, in the inner-layer case the solution is 'converging' to the delta function, but the L_2 norm of the delta function is not defined. Thus inner-layer behaviour about $u \equiv 0$ can and does occur.

3. Numerical results

3.1. Numerical method As q^2 becomes large, we find that the solutions tend to non-$L_2(0, \pi)$ functions with exponentially small tails. We find also that blow-up behaviour occurs for a finite value of $u(0)$, say, $\underline{\varrho}$. The blow-up behaviour is like $(x_0 - x)^{-2}$. As q^2 becomes large $\underline{\varrho}$ behaves like $(3/2)e^{-q\pi/2}$, so that the symmetric solution to the problem has a value of $u(0)$ between 0 and $(3/2)e^{-q\pi/2}$. For example, at $q^2 = 100$ a solution occurs at $u(0) = 8.92 \times 10^{-5}$, and blow-up occurs at $u(0) = 8.96 \times 10^{-5}$. We find also that the bifurcation diagram is disconnected (in the real plane). Peninsulas containing solutions occur within regions of blow-up. Because of these features of the problem it seems that the collocation methods with an adaptive mesh used in packages such as AUTO97 often do not produce reliable results. The cruder but more robust method of shooting for a solution is therefore employed. Due to the large increase in time required when using the method of shooting to compute our bifurcation diagram as q^2 increases, it was decided to calculate it in the interval $q^2 \in [0, 10]$. In particular, for q^2 large the solutions continue to bifurcate and the singularities become worse. It is useful to calculate the symmetric solution in the case $q^2 = 100$, and this can be done with a relatively efficient use of computer time.

A 4th-order Runge-Kutta method was used to shoot for a solution; and the methodology was checked against a similar problem with a known answer. Specifically, we use ODE45, a Matlab sub-routine for the numerical solution of ordinary differential equations (see [8, Section 3, p. 138]). It employs automatic step-size Runge-Kutta-Fehlberg integration methods and uses a 4th and 5th-order pair of formulae for higher accuracy. When the solution is more slowly changing, the automatic step-size algorithm takes larger steps; and since it uses higher-order formulae, it usually takes fewer integration steps and gives a solution more rapidly. The Matlab sub-routine is an implementation of an algorithm given in Forsythe, Malcolm and Moler [3].

3.2. Symmetry-breaking bifurcations We obtain a bifurcation diagram which is given in Figure 2. This is a bifurcation diagram of q^2 on the x-axis against the $\max|u(x)|$ of the solution to (2) on the y-axis. From Figure 2 we see that there is a unique non-zero solution to (2) for $q^2 \in (0, 1.75)$. It appears that for all $q^2 \in (0, 1.75)$ the solution of (2) is symmetric with the profile given in Figure 1. As q^2 tends to zero, the non-trivial solution converges, in every sense, to the zero solution.

Near $q^2 = 1.75$ two non-symmetric solutions bifurcate off the main symmetric branch. The next bifurcation from the main symmetric branch occurs at $q^2 = 3.6$, in a similar way to that at $q^2 = 1.75$. Again the two solutions bifurcating off the symmetric branch are non-symmetric.

We plot the five non-trivial solutions at $q^2 = 4$, in Figure 5. Note that the pair of solutions corresponding to the bifurcation at $q^2 = 1.75$ are a reflection of each other in the $x = \pi/2$-axis. Similarly, the pair of solutions corresponding to the bifurcation at $q^2 = 3.6$, are also a reflection of each other in the $x = \pi/2$-axis.

Near $q^2 = 4.64$, the four non-symmetric solutions vanish. Near $q^2 = 6.55$ four non-symmetric solutions appear. This seems in some sense to be a re-appearance of the four non-symmetric solutions that occur in the range $q^2 \in (3.6, 4.64)$.

For $q^2 = 100$ we find that $\underline{\varrho} = 8.96 \times 10^{-5}$. We now study the behaviour of solutions to (2) for $\varrho \in (0, \underline{\varrho})$. For $\varrho \in (0, 10^{-11})$ we find that $d(u_x(\pi))/d\varrho$ is a positive constant. This indicates that there are no solutions for (2) with $q^2 = 100$ and $\varrho \in (0, 10^{-11})$. The symmetric solution of (2) occurs at $\varrho = 8.9150869 \times 10^{-5}$, and is plotted in Figure 3. Note that the maximum height of the symmetric solution when $q^2 = 100$ is 75, which corresponds to our earlier asymptotic prediction.

3.3. Linearized problem We consider the linearized version of (2),

$$
\begin{aligned}
&-v_{xx} + q^2 v = 2uv(1 + \sin x), \quad \text{for } 0 < x < \pi,\\
&\text{subject to} \qquad v_x(0) = 0, \quad v_x(\pi) = 0,
\end{aligned} \tag{21}
$$

where u is a solution of (2) corresponding to q. The values of $q^2 \in [0, 10]$ for which the linearized problem (21) has non-trivial solutions are approximately $q^2 = 0$, $q^2 = 1.75$, $q^2 = 3.6$, $q^2 = 4.64$, $q^2 = 6.55$. That is, the linearized problem has non-trivial solutions v at the bifurcation points of (2)—as predicted by local bifurcation theory. The solution of the linearized problem was computed using a finite centralized second-difference method. Since the bifurcation results correspond exactly with our previous results, this acts as a further verification of our earlier Runge-Kutta method.

3.4. Blow-up behaviour In order to shoot for a solution of the BVP (2) we consider the initial value problem (IVP) formulation. In other words, we replace the condition

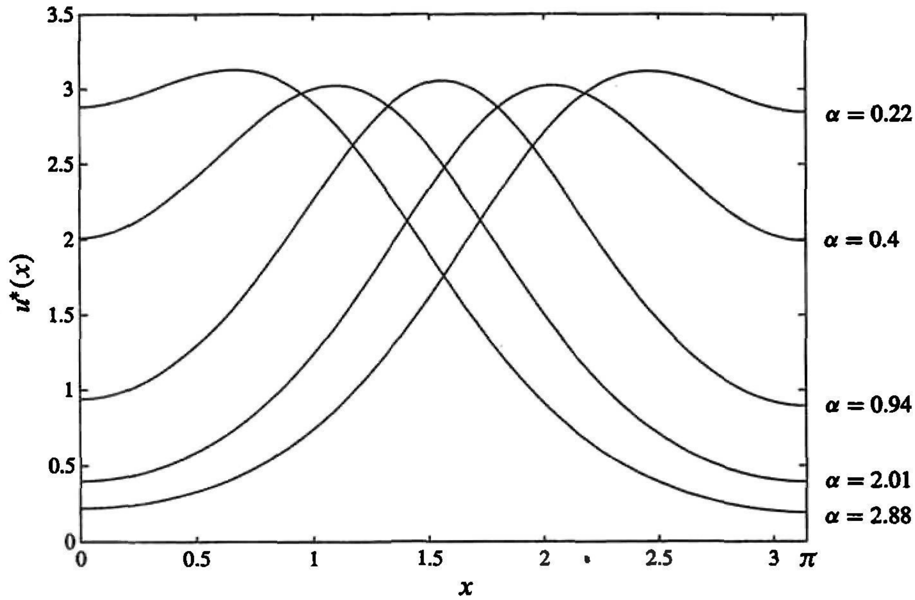

FIGURE 5. The five non-trivial solutions $u^*(x)$ of (2) for $q^2 = 4$.

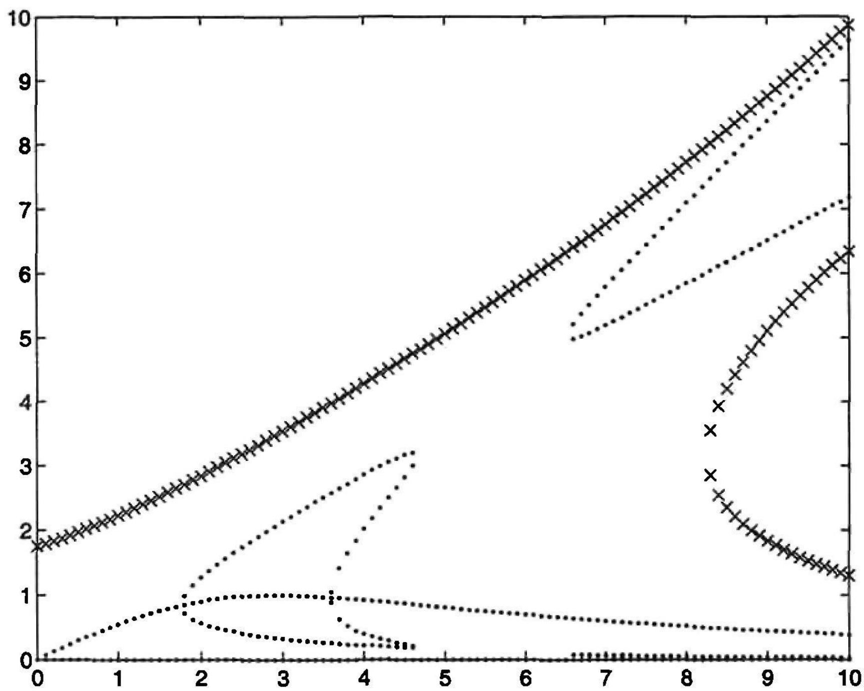

FIGURE 6. Bifurcation diagram for $u^*(0)$ against $q^2 \in [0, 10]$; "blow up" occurs in hatched regions.

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that $u_x(\pi) = 0$ in (2) with $u(0) = \varrho$ for some $\varrho \in \Re$, and vary ϱ so that $u_x(\pi) = 0$. By Reid [9, p. 34] we know that for each $\varrho \in \Re$ there exists an $x^* > 0$ such that a unique solution to the IVP exists for $0 < x < x^*$. However, it appears that for a certain value of ϱ, say ϱ^*, corresponding to a given q, the solution becomes extremely large and negative when x increases, and remains so for all $\varrho > \varrho^*$. That is, for $\varrho > \varrho^*$ it appears that no solutions to (2) exist; and the solutions of the IVP appear to "blow up" in the sense that $\lim_{x \to x_0} |u(x)| = \infty$, where $x_0 \le \pi$, so that the boundary condition at $x = \pi$ in (2) no longer holds.

For $q^2 > 8.3$ there are numbers $\underline{\varrho}$ and $\overline{\varrho}$, with $0 < \underline{\varrho} < \overline{\varrho} < \varrho^*$, such that the solutions of the IVP "blow up" both for ϱ in the interval $(\underline{\varrho}, \overline{\varrho})$, as well as for $\varrho > \varrho^*$. The regions of "blow-up" behaviour are indicated in Figure 6 by $\times$'s and hatching.

It seems that the symmetric solution (2) occurs for $\varrho \in (0, \underline{\varrho})$. As q^2 gets larger $\underline{\varrho}$ behaves like $(3/2)e^{-q\pi/2}$.

Given ϱ, let $\underline{q}, \overline{q}, q^*$ be the values of q for which $\varrho = \underline{\varrho}$, $\varrho = \overline{\varrho}$ and $\varrho = \varrho^*$, respectively. The pair $(\overline{q}^2, \overline{\varrho})$ may be regarded as a critical value such that as (q^2, ϱ) tends to $(\overline{q}^2, \overline{\varrho})$, the solution blows up.

4. Conclusion

By considering the Green's operator formulation of the problem we first show that all non-zero solutions of our problem must be positive. When $q^2 = 0$ we show that the only possible solution is $u \equiv 0$. In the q^2-small case we find by a regular approximation that to $O(q^6)$ $u(x) = q^2\overline{u} + q^4[2^{-1}x^2(\overline{u} - \overline{u}^2) - x\overline{u}^2 + \overline{u}^2 \sin x + D]$, where $\overline{u} = \pi(\pi + 2)^{-1}$ and $D = 6^{-1}\pi(\pi^3 - 8\pi^2 - 12\pi + 72)(\pi^3 + 6\pi^2 + 12\pi + 8)^{-1}$. In the case where q^2 is large we are able to obtain $u^*(x) = 3q^2 e^{-q(x-\pi/2)}[1 + e^{-q(x-\pi/2)}]^{-2} \approx 3q\delta(x - \pi/2)$ as an asymptotic solution. We study the behaviour of these approximations for q^2 small and q^2 large and ask if they "match up" well. We find that there is indeed a reasonable "match-up".

We show using the monotone convergence theorem for quadratic forms that the inverse of the operator on the left-hand side of (1) is strongly convergent as $q^2 \to \infty$. We then argue that were boundary-layer behaviour to occur, a contradiction would follow. That is, strong convergence of the operator is sufficient to stop outer-layer behaviour occurring. However, the argument does not stop inner-layer behaviour occurring, which can and does, for example, as a delta function centred on the $\pi/2$-axis. This behaviour also occurs when the Neumann boundary conditions are replaced by Dirichlet ones.

Because for q^2 small the Neumann boundary conditions become important, prohibiting any solution other than the trivial one at $q^2 = 0$; and because for q^2 large the strong convergence of the Green's operator forces the solutions to tend to non-L_2

functions, the numerical picture is more complex and more difficult to compute than for what at first glance appears to be such an extremely simple problem. We obtain a bifurcation diagram for $q^2 \in [0, 10]$ which confirms that there does indeed exist a positive, unique, symmetric, non-zero solution for $q^2 \in (0, 1.75)$. This indicates, as might be expected, that the existence results given in Mays and Norbury [7] are somewhat conservative. In particular, the non-trivial solution is both unique and symmetric on $q^2 \in (0, 1.226)$, properties not given by our existence method. At $q^2 = 1.75$ a symmetry-breaking bifurcation occurs. Whilst the symmetric branch appears to remain for all q^2, the picture becomes considerably more complex with regard to non-symmetric solutions. We check that numerically at $q^2 = 100$ our asymptotic results for q^2 large are a good approximation.

The linearized problem is examined to confirm the accuracy of our earlier bifurcation results.

Our results suggest that in Benjamin's problem one must be very careful to choose the "correct" solution for ϵ^2 small, that is, for q^2 large in our formulation of the problem. It is known that solitary waves occur in Benjamin's model: our results also suggest that it is possible that other unusual long waves may exist in his model; although whether these would ever be found in reality is questionable.

Appendix A

To show that B_n converges strongly to B as $n \to \infty$ we first quote the following result of Kato [4, Ch. VIII, Th. 3.6].

THEOREM 1 (Monotone convergence theorem for quadratic forms). *Let t_n, t be densely defined, closed, nonnegative quadratic forms in a Hilbert space $\mathscr{H}$, where $n = 1, 2, \ldots$. Suppose*

(i) *$\mathscr{D}(t_n) \subset \mathscr{D}(t)$ for all n;*
(ii) *there is a core $\mathscr{D}$ of t such that $\mathscr{D} \subset \mathscr{D}(t_n)$ for sufficiently large n;*
(iii) *$\lim_{n\to\infty} t_n(u) = t(u)$, if $u \in \mathscr{D}$;*
(iv) *$t_n(u) \geq t(u)$ for all $u \in \mathscr{D}$ and for all $n = 1, 2, \ldots$.*

Let T_n, T be the operators associated with t_n, t, respectively. Then the resolvents $R(\lambda; T_n) \to R(\lambda; T)$ strongly as $n \to \infty$ for all $\lambda < \kappa$, where κ is the lower bound of t.

THEOREM 2. *The operator B_n converges strongly to B as $n \to \infty$.*

PROOF. Let $\mathscr{H} = L_2(0, \pi)$; let $\mathscr{D}' = W^{1,2}(0, \pi)$; let $\{\epsilon_n\}$ be any sequence for which $\epsilon_n \to 0$ as $n \to \infty$; and assume $1 > \epsilon_n^2 > 0$ for all n. Let

$$t_n(u, v) = \int_0^\pi \{uv + \epsilon_n^2 u_x v_x\}\, dx \quad \text{and let} \quad t(u, v) = \int_0^\pi \{uv\}\, dx.$$

Then the domains of the forms t_n and t are $\mathscr{D}(t_n) = W^{1,2}(0, \pi)$, and $\mathscr{D}(t) = L_2(0, \pi)$, respectively. Thus t_n and t are densely defined on $\mathscr{H}$. Now

$$t_n(u) \equiv t_n(u, u) = \int_0^\pi \{u^2 + \epsilon_n^2 u_x^2\} dx \geq \int_0^\pi u^2 dx = \|u\|^2,$$

and $t(u) \equiv t(u, u) = \|u\|^2$. Therefore both t_n and t are bounded below by 1. We now show that t_n and t are both closed.

We construct the pre-Hilbert spaces $\mathscr{H}_{t_n}$ and $\mathscr{H}_t$ as defined in Kato [4, Ch. VI, § 1.3] with inner products

$$(u, v)_{t_n} = t_n(u, v) + (u, v) = \int_0^\pi \{2uv + \epsilon_n^2 u_x v_x\}\, dx$$

and

$$(u, v)_t = t(u, v) + (u, v) = \int_0^\pi \{2uv\}\, dx.$$

Thus $\mathscr{H}_{t_n}$ and $\mathscr{H}_t$ are merely the spaces $W^{1,2}(0, \pi)$ and $L_2(0, \pi)$, respectively, weighted with constants, and are consequently Hilbert spaces. Hence by Kato [4, Ch. VI, Th. 1.11], t_n and t are closed forms. Condition (i) is satisfied, since $\mathscr{D}(t_n) = W^{1,2}(0, \pi) \subset L_2(0, \pi) = \mathscr{D}(t)$. We note that $\mathscr{D}' = W^{1,2}(0, \pi)$ is dense in $\mathscr{H}_t = L_2(0, \pi)$. Thus by Kato [4, Ch. VI, Th. 1.21], $\mathscr{D}'$ is a core of t. Since by definition $\mathscr{D}' = W^{1,2}(0, \pi) = \mathscr{D}(t_n)$, condition (ii) is satisfied. Conditions (iii) and (iv) are satisfied immediately.

It is straightforward to show that t_n is the associated quadratic form of the formal partial differential operator $-\epsilon_n^2 u_{xx} + u$, subject to Neumann boundary conditions. It is immediate that t is the associated quadratic form of u.

We denote by T_n, T the operators associated with t_n, t, respectively. From the monotone convergence theorem for quadratic forms, and since 1 is a lower bound for t, the resolvent $R(\lambda; T_n)$ converges strongly to $R(\lambda; T)$ in $L_2(0, \pi)$ for $\lambda < 1$. Let $\lambda = 0$. Then $B_n \equiv T_n^{-1}$ converges strongly to $B \equiv T^{-1}$ as $n \to \infty$.

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