BULL. AUSTRAL. MATH. SOC. MOS 42A32
VOL. 5 (1971), 205-219.

Integrability of trigonometric series III

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Ralph P. Boas, Jr, proved the following theorem: Let g be an odd function, integrable on $(0, \pi)$ and periodic with period 2π, and its Fourier series be $\sum b_n \sin nt$. If $0 < r < 1$ and $b_n \geq 0$ for all n, then $t^{-r}g(t) \in L(0, \pi)$ if and only if the series $\sum b_n/n^{1-r}$ converges. Philip Heywood asked whether the conditions $g(t) \in L(0, \pi)$ and $b_n \geq 0$ can be replaced by $tg(t) \in L(0, \pi)$ and $b_n \geq -A/n^r$ or not. We prove this problem affirmatively.

1.

Our object is to prove the following

THEOREM 1. *Let* $0 < r < 1$ *and let* g *be an odd function satisfying the conditions:*

(i) $tg(t) \in L(0, \pi)$, *and*

(ii) $b_n(g) \geq -c/n^r$ *for all* $n \geq 1$ *and a positive constant* c, *where* $b_n(g)$ *is the* n*-th generalized sine coefficient of* g.

Then $\int_{+0}^{\pi} t^{-r}g(t)dt$ *exists, if and only if* $\sum b_n(g)/n^{1-r}$ *converges.*

Received 31 March 1971. Papers I and II in this series are not referred to in this paper.

https://doi.org/10.1017/S0004972700047080 Published online by Cambridge University Press

This theorem was conjectured by Heywood [3] and is a generalization of a theorem of Boas [1] (*cf.* [2]) where $g \in L$ and $b_n(g) \geq 0$ instead of *(i)* and *(ii)*, respectively. We can prove the $\int \to \sum$ part of Theorem 1 without the assumption *(ii)*, that is,

THEOREM 1'. *Let* $0 < r < 1$ *and let* g *be an odd function. If* $tg(t) \in L(0, \pi)$ *and* $b_n(g)$ *is the* n*-th generalized sine coefficient, then if* $\int_{+0}^{\pi} t^{-r} g(t)dt$ *exists, then* $\sum b_n(g)/n^{1-r}$ *converges.*

2.

We shall transform Theorems 1 and 1'. By the definition,

$$b_n(g) = \frac{2}{\pi}\int_0^{\pi} g(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi} 2\operatorname{tg}t/2\cdot g(t)\,\frac{\sin nt}{2\operatorname{tg}t/2}\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} f(t)\,\frac{\sin nt}{2\operatorname{tg}t/2}\,dt = s_n^*(f)$$

where f is an even function defined by

$$f(t) = 2\operatorname{tg}t/2\cdot g(t) \quad \text{on} \quad (0, \pi)$$

and $s_n^*(f)$ is the modified n-th partial sum of the Fourier series of f at the origin. Using f and $s_n^*(f)$ instead of g and $b_n(g)$, Theorems 1 and 1' can be stated in the following equivalent form.

THEOREM 2. *Let* $0 < r < 1$ *and let* f *be an even function such that*

(i) $f \in L(0, \pi)$, *and*

(ii) $s_n^*(f) \geq -c/n^r$ *for all* $n \geq 1$ *and a positive constant* c .

Then $\int_{+0}^{\pi} t^{-r-1} f(t)dt$ *exists, if and only if* $\sum s_n^*(f)/n^{1-r}$ *converges.*

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THEOREM 2'. *Let* $0 < r < 1$ *and let* f *be an even function integrable on* $(0, \pi)$. *Then if* $\int_{+0}^{\pi} t^{-r-1} f(t)dt$ *exists, then* $\sum s_n^*(f)/n^{1-r}$ *converges.*

We can prove the following similar theorems.

THEOREM 3. *Let* $0 < r < 1$ *and let* f *be an even function such that*

(i) $f \in L(0, \pi)$, *and*

(ii) $s_n(f) \ge -c/n^r$ *for all* $n \ge 1$ *and a positive constant* c, *where* $s_n(f)$ *is the* n-*th partial sum of the Fourier series of* f *at the origin.*

Then $\int_{+0}^{\pi} t^{-r-1} f(t)dt$ *exists, if and only if* $\sum s_n(f)/n^{1-r}$ *converges.*

THEOREM 3'. *Let* $0 < r < 1$ *and let* f *be an even function integrable on* $(0, \pi)$. *Then if* $\int_{+0}^{\pi} t^{-r-1} f(t)dt$ *exists, then* $\sum s_n(f)/n^{1-r}$ *converges.*

3. Proof of the $\sum \to \int$ part of the theorems

3.1. We shall first prove Theorem 3. We write $s_n = s_n(f)$ and

$$f(t) \sim \sum_{n=1}^{\infty} a_n \cos nt .$$

We can take $c = 1$, so that $s_n + 1/n^r \ge 0$ for all $n \ge 1$. Then

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$$(1)\quad \int_x^\pi \frac{f(t)}{t^r\cdot 2\sin t/2}\,dt$$

$$= \sum_{n=1}^\infty a_n \int_x^\pi \frac{\cos nt}{t^r\cdot 2\sin t/2}\,dt$$

$$= \sum_{n=1}^\infty (s_n - s_{n-1}) \int_x^\pi \frac{\cos nt}{t^r\cdot 2\sin t/2}\,dt$$

$$= \sum_{n=1}^\infty s_n \int_x^\pi \frac{\sin(n+1/2)t}{t^r}\,dt = \sum_{n=1}^\infty \frac{s_n}{(n+1/2)^{1-r}} \int_{(n+1/2)x}^{(n+1/2)\pi} \frac{\sin t}{t^r}\,dt$$

$$= \sum_{n=1}^\infty \frac{s_n}{(n+1/2)^{1-r}} \int_{(n+1/2)x}^{\infty} \frac{\sin t}{t^r}\,dt - \sum_{n=1}^\infty \frac{s_n}{(n+1/2)^{1-r}} \int_{(n+1/2)\pi}^{\infty} \frac{\sin t}{t^r}\,dt$$

where the last series on the right side is a finite constant, since

$$\int_{(n+1/2)\pi}^\infty \frac{\sin t}{t^r}\,dt = \left[-\frac{\cos t}{t^r}\right]_{t=(n+1/2)\pi}^\infty - r\int_{(n+1/2)\pi}^\infty \frac{\cos t}{t^{1+r}}\,dt$$

$$= O\left(1/n^{1+r}\right)$$

and $s_n = o\left(n^{1-r}\right)$ by the convergence of the series $\sum s_n/n^{1-r}$.

Let ε be a positive number < 1 and we write

$$\sum_{n=1}^\infty \frac{s_n}{(n+1/2)^{1-r}} \int_{(n+1/2)x}^\infty \frac{\sin t}{t^r}\,dt = \sum_{n=1}^{[\varepsilon/x]} + \sum_{n=[\varepsilon/x]+1}^{[1/\varepsilon x]} + \sum_{n=[1/\varepsilon x]+1}^{\infty}$$

$$= P + Q + R\ .$$

Putting $[\varepsilon/x] = y$, we get

$$P = \sum_{n=1}^{y} \frac{s_n}{(n+1/2)^{1-r}} \int_{(n+1/2)x}^\infty \frac{\sin t}{t^r}\,dt$$

$$= \frac{\pi}{2\Gamma(r)\sin(r\pi/2)} \sum_{n=1}^{y} \frac{s_n}{(n+1/2)^{1-r}} - \sum_{n=1}^{y} \frac{s_n}{(n+1/2)^{1-r}} \int_0^{(n+1/2)x} \frac{\sin t}{t^r}\,dt$$

$$= A \sum_{n=1}^{y} \frac{s_n}{(n+1/2)^{1-r}} - \sum_{n=1}^{y} \frac{s_n + 1/n^r}{(n+1/2)^{1-r}} \int_0^{(n+1/2)x} \frac{\sin t}{t^r}\,dt$$

$$+ \sum_{n=1}^{y} \frac{1}{n^r(n+1/2)^{1-r}} \int_0^{(n+1/2)x} \frac{\sin t}{t^r}\,dt$$

$$= S - T + U\ ,$$

where S tends to a constant as $x \to 0$, by the assumption, and

$$T \le A \sum_{n=1}^{y} \frac{s_n+1/n^r}{n^{1-r}} (nx)^{2-r}$$

$$= Ax^{2-r} \sum_{n=1}^{y} ns_n + Ax^{2-r} \sum_{n=1}^{y} n^{1-r}$$

$$= Ax^{2-r} \left(\sum_{n=1}^{y-1} \Delta n^{2-r} \sum_{m=1}^{n} \frac{s_m}{m^{1-r}} + y^{2-r} \sum_{m=1}^{y} \frac{s_m}{m^{1-r}} \right) + A\varepsilon^{2-r}$$

$$\le A\varepsilon^{2-r} ,$$

and further

$$U \le A \sum_{n=1}^{y} \frac{1}{n} (nx) \le A\varepsilon .$$

Therefore, collecting the above estimations, we get

$$\limsup_{x \to 0} |P-A| \le A\varepsilon^{2-r} . \tag{2}$$

Writing $z = [1/\varepsilon x]$ and using integration by parts and Abel's transformation, we get

$$R = \sum_{n=z+1}^{\infty} \frac{s_n}{(n+1/2)^{1-r}} \int_{(n+1/2)x}^{\infty} \frac{\sin t}{t^r} dt$$

$$= \sum_{n=z+1} \frac{s_n+1/n^r}{(n+1/2)^{1-r}} \int_{(n+1/2)x}^{\infty} \frac{\sin t}{t^r} dt$$

$$- \sum_{n=z+1}^{\infty} \frac{1}{n^r (n+1/2)^{1-r}} \int_{(n+1/2)x}^{\infty} \frac{\sin t}{t^r} dt$$

$$= \sum_{n=z+1} \frac{s_n+1/n^r}{(n+1/2)^{1-r}} \left(\frac{\cos(n+1/2)x}{(n+1/2)^r x^r} - r \int_{(n+1/2)x}^{\infty} \frac{\cos t}{t^{1+r}} dt \right) + O(\varepsilon^r)$$

$$= \frac{1}{x^r} \left(S_{z+1} \cos(z+3/2)x - 2\sin x/2 \sum_{n=z+2}^{\infty} S_n \sin nx \right)$$

$$+ O\left(\frac{1}{x^{1+r}} \sum_{n=z+1}^{\infty} \frac{s_n+1/n^r}{n^2} \right) + O(\varepsilon^r)$$

$$= O(\varepsilon^r) + O(\varepsilon^{r+1}) + O(\varepsilon^r) = O(\varepsilon^r) ,$$

where

$$S_k = \sum_{n=k}^{\infty} \frac{s_n+1/n^r}{n+1/2} \downarrow 0 \quad \text{and} \quad S_k = O(1/k^r) \quad \text{as} \quad k \to \infty .$$

Therefore,

$$\text{(3)} \qquad \limsup_{x\to 0} |R| \le A\varepsilon^r .$$

Finally, using the expansion of sine series,

$$\begin{aligned}
\text{(4)} \qquad Q &= \sum_{n=y+1}^{z} \frac{s_n}{(n+1/2)^{1-r}} \int_{(n+1/2)x}^{\infty} \frac{\sin t}{t^r}\, dt \\
&= \sum_{n=y+1}^{z} \frac{s_n}{(n+1/2)^{1-r}} \int_{0}^{(n+1/2)x} \frac{\sin t}{t^r}\, dt + o(1) \\
&= \sum_{n=y+1}^{z} \frac{s_n}{(n+1/2)^{1-r}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_{0}^{(n+1/2)x} t^{2k+1-r} dt + o(1) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2-r}}{(2k+1)!(2k+2-r)} \sum_{n=y+1}^{z} (n+1/2)^{2k+1} s_n + o(1) \\
&= o\left(\sum_{k=0}^{\infty} \frac{\varepsilon^{-2k-2+r}}{(2k+1)!(2k+2-r)} \right) + o(1) \\
&= o(1) \quad \text{as} \quad x \to 0 ,
\end{aligned}$$

since, putting $t_n = \sum_{m=y+1}^{\infty} s_m/(m+1/2)^{1-r}$, we have

$$\begin{aligned}
\sum_{n=y+1}^{z} (n+1/2)^{2k+1} s_n &= \sum_{n=y+1}^{z} \frac{s_n}{(n+1/2)^{1-r}} (n+1/2)^{2k+2-r} \\
&= \sum_{n=y+1}^{z} \left(t_n - t_{n-1}\right)(n+1/2)^{2k+2-r} \\
&= t_z (z+1/2)^{2k+2-r} + \sum_{n=y+1}^{z-1} t_n \Delta\left((n+1/2)^{2k+2-r}\right) \\
&= o\left(1/(\varepsilon x)^{2k+2-r}\right) \quad \text{as} \quad x \to 0 .
\end{aligned}$$

Combining (2), (3) and (4), we get

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$$\limsup_{x\to 0} \left| \int_x^{\pi} \frac{f(t)}{t^r \cdot 2\sin t/2}\, dt - A \right| \le A\varepsilon^r .$$

Letting $\varepsilon \to 0$, we get the $\sum \to \int$ part of Theorem 3.

3.2. We shall prove Theorems 1 and 2. If we show that

$$(5) \qquad \sum_{n=1}^{\infty} a_n \int_x^{\pi} \frac{\sin(n+1/2)t}{t^r}\, dt = A + o(1) \quad \text{as} \quad x \to 0 ,$$

then, by (1),

$$\int_x^{\pi} \frac{f(t)}{t^r \cdot 2\sin t/2}\, dt = \sum_{n=1}^{\infty} \frac{s_n^*}{(n+1/2)^{1-r}} \int_{(n+1/2)x}^{(n+1/2)\pi} \frac{\sin t}{t^r}\, dt + A + o(1) \quad \text{as} \quad x \to 0 .$$

We can apply the method in §3.1 to the right side integral under the assumption *(ii)* of Theorem 2, so we can complete the proof parallel to §3.1.

Now, the left side series of (5) is

$$\sum_{n=1}^{\infty} s_n \int_x^{\pi} \frac{\sin(n+1/2)t - \sin(n-1/2)t}{t^r}\, dt$$

$$= 2 \sum_{n=1}^{\infty} s_n \int_x^{\pi} \frac{\sin t/2}{t^r} \cos nt\, dt$$

$$= 2 \frac{\sin x/2}{x^r} \sum_{n=1}^{\infty} \frac{s_n}{n} \sin nx + 2 \sum_{n=1}^{\infty} \frac{s_n}{n} \int_x^{\pi} \frac{d}{dt}\left(\frac{\sin t/2}{t^r}\right) \cos nt\, dt$$

$$= 2V + 2W$$

and

$$V = \frac{\sin x/2}{x^r} \sum_{n=1}^{\infty} \frac{s_n^*}{n} \sin nx + \frac{\sin x/2}{2x^r} \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx ,$$

where the series of the last term converges uniformly, since it is the termwise integrated series of the Fourier series of f and then the last term of the right side tends to zero as $x \to 0$. On the other hand, putting

$$S_n^* = \sum_{k=n}^{\infty} \left(s_k^* + 1/k^r\right) / k^{1-r} ,$$

we have

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$$\sum_{n=1}^{\infty} \frac{s_n^*}{n} \sin nx = \sum_{n=1}^{\infty} \frac{s_n^*+1/n^r}{n^{1-r}} \frac{\sin nx}{n^r} - \sum_{n=1}^{\infty} \frac{\sin nx}{n^{1+r}}$$

$$= - \sum_{n=1}^{\infty} S_n^* \Delta\left(\frac{\sin nx}{n^r}\right) + o(1) ,$$

and then V tends to zero as $x \to 0$.

Finally, putting

$$S_n = \sum_{k=n}^{\infty} s_k/k^{1-r}$$

and using Abel's transformation,

$$W = \sum_{n=1}^{\infty} S_n \Delta\left[\frac{1}{n^r} \int_x^{\pi} \frac{d}{dt}\left(\frac{\sin t/2}{t^r}\right) \cos nt dt\right]$$

$$= 2 \sum_{n=1}^{\infty} \frac{S_n}{n^r} \int_x^{\pi} \frac{d}{dt}\left(\frac{\sin t/2}{t^r}\right) \sin t/2 \sin(n+1/2)t dt + A + o(1) ,$$

which tends to a limit as $x \to 0$, since the series on the right side is absolutely convergent. Thus we have proved the required (5).

4. Proof of the $\int \to \sum$ part of the theorems

We shall prove Theorem 1'. We can write

$$\sum_{n=M}^{N} \frac{b_n(g)}{n^{1-r}} = \frac{1}{\pi} \int_0^{\pi} g(t) \left(\sum_{n=M}^{N} \frac{\sin nt}{n^{1-r}}\right) dt$$

$$= \frac{1}{\pi} \int_0^{\pi} g(t) dt \left[\int_{M-1/2}^{N+1/2} \frac{\sin ut}{u^{1-r}} du + \int_{M-1/2}^{N+1/2} \frac{\sin ut}{u^{1-r}} dj(u)\right]$$

$$= \frac{1}{\pi}(P+Q) ,$$

where

$$j(u) = - u + [u] + 1/2 \sim \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin 2\pi m u}{m} .$$

Now

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$$P = \int_0^{\pi} \frac{g(t)}{t^r}\,dt \int_{(M-1/2)t}^{(N+1/2)t} \frac{\sin v}{v^{1-r}}\,dv$$
$$= \int_0^{(M-1/2)t} \frac{\sin v}{v^{1-r}}\,dv \int_{v/(N+1/2)}^{v/(M-1/2)} \frac{g(t)}{t^r}\,dt$$
$$+ \int_{(M-1/2)\pi}^{(N+1/2)\pi} \frac{\sin v}{v^{1-r}}\,dv \int_{v/(N+1/2)}^{\pi} \frac{g(t)}{t^r}\,dt$$
$$= R + S$$

and

$$R = \left\{\sum_{k=0}^{M-1} \int_{k\pi}^{(k+1)\pi} dv - \int_{(M-1/2)\pi}^{M\pi} dv\right\} \int_{v/(N+1/2)}^{v/(M-1/2)} dt$$
$$= \sum_{k=1}^{[(M-1/2)]} \left(\int_{(2k-1)\pi}^{2k\pi} + \int_{2k\pi}^{(2k+1)\pi}\right) dv \int_{v/(N+1/2)}^{v/(M-1/2)} dt + o(1)$$
$$= R' + o(1) \quad \text{as} \quad M,\ N \to \infty .$$

Writing $[(M-1)/2] = M'$,

$$R' = -\sum_{k=1}^{M'} \int_0^{\pi} \sin v\,dv \left\{\frac{1}{\{(2k-1)\pi+v\}^{1-r}} \int_{\{(2k-1)\pi+v\}/(N+1/2)}^{\{(2k-1)\pi+v\}/(M-1/2)} \frac{g(t)}{t^r}\,dt\right.$$
$$\left. - \frac{1}{(2k\pi+v)^{1-r}} \int_{(2k\pi+v)/(N+1/2)}^{(2k\pi+v)/(M-1/2)} \frac{g(t)}{t^r}\,dt\right\}$$
$$= -\int_0^{\pi} \sin v\,dv \left\{\sum_{k=1}^{M'} \left(\frac{1}{\{(2k-1)\pi+v\}^{1-r}}\right.\right.$$
$$\left. - \frac{1}{(2k\pi+v)^{1-r}}\right) \int_{\{(2k-1)\pi+v\}/(N+1/2)}^{\{(2k-1)\pi+v\}/(M-1/2)} \frac{g(t)}{t^r}\,dt$$
$$\left. + \sum_{k=1}^{M'} \frac{1}{(2k\pi+v)^{1-r}} \left(\int_{\{(2k-1)\pi+v\}/(N+1/2)}^{\{(2k-1)\pi+v\}/(M-1/2)} - \int_{(2k\pi+v)/(N+1/2)}^{(2k\pi+v)/(M-1/2)}\right) \frac{g(t)}{t^r}\,dt\right\}$$
$$= T_1 - T_2 .$$

We can easily see that $T_1 = o(1)$ as $M,\ N \to \infty$ and

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$$T_2 = \int_0^\pi \sin v dv \left\{ \sum_{k=1}^{M'} \frac{1}{(2k\pi+v)^{1-r}} \int_{((2k-1)\pi+v)/(N+1/2)}^{(2k\pi+v)/(N+1/2)} \frac{g(t)}{t^r} dt - \sum_{k=1}^{M'} \frac{1}{(2k\pi+v)^{1-r}} \int_{((2k-1)\pi+v)/(M-1/2)}^{(2k\pi+v)/(M-1/2)} \frac{g(t)}{t^r} dt \right\}$$

$$= T_2' - T_2'' ,$$

where

$$T_2' = \int_0^\pi \sin v dv \int_{1/2}^{M'+1/2} \frac{d\omega}{(2\pi\omega+v)^{1-r}} \int_{(2\pi\omega-\pi+v)/(N+1/2)}^{(2\pi\omega+v)/(N+1/2)} \frac{g(t)}{t^r} dt + \int_0^\pi \sin v dv \int_{1/2}^{M'+1/2} \frac{dj(\omega)}{(2\pi\omega+v)^{1-r}} \int_{(2\pi\omega-\pi+v)/(N+1/2)}^{(2\pi\omega+v)/(N+1/2)} \frac{g(t)}{t^r} dt$$

$$= U_1 + U_2 .$$

Writing $2\pi\omega/(N+1/2) = \omega'$, $v/(N+1/2) = v'$, we have

$$U_1 = \frac{(N+1/2)^{1+r}}{2} \int_0^{\pi/(N+1/2)} \sin(N+1/2)v' dv' \cdot \int_{1/2(N+1/2)}^{(M'+1/2)/(N+1/2)} \frac{d\omega'}{(\omega'+v')^{1-r}} \int_{\omega'+v'-\pi/(N+\frac{1}{2})}^{\omega'+v'} \frac{g(t)}{t^r} dt$$

$$= \frac{(N+1/2)^{1+r}}{2} \int_0^{\pi/(N+1/2)} \sin(N+1/2)v' dv' \cdot \int_{\pi/(N+1/2)}^{(M'+1/2)/(N+1/2)} \frac{g(t)}{t^r} dt \int_{t-v'}^{t-v'+\pi/(N+1/2)} \frac{d\omega'}{(\omega'+v')^{1-r}} + o(1) .$$

For any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left| \int_b^c g(t) t^{-r} dt \right| < \varepsilon \quad \text{for any} \quad 0 < b < c < \delta ;$$

then, if $\pi/(N+1/2) < \delta < (M'+1/2)/(N+1/2)$, we get

$$U_1 = \frac{(N+1/2)^{1+r}}{2\pi} \int_0^{\pi/(N+1/2)} \sin(N+1/2)v'dv' \cdot \left\{\int_{\pi/(N+1/2)}^{\delta} \frac{g(t)}{t^r}dt + \int_{\delta}^{(M'+1/2)/(N+1/2)} \frac{g(t)}{t^r}dt\right\} \cdot \int_{t-v'}^{t-v'+\pi/(N+1/2)} \frac{dw'}{(w'+v')^{1-r}} + o(1)$$

$$= U_1' + U_1'' + o(1) ,$$

where

$$|U_1'| \le A(N+1/2)^{1+r} \frac{A}{N+1/2} \frac{(N+1/2)^{1-r}}{N+1/2} \varepsilon \le A\varepsilon$$

and

$$|U_1''| \le A(N+1/2)^{1+r} \frac{1}{N+1/2} \frac{A}{N+1/2} = o(1) \quad \text{as} \quad M, N \to \infty .$$

Thus we have $|U_1| \le A\varepsilon$ which holds also for all cases of δ and then $U_1 = o(1)$ as $M, N \to \infty$. Now, by integration by parts,

$$U_2 = \int_0^{\pi} \sin v dv \int_{1/2}^{M'+1/2} \frac{j(w)}{(2\pi w+v)^{2-r}} dw \cdot \int_{(2\pi w-\pi+v)/(N+1/2)}^{(2\pi w+v)/(N+1/2)} \frac{g(t)}{t^r} dt$$
$$+ \int_0^{\pi} \sin v dv \int_{1/2}^{M'+1/2} \frac{j(w)}{(2\pi w+v)^{1-r}} \frac{g\{(2\pi w+v)/(N+1/2)\}}{\{(2\pi w+v)/(N+1/2)\}^r} \cdot \frac{2\pi}{N+1/2} dw$$
$$- \int_0^{\pi} \sin v dv \int_{1/2}^{M'+1/2} \frac{j(w)}{(2\pi w+v)^{1-r}} \frac{g\{(2\pi w-\pi+v)/(N+1/2)\}}{\{(2\pi w-\pi+v)/(N+1/2)\}^r} \cdot \frac{2\pi}{N+1/2} dw$$
$$= U_2' + U_2'' - U_2'''$$

where, substituting the Fourier series of $j(w)$,

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$$U_2' = A \sum_{m=1}^{\infty} \frac{1}{m} \int_0^{\pi} \sin v dv \int_{1/2}^{M'+1/2} \frac{\sin 2\pi m w}{(2\pi w+v)^{2-r}} dw \int_{(2\pi w-\pi+v)/(N+1/2)}^{(2\pi w+v)/(N+1/2)} \frac{g(t)}{t^r} dt$$

$$= A(N+1/2)^r \sum_{m=1}^{\infty} \frac{1}{m} \int_0^{\pi/(N+1/2)} \sin(N+1/2)v dv$$

$$\cdot \int_{\pi/(N+1/2)}^{2\pi(M'+1/2)/(N+1/2)} \frac{\sin m(N+1/2)w'}{(w'+v')^{2-r}} dw' \cdot \int_{w'+v'-\pi/(N+1/2)}^{w'+v'} \frac{g(t)}{t^r} dt$$

$$= A(N+1/2)^r \sum_{m=1}^{\infty} \frac{1}{m} \int_0^{\pi/(N+1/2)} \sin(N+1/2)v dv$$

$$\cdot \int_{\pi/(N+1/2)}^{2\pi(M'+1/2)(N+1/2)} \frac{g(t)}{t^r} dt \int_{t-v'}^{t-v'+\pi/(N+1/2)} \frac{\sin m(N+1/2)w'}{(w'+v')^{2-r}} dw' + o(1)$$

$$= o(1) \text{ , as } M, N \to \infty ;$$

and, by the second mean value theorem,

$$U_2'' = \frac{2\pi}{N+1/2} \int_0^{\pi} \sin v dv \sum_{k=1}^{[M'+1/2]} \int_k^{k+1} \frac{j(w)}{(2\pi w+v)^{1-r}} \frac{g((2\pi w+v)/(N+1/2))}{((2\pi w+v)/(N+1/2))^r} dw + o(1)$$

$$= \frac{2\pi}{N+1/2} \int_0^{\pi} \sin v dv \sum_{k=1}^{[M'+1/2]} \frac{1}{(2\pi k+v)^{1-r}}$$

$$\cdot \left(\int_k^{k'} - \int_{k''}^{k'''}\right) \frac{g((2\pi w+v)/(N+1/2))}{((2\pi w+v)/(N+1/2))^r} dw + o(1)$$

$$= o(1) \text{ , as } M, N \to \infty ,$$

where $k < k' < k'' < k''' < k+1$, and similarly $U_2''' = o(1)$. Therefore $U_2 = o(1)$. Thus we have proved that $T_2' = U_1 + U_2 = o(1)$. Similarly $T_2'' = o(1)$, and then $T_2 = T_2' - T_2'' = o(1)$. Thus $R = o(1)$. Estimation of S is similar to R , and then $P = o(1)$.

Now, we shall estimate Q .

$$Q = \int_0^{\pi} g(t)dt \int_{M-1/2}^{N+1/2} \frac{\sin ut}{u^{1-r}}\, dj(u)$$

$$= - \int_0^{\pi} g(t)dt \int_{M-1/2}^{N+1/2} j(u)\frac{d}{du}\left(\frac{\sin ut}{u^{1-r}}\right) du$$

$$= - \int_0^{\pi} tg(t)dt \int_{M-1/2}^{N+1/2} j(u)\, \frac{\cos ut}{u^{1-r}}\, du$$

$$+ (1-r) \int_0^{\pi} g(t)dt \int_{M-1/2}^{N+1/2} j(u)\frac{\sin ut}{u^{2-r}}\, du$$

$$= - Q_1 + (1-r)Q_2 .$$

Using the Fourier expansion of j ,

$$\int_{M-1/2}^{N+1/2} j(u)\, \frac{\cos ut}{u^{1-r}}\, du = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \int_{M-1/2}^{N+1/2} \frac{\cos ut \sin 2\pi mu}{u^{1-r}}\, du$$

$$= \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{1}{m} \int_{M-1/2}^{N+1/2} \frac{\sin(2\pi m+t)u+\sin(2\pi m-t)u}{u^{1-r}}\, du$$

$$= O\left(\frac{1}{M^{1-r}} \sum_{m=1}^{\infty} \frac{1}{m^2}\right) = O\left(\frac{1}{M^{1-r}}\right)$$

and then

$$|Q_1| \quad \frac{A}{M^{1-r}} \int_0^{\pi} t|g(t)|dt = o(1) \quad \text{as} \quad M \to \infty .$$

On the other hand

$$Q_2 = \int_0^{\pi} g(t)dt \int_{M-1/2}^{N+1/2} j(u)\, \frac{\sin ut}{u^{2-r}}\, du$$

$$= \int_0^{1/(N+1/2)} g(t)dt \int_{M-1/2}^{N+1/2} j(u)\, \frac{\sin ut}{u^{2-r}}\, du$$

$$+ \int_{1/(N+1/2)}^{1/(M-1/2)} g(t)dt \left(\int_{M-1/2}^{1/t} + \int_{1/t}^{N+1/2}\right) j(u)\, \frac{\sin ut}{u^{2-r}}\, du$$

$$+ \int_{1/(M-1/2)}^{\pi} g(t)dt \int_{M-1/2}^{N+1/2} j(u)\, \frac{\sin ut}{u^{2-r}}\, du$$

$$= V_1 + V_2 + V_3 + V_4 .$$

https://doi.org/10.1017/S0004972700047080 Published online by Cambridge University Press

We have

$$V_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^{1/(N+1/2)} t^{2k+1} g(t)dt \int_{M-1/2}^{N+1/2} j(u)u^{2k-1+r} du$$

and then

$$|V_1| \le \int_0^{1/(N+1/2)} t|g(t)|dt \left|\int_{M-1/2}^{N+1/2} \frac{j(u)}{u^{1-r}} du\right|$$
$$+ \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \int_0^{1/(N+1/2)} t^{2k+1}|g(t)|dt \cdot \left|\int_{M-1/2}^{N+1/2} j(u)u^{2k-1+r} du\right|$$
$$\le \frac{A}{m^{1-r}} \int_0^{1/(N+1/2)} t|g(t)|dt$$
$$+ \sum_{k=1}^{\infty} \frac{(N+1/2)^{2k-1+r}}{(2k+1)!(N+1/2)^{2k}} \int_0^{1/(N+1/2)} t|g(t)|dt$$
$$= o(1) \text{ , as } M, N \to \infty .$$

Secondly,

$$V_2 = \int_{1/(N+1/2)}^{1/(M-1/2)} g(t)dt \int_{M-1/2}^{1/t} j(u) \frac{\sin ut}{u^{2-r}} du$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_{1/(N+1/2)}^{1/(M-1/2)} t^{2k+1} g(t)dt \int_{M-1/2}^{1/t} j(u)u^{2k-1+r} du$$

and then

$$|V_2| \le \frac{A}{M^{1-r}} \int_{1/(N+1/2)}^{1/(M-1/2)} t|g(t)|dt + A \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \int_{1/(N+1/2)}^{1/(M-1/2)} t^{2-r}|g(t)|dt$$
$$= o(1) \text{ , as } M, N \to \infty .$$

Finally,

$$V_3 = \int_{1/(N+1/2)}^{1/(M-1/2)} g(t)dt \int_{1/t}^{N+1/2} j(u) \frac{\sin ut}{u^{2-r}} du$$
$$= O\left(\int_{1/(N+1/2)}^{1/(M-1/2)} t^{2-r}|g(t)|dt\right)$$
$$= o(1) \text{ , as } M, N \to \infty ,$$

since

$$\int_{1/t}^{N+1/2} j(u)\, \frac{\sin ut}{u^{2-r}}\, du = \frac{1}{\pi} \sum_{m}^{\infty} \frac{1}{m} \int_{1/t}^{N+1/2} \frac{\sin 2\pi m u \cdot \cos ut}{u^{2-r}}\, du$$

$$= O\left(t^{2-r} \sum_{m=1}^{\infty} \frac{1}{m^2}\right) = O\left(t^{2-r}\right) ;$$

and similarly V_4 is also $o(1)$. Thus we have proved the theorem.

The proofs of Theorems 2' and 3' are now immediate.

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