

## $L^p$ – $L^q$ MULTIPLIERS ON COMMUTATIVE HYPERGROUPS

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### Abstract

The main purpose of this paper is to prove Hörmander’s  $L^p$ – $L^q$  boundedness of Fourier multipliers on commutative hypergroups. We carry out this objective by establishing the Paley inequality and Hausdorff–Young–Paley inequality for commutative hypergroups. We show the  $L^p$ – $L^q$  boundedness of the spectral multipliers for the generalised radial Laplacian by examining our results on Chébli–Trimèche hypergroups. As a consequence, we obtain embedding theorems and time asymptotics for the  $L^p$ – $L^q$  norms of the heat kernel for generalised radial Laplacian.

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### 1. Introduction

In this paper, we establish  $L^p$ – $L^q$  boundedness of Fourier multipliers on commutative hypergroups. As an application, we prove the boundedness result for spectral multipliers in the context of Chébli–Trimèche hypergroups. Bloom and Xu [15] proved the Hörmander multiplier theorem concerning the  $L^p$ -boundedness of Fourier multipliers on Chébli–Trimèche hypergroups by using the techniques developed by Anker in his seminal paper [6]. In particular, they extended the result of Stempak [40] proved for Bessel–Kingman hypergroups, a particular class of Chébli–Trimèche hypergroups of polynomial growth. After that, Gosselin and Stempak [25] proved an improved version of the result in [40] using a similar method to the one used by Hörmander in his classical paper [26] for Bessel–Kingman hypergroups, which was later studied by several researchers in many different settings, for example, [8, 23, 24, 35, 39]. Here, we deal with  $L^p$ – $L^q$  multipliers as opposed to the  $L^p$ -multipliers for which theorems of Mihlin and Hörmander or Marcinkiewicz type provide results for Fourier multipliers

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in different settings based on the regularity of the symbol. In the Lie group setting, the Mihlin–Hörmander multiplier theorem was studied by many researchers. We cite here [6, 20, 22, 26, 37] to mention a few of them.

The Paley-type inequality describes the growth of the Fourier transform of a function in terms of its  $L^p$ -norm. Interpolating the Paley-inequality with the Hausdorff–Young inequality, one can obtain the following Hörmander’s version of the Hausdorff–Young–Paley inequality:

$$\left( \int_{\mathbb{R}^n} |(\mathcal{F}_{\mathbb{R}^n} f)(\xi)\phi(\xi)|^{1/r-1/p'} |r d\xi \right)^{1/r} \leq \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leq r \leq p' < \infty, \quad 1 < p < 2.$$

Also, as a consequence of the Hausdorff–Young–Paley inequality, Hörmander [26, page 106] proved that the condition

$$\sup_{t>0} t^b |\{\xi \in \mathbb{R}^n : m(\xi) \geq t\}| < \infty, \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{b},$$

where  $1 < p \leq 2 \leq q < \infty$  implies the existence of a bounded extension of a Fourier multiplier  $T_m$  with symbol  $m$  from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ . Recently, the second author with R. Akyzhanov studied classical Hörmander results for unimodular locally compact groups and homogeneous spaces [3, 4]. In [4], the key idea behind the extension of Hörmander theorem is the reformulation of this theorem as

$$\|T_m\|_{L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} \lesssim \sup_{s>0} s \left( \int_{\{\xi \in \mathbb{R}^n : m(\xi) \geq s\}} d\xi \right)^{1/p-1/q} \simeq \|m\|_{L^{r,\infty}(\mathbb{R}^n)} \simeq \|T_m\|_{L^{r,\infty}(\text{VN}(\mathbb{R}^n))},$$

where  $1/r = 1/p - 1/q$ ,  $\|m\|_{L^{r,\infty}(\mathbb{R}^n)}$  is the Lorentz norm of  $m$ , and  $\|T_m\|_{L^{r,\infty}(\text{VN}(\mathbb{R}^n))}$  is the norm of the operator  $T_m$  in the Lorentz space on the group von Neumann algebra  $\text{VN}(\mathbb{R}^n)$  of  $\mathbb{R}^n$ . They use the Lorentz spaces and group von Neumann algebra technique for extending it to general locally compact unimodular groups. The unimodularity assumption has its own advantages such as existence of the canonical trace on the group von Neumann algebra, and consequently the Plancherel formula and the Hausdorff-Young inequality.

In this paper, we prove the Paley-type inequality, Hausdorff–Young–Paley inequality and Hörmander multiplier theorem for commutative hypergroups. We refer to Section 2 for more details and all the notation used here. The following theorem is an analogue of the Paley inequality for commutative hypergroups.

**THEOREM 1.1.** *Suppose that  $\psi$  is a positive function on  $\widehat{H}$  satisfying the condition*

$$M_\psi := \sup_{t>0} t \int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \geq t}} d\pi(\chi) < \infty.$$

Then for  $f \in L^p(H, d\lambda)$ ,  $1 < p \leq 2$ ,

$$\left( \int_{\widehat{H}} |\widehat{f}(\chi)|^p \psi(\chi)^{2-p} d\pi(\chi) \right)^{1/p} \lesssim M_\psi^{(2-p)/p} \|f\|_{L^p(H, d\lambda)}.$$

The following Hausdorff–Young–Paley inequality is shown for commutative hypergroups.

**THEOREM 1.2.** *Let  $1 < p \leq 2$  and let  $1 < p \leq b \leq p' \leq \infty$ , where  $p' = p/(p - 1)$ . If  $\psi(\chi)$  is a positive function on  $\widehat{H}$  such that*

$$M_\psi := \sup_{t>0} t \int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \geq t}} d\pi(\chi)$$

is finite, then for every  $f \in L^p(H, d\lambda)$ ,

$$\left( \int_{\widehat{H}} (|\widehat{f}(\chi)|\psi(\chi)^{1/b-1/p'})^b d\pi(\chi) \right)^{1/b} \lesssim M_\psi^{1/b-1/p'} \|f\|_{L^p(H, d\lambda)}.$$

Next, we establish the following  $L^p$ - $L^q$  boundedness result for Fourier multipliers on commutative hypergroups.

**THEOREM 1.3.** *Let  $1 < p \leq 2 \leq q < \infty$  and let  $H$  be a commutative hypergroup. Suppose that  $T$  is a Fourier multiplier with symbol  $h$ . Then,*

$$\|T\|_{L^p(H, d\lambda) \rightarrow L^q(H, d\lambda)} \lesssim \sup_{s>0} s \left[ \int_{\{\chi \in \widehat{H}; |h(\chi)| \geq s\}} d\pi(\chi) \right]^{1/p-1/q}.$$

Now, we establish a spectral multiplier theorem for the operator  $L$ , called the generalised radial Laplacian, given by

$$L = L_{A,x} := -\frac{d^2}{dx^2} - \frac{A'(x)}{A(x)} \frac{d}{dx},$$

where  $A$  is the Chébli–Trimèche function. Define  $\rho := \frac{1}{2} \lim_{x \rightarrow \infty} A'(x)/A(x)$ . It is known that this limit exists and is nonnegative (see Section 4.1 for more details and notation). For some particular choices of the Chébli–Trimèche function, the operator  $L$  arises as the radial part of Laplace–Beltrami operators on symmetric spaces of rank one and on Euclidean space. It has been recently observed in [11] that  $L$  appeared as the radial part of Laplace–Beltrami operators on simply connected harmonic manifolds of purely exponential volume growth with  $A$  as the density function on the harmonic manifold.

**THEOREM 1.4.** *Let  $L$  be the generalised radial Laplacian and  $1 < p \leq 2 \leq q < \infty$ . Suppose that  $\varphi$  is a monotonically decreasing continuous function on  $[\rho^2, \infty)$  such that  $\lim_{u \rightarrow \infty} \varphi(u) = 0$ . Then,*

$$\|\varphi(L)\|_{op} \lesssim \sup_{u>\rho^2} \varphi(u) \begin{cases} (u - \rho^2)^{(a+1)(1/p-1/q)} & \text{if } (u - \rho^2)^{1/2} \leq K, \\ [K^{2a+2} - K^{2\alpha+2} + (u - \rho^2)^{(\alpha+1)}]^{1/p-1/q} & \text{if } (u - \rho^2)^{1/2} > K, \end{cases}$$

where  $K, \alpha$  and  $a$  are constants appearing in the estimate of the  $c$ -function, in the definition of Chébli–Trimèche function  $A$  and in Condition (P) on  $A$ , respectively, and  $\|\cdot\|_{op}$  denotes the operator norm from  $L^p(\mathbb{R}_+, A dx)$  to  $L^q(\mathbb{R}_+, A dx)$ .

The precise values of  $K$ ,  $\alpha$  and  $a$  and their appearance in the context of analysis on Chébli–Triméche hypergroups is given in Section 4.1.

The above spectral multipliers theorem leads to very interesting results concerning the  $L^p$ – $L^q$  norm of the heat kernel of  $L$  and embedding theorems for  $L$ , by a particular choice of the function  $\varphi$ .

**COROLLARY 1.5.** *Let  $L$  be the generalised radial Laplacian. For any  $1 < p \leq 2 \leq q < \infty$ , there exists a positive constant  $C = C_{\alpha,a,p,q,K}$  such that*

$$\|e^{-tL}\|_{op} \lesssim \begin{cases} t^{-(\max\{\alpha,a+1\}+1)(1/p-1/q)} & \text{if } 0 < t < \frac{\max\{\alpha,a+1\}}{K^2} \left(\frac{1}{p} - \frac{1}{q}\right), \\ e^{-t\rho^2} t^{-(a+1)(1/p-1/q)} & \text{if } t \geq \frac{a+1}{K^2} \left(\frac{1}{p} - \frac{1}{q}\right), \end{cases}$$

where  $\|\cdot\|_{op}$  denotes the operator norm from  $L^p(\mathbb{R}_+, A dx)$  to  $L^q(\mathbb{R}_+, A dx)$ .

We also have the embeddings

$$\|f\|_{L^q(\mathbb{R}_+, A dx)} \leq C\|(1+L)^b f\|_{L^q(\mathbb{R}_+, A dx)}$$

provided that  $b \geq (\alpha+1)(1/p-1/q)$ ,  $1 < p \leq 2 \leq q < \infty$ .

We organise the paper as follows: in Section 2, we present the basics of Fourier analysis on commutative hypergroups. Section 3 is devoted to the study of  $L^p$ – $L^q$  Fourier multipliers on commutative hypergroups along with the proof of the key inequalities needed to establish our main results. In Section 4, we analyse the results of the previous section in the setting of Chébli–Triméche hypergroups and then use them to obtain the main result of this section on  $L^p$ – $L^q$  boundedness of spectral multipliers of a generalised radial Laplacian and its important consequences.

Throughout the paper, we use the notation  $A \lesssim B$  to indicate  $A \leq cB$  for a suitable constant  $c > 0$ .

## 2. Fourier analysis on commutative hypergroups

In this section, we present the essentials of Fourier analysis on commutative hypergroups. We begin this section with the definition of a hypergroup. In [27], Jewett refers to hypergroups as convos.

**DEFINITION 2.1.** A *hypergroup* is a nonempty locally compact Hausdorff space  $H$  with a weakly continuous, associative convolution  $*$  on the Banach space  $M(H)$  of all bounded regular Borel measures on  $H$  such that  $(M(H), *)$  becomes a Banach algebra and the following properties hold.

- (i) For any  $x, y \in H$ , the convolution  $\delta_x * \delta_y$  is a probability measure with compact support, where  $\delta_x$  is the point mass measure at  $x$ . Also, the mapping  $(x, y) \mapsto \text{supp}(\delta_x * \delta_y)$  is continuous from  $H \times H$  to the space  $C(H)$  of all nonempty compact subsets of  $H$  equipped with the Michael (Vietoris) topology (see [36] for details).

- (ii) There exists a unique element  $e \in H$  such that  $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$  for every  $x \in H$ .
- (iii) There is a homeomorphism  $x \mapsto \check{x}$  on  $H$  of order two that induces an involution on  $M(H)$  where  $\check{\mu}(E) = \mu(\check{E})$  for any Borel set  $E$ , and  $e \in \text{supp}(\delta_x * \delta_y)$  if and only if  $x = \check{y}$ .

Note that the weak continuity assures that the convolution of bounded measures on a hypergroup is uniquely determined by the convolution of point measures. A hypergroup is called a *commutative hypergroup* if the convolution is commutative. A hypergroup  $H$  is called *hermitian* if the involution on  $H$  is the identity map, that is,  $\check{x} = x$  for all  $x \in H$ . Note that a hermitian hypergroup is commutative. Every locally compact abelian group is a trivial example of a commutative hypergroup. Other important examples include Gelfand pairs, Bessel–Kingman hypergroups, Jacobi hypergroups and Chébli–Trimèche hypergroups.

A *left Haar measure*  $\lambda$  on  $H$  is a nonzero positive Radon measure  $\lambda$  such that

$$\int_H f(x * y) d\lambda(y) = \int_H f(y) d\lambda(y) \quad (\text{for all } x \in H, f \in C_c(H)),$$

where we use the notation  $f(x * y) = (\delta_x * \delta_y)(f)$ . It is well known that a Haar measure is unique (up to scalar multiple) if it exists [27]. Throughout this article, a left Haar measure is simply called a Haar measure. We would like to make a remark here that it is still not known if a general hypergroup has a Haar measure but several important classes of hypergroups including commutative hypergroups, compact hypergroups, discrete hypergroups and nilpotent hypergroups possess a Haar measure; see, for example, [5, 12, 29–31] for more details on harmonic analysis on hypergroups and its applications.

Denote the space of all bounded continuous complex valued functions on  $H$  by  $C^b(H)$ . The dual space of a commutative hypergroup  $H$  is defined by

$$\widehat{H} = \{\chi \in C^b(H) : \chi \neq 0, \chi(\check{x}) = \overline{\chi(x)} \text{ and } \chi(x * y) = \chi(x)\chi(y) \text{ for all } x, y \in H\}.$$

The elements of  $\widehat{H}$  are called *characters* of  $H$ . We equip  $\widehat{H}$  also with the compact-open topology so that  $H$  becomes a locally compact Hausdorff space. Note that  $\widehat{H}$  need not possess a hypergroup structure. For example, see [27, Example 9.3C].

The Fourier transform of  $f \in L^1(H, \lambda)$  is defined by

$$\widehat{f}(\chi) = \int_H f(x) \overline{\chi(x)} d\lambda(x) \quad \text{for all } \chi \in \widehat{H}.$$

By [12, Theorem 2.2.4], the mapping  $f \mapsto \widehat{f}$  is a norm-decreasing  $*$ -algebra homomorphism from  $L^1(H, \lambda)$  into  $L^\infty(\widehat{H}, \pi)$ . Furthermore,  $\widehat{f}$  vanishes at infinity. Also, as in the case of locally compact abelian groups, there exists a unique positive Borel measure  $\pi$  on  $\widehat{H}$  such that

$$\int_H |f(x)|^2 d\lambda(x) = \int_{\widehat{H}} |\widehat{f}(\chi)|^2 d\pi(\chi) \quad \text{for all } f \in L^2(H, \lambda) \cap L^1(H, \lambda).$$

In fact, the Fourier transform extends to a unitary operator from  $L^2(H, \lambda)$  onto  $L^2(\widehat{H}, \pi)$ . We would like to remark here that the support of  $\pi$ , denoted  $\mathcal{S}$ , need not be equal to  $\widehat{H}$  (see [12, Example 2.2.49]). When  $\mathcal{S} = \widehat{H}$ ,  $H$  is called a *strong hypergroup*.

The following theorem is the Hausdorff–Young inequality for commutative hypergroups [21].

**THEOREM 2.2.** *Let  $p, p'$  be such that  $1 \leq p \leq 2$  and  $1/p + 1/p' = 1$ . Then for  $f \in L^2(H, d\lambda) \cap L^1(H, d\lambda)$ , we have the inequality*

$$\| \widehat{f} \|_{L^{p'}(\widehat{H}, d\pi)} \leq \| f \|_{L^p(H, d\lambda)}.$$

### 3. Fourier multipliers for commutative hypergroups

**3.1. Paley inequality for commutative hypergroups.** In this subsection, we prove the Paley inequality for commutative hypergroups. The Paley inequality has been proved for many different settings, for example, compact homogeneous spaces [2, 3], compact quantum groups [1, 43], compact hypergroups [32], generalised Fourier transforms [33] and locally compact groups [4].

The Paley inequality is an important inequality in itself but also plays a vital role in obtaining the Hausdorff–Young–Paley inequality for commutative hypergroups. We follow the strategy of the proof of the Paley inequality in [4] in the noncommutative setting.

**THEOREM 3.1.** *Let  $H$  be a commutative hypergroup equipped with a Haar measure  $\lambda$  and let  $\widehat{H}$  be the dual of  $H$  equipped with measure  $\pi$ . Suppose that  $\psi$  is a positive function on  $\widehat{H}$  satisfying the condition*

$$M_\psi := \sup_{t>0} t \int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \geq t}} d\pi(\chi) < \infty.$$

Then for  $f \in L^p(H, d\lambda)$ ,  $1 < p \leq 2$ ,

$$\left( \int_{\widehat{H}} |\widehat{f}(\chi)|^p \psi(\chi)^{2-p} d\pi(\chi) \right)^{1/p} \lesssim M_\psi^{(2-p)/p} \| f \|_{L^p(H, d\lambda)}. \tag{3-1}$$

**PROOF.** Let us consider a measure  $\nu$  on  $\widehat{H}$  given by

$$\nu(\chi) = \psi(\chi)^2 d\pi(\chi). \tag{3-2}$$

We define the corresponding  $L^p(\widehat{H}, \nu)$ -space,  $1 \leq p < \infty$ , as the space of all complex-valued functions  $f$  defined by  $\widehat{H}$  such that

$$\| f \|_{L^p(\widehat{H}, \nu)} := \left( \int_{\widehat{H}} |f(\chi)|^p \psi(\chi)^2 d\pi(\chi) \right)^{1/p} < \infty.$$

We define a sublinear operator  $T$  for  $f \in L^p(H, d\lambda)$  by

$$Tf(\chi) := \frac{|\widehat{f}(\chi)|}{\psi(\chi)} \in L^p(\widehat{H}, \nu).$$

We show that  $T$  is well-defined and bounded from  $L^p(H, d\lambda)$  to  $L^p(\widehat{H}, \nu)$  for any  $1 < p \leq 2$ . In other words, we claim the following estimate:

$$\|Tf\|_{L^p(\widehat{H}, \nu)} = \left( \int_{\widehat{H}} \frac{|\widehat{f}(\chi)|^p}{\psi(\chi)^p} \psi(\chi)^2 d\pi(\chi) \right)^{1/p} \lesssim M_\psi^{(2-p)/p} \|f\|_{L^p(H, d\lambda)}, \tag{3-3}$$

which gives us the required inequality (3-1) with  $M_\psi := \sup_{t>0} \int_{\substack{\chi \in \widehat{H} \\ \frac{|\widehat{f}(\chi)|}{\psi(\chi)} \geq t}} d\pi(\lambda)$ . We show that  $T$  is weak-type (2, 2) and weak-type (1, 1). More precisely, with the distribution function,

$$\nu(y; Tf) = \int_{\substack{\chi \in \widehat{H} \\ \frac{|\widehat{f}(\chi)|}{\psi(\chi)} \geq y}} \psi(\chi)^2 d\pi(\chi),$$

where  $\nu$  is given by (3-2), we show that

$$\nu(y; Tf) \leq \left( \frac{M_2 \|f\|_{L^2(H, d\lambda)}}{y} \right)^2 \quad \text{with norm } M_2 = 1, \tag{3-4}$$

$$\nu(y; Tf) \leq \frac{M_1 \|f\|_{L^1(H, d\lambda)}}{y} \quad \text{with norm } M_1 = M_\psi. \tag{3-5}$$

Then the estimate (3-3) follows from the Marcinkiewicz interpolation theorem. Now, we show (3-4). Using the Plancherel identity,

$$\begin{aligned} y^2 \nu(y; Tf) &\leq \sup_{y>0} y^2 \nu(y; Tf) =: \|Tf\|_{L^{2,\infty}(\widehat{H}, \nu)}^2 \leq \|Tf\|_{L^2(\widehat{H}, \nu)}^2 \\ &= \int_{\widehat{H}} \left( \frac{|\widehat{f}(\chi)|}{\psi(\chi)} \right)^2 \psi(\chi)^2 d\pi(\chi) \\ &= \int_{\widehat{H}} |\widehat{f}(\chi)|^2 d\pi(\chi) = \|f\|_2^2. \end{aligned}$$

Thus,  $T$  is type (2, 2) with norm  $M_2 \leq 1$ . Further, we show that  $T$  is of weak type (1, 1) with norm  $M_1 = M_\psi$ ; more precisely, we show that

$$\nu\left\{ \chi \in \widehat{H} : \frac{|\widehat{f}(\chi)|}{\psi(\chi)} > y \right\} \lesssim M_\psi \frac{\|f\|_{L^1(H, d\lambda)}}{y}. \tag{3-6}$$

The left-hand side is an integral  $\int \psi(\chi) d\pi(\chi)$  taken over all those  $\chi \in \widehat{H}$  for which  $|\widehat{f}(\chi)|/\psi(\chi) > y$ . Since  $|\widehat{f}(\chi)| \leq \|f\|_1$  for all  $\chi \in \widehat{H}$ ,

$$\left\{ \chi \in \widehat{H} : \frac{|\widehat{f}(\chi)|}{\psi(\chi)} > y \right\} \subset \left\{ \chi \in \widehat{H} : \frac{\|f\|_1}{\psi(\chi)} > y \right\}$$

for any  $y > 0$  and, therefore,

$$\nu\left\{\chi \in \widehat{H} : \frac{|\widehat{f}(\chi)|}{\psi(\chi)} > y\right\} \leq \nu\left\{\chi \in \widehat{H} : \frac{\|f\|_1}{\psi(\chi)} > y\right\}.$$

Now by setting  $w := \|f\|_1/y$ ,

$$\nu\left\{\chi \in \widehat{H} : \frac{\|f\|_1}{\psi(\chi)} > y\right\} \leq \int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \leq w}} \psi(\chi)^2 d\pi(\chi).$$

Now we claim that

$$\int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \leq w}} \psi(\chi)^2 d\pi(\chi) \lesssim M_\psi w. \tag{3-7}$$

Indeed, first we notice that

$$\int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \leq w}} \psi(\chi)^2 d\pi(\chi) = \int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \leq w}} d\pi(\chi) \int_0^{\psi(\chi)^2} d\tau.$$

By interchanging the order of integration,

$$\int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \leq w}} d\pi(\chi) \int_0^{\psi(\chi)^2} d\tau = \int_0^{w^2} d\tau \int_{\substack{\chi \in \widehat{H} \\ \tau^{1/2} \leq \psi(\chi) \leq w}} d\pi(\chi).$$

Further, by making substitution  $\tau = t^2$ ,

$$\begin{aligned} \int_0^{w^2} d\tau \int_{\substack{\chi \in \widehat{H} \\ \tau^{1/2} \leq \psi(\chi) \leq w}} d\pi(\chi) &= 2 \int_0^w t dt \int_{\substack{\chi \in \widehat{H} \\ t \leq \psi(\chi) \leq w}} d\pi(\chi) \\ &\leq 2 \int_0^w t dt \int_{\substack{\chi \in \widehat{H} \\ t \leq \psi(\chi)}} d\pi(\chi). \end{aligned}$$

Since

$$t \int_{\substack{\chi \in \widehat{H} \\ t \leq \psi(\chi)}} d\pi(\chi) \leq \sup_{t>0} t \int_{\substack{\chi \in \widehat{H} \\ t \leq \psi(\chi)}} d\pi(\chi) = M_\psi$$

is finite by the assumption that  $M_\psi < \infty$ ,

$$2 \int_0^w t dt \int_{\substack{\chi \in \widehat{H} \\ t \leq \psi(\chi)}} d\pi(\chi) \lesssim M_\psi w.$$



This establishes our claim (3-7) and eventually proves (3-6). So, we have proved (3-4) and (3-5). Then by using the Marcinkiewicz interpolation theorem with  $p_1 = 1$  and  $p_2 = 2$ , and  $1/p = 1 - \theta + \theta/2$ , we now obtain

$$\left( \int_{\widehat{H}} \left( \frac{|\widehat{f}(\chi)|}{\psi(\chi)} \right)^p \psi(\chi)^2 d\pi(\chi) \right)^{1/p} = \|Tf\|_{L^p(\widehat{H}, \nu)} \lesssim M_\psi^{(2-p)/p} \|f\|_{L^p(H, d\lambda)}.$$

This completes the proof of the theorem. □

**3.2. Hausdorff–Young–Paley inequality for commutative hypergroups.** In this subsection, we prove the Hausdorff–Young–Paley inequality for commutative hypergroups. The Hausdorff–Young–Paley inequality is an important inequality in itself but it serves as an essential tool to prove the  $L^p$ – $L^q$  Fourier multiplier for commutative hypergroups. The following theorem [9] is useful to obtain our result.

**THEOREM 3.2.** *Let  $d\mu_0(x) = \omega_0(x)d\mu'(x)$ ,  $d\mu_1(x) = \omega_1(x)d\mu'(x)$  and write  $L^p(\omega) = L^p(\omega d\mu')$  for the weight  $\omega$ . Suppose that  $0 < p_0, p_1 < \infty$ . Then,*

$$(L^{p_0}(\omega_0), L^{p_1}(\omega_1))_{\theta, p} = L^p(\omega),$$

where  $0 < \theta < 1$ ,  $1/p = (1 - \theta)/p_0 + \theta/p_1$  and  $\omega = \omega_0^{p(1-\theta)/p_0} \omega_1^{p\theta/p_1}$ .

The following corollary is immediate.

**COROLLARY 3.3.** *Let  $d\mu_0(x) = \omega_0(x)d\mu'(x)$ ,  $d\mu_1(x) = \omega_1(x)d\mu'(x)$ . Suppose that  $0 < p_0, p_1 < \infty$ . If a continuous linear operator  $A$  admits bounded extensions,  $A : L^p(Y, \mu) \rightarrow L^{p_0}(\omega_0)$  and  $A : L^p(Y, \mu) \rightarrow L^{p_1}(\omega_1)$ , then there exists a bounded extension  $A : L^p(Y, \mu) \rightarrow L^b(\omega)$  of  $A$ , where  $0 < \theta < 1$ ,  $1/b = (1 - \theta)/p_0 + \theta/p_1$  and  $\omega = \omega_0^{b(1-\theta)/p_0} \omega_1^{b\theta/p_1}$ .*

Using the above corollary, we now present the Hausdorff–Young–Paley inequality.

**THEOREM 3.4 (Hausdorff–Young–Paley inequality).** *Let  $H$  be a commutative hypergroup equipped with a Haar measure  $\lambda$  and let  $\widehat{H}$  be the dual of  $H$  equipped with measure  $\pi$ . Let  $1 < p \leq 2$  and let  $1 < p \leq b \leq p' < \infty$ , where  $p' = p/(p - 1)$ . If  $\psi(\chi)$  is a positive function on  $\widehat{H}$  such that*

$$M_\psi := \sup_{t>0} t \int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \geq t}} d\pi(\chi)$$

is finite, then for every  $f \in L^p(H, d\lambda)$ ,

$$\left( \int_{\widehat{H}} (|\widehat{f}(\chi)|\psi(\chi)^{1/b-1/p'})^b d\pi(\chi) \right)^{1/b} \lesssim M_\psi^{1/b-1/p'} \|f\|_{L^p(H, d\lambda)}.$$

This naturally reduces to the Hausdorff–Young inequality when  $b = p'$  and to the Paley inequality (3-1) when  $b = p$ .

**PROOF.** From Theorem 3.1, the operator defined by

$$Af(\chi) = \widehat{f}(\chi), \quad \chi \in \widehat{H}$$

is bounded from  $L^p(H, d\lambda)$  to  $L^p(\widehat{H}, \omega_0 d\mu')$ , where  $d\mu'(\chi) = d\pi(\chi)$  and  $\omega_0(\chi) = \psi(\chi)^{2-p}$ . From Theorem 2.2, we deduce that  $A : L^p(H, d\lambda) \rightarrow L^{p'}(\widehat{H}, \omega_1 d\mu')$  with  $d\mu'(\chi) = d\pi(\chi)$ , and  $\omega_1(\chi) = 1$  admits a bounded extension. By using the real interpolation (Corollary 3.3), we prove that  $A : L^p(H, d\lambda) \rightarrow L^b(\widehat{H}, \omega d\mu')$ ,  $p \leq b \leq p'$ , is bounded, where the space  $L^b(\widehat{H}, \omega d\mu')$  is defined by the norm

$$\|\sigma\|_{L^b(\widehat{H}, \omega d\mu')} := \left( \int_{\widehat{H}} |\sigma(\chi)|^b w(\chi) d\mu'(\chi) \right)^{1/b} = \left( \int_{\widehat{H}} |\sigma(\chi)|^b w(\chi) d\pi(\chi) \right)^{1/b}$$

and  $\omega(\chi)$  is a positive function over  $\widehat{H}$  to be determined. To compute  $\omega$ , we can use Corollary 3.3 by fixing  $\theta \in (0, 1)$  such that  $1/b = (1 - \theta)/p + \theta/p'$ . In this case,  $\theta = (p - b)/b(p - 2)$  and

$$\omega = \omega_0^{b(1-\theta)/p} \omega_1^{b\theta/p'} = \psi(\chi)^{1-b/p'}.$$

Thus, we finish the proof. □

**3.3.  $L^p$ - $L^q$  boundedness of Fourier multipliers on commutative hypergroups.**

We begin with the definition of Fourier multipliers on commutative hypergroups. For a function  $h \in L^\infty(\widehat{H}, d\pi)$ , define the operator  $T_h$  as

$$\widehat{T_h f}(\chi) = h(\chi)\widehat{f}(\chi), \quad \chi \in \widehat{H},$$

for all  $f$  belonging to a suitable function space on  $\widehat{H}$ . The operator  $T_h$  is called the Fourier multiplier on  $H$  with symbol  $h$ . It is clear that  $T_h$  is a bounded operator on  $L^2(H, d\lambda)$  by the Plancherel theorem. A Fourier multiplier commutes with the translation operators. In fact, the Fourier multipliers can be characterised using translation operators [12, 34, 38].

In the next result, we show that if the symbol  $h$  of a Fourier multiplier  $A$  defined on  $C_c(H)$  satisfies a certain Hörmander’s condition, then  $T_h$  can be extended as a bounded linear operator from  $L^p(H, d\lambda)$  to  $L^q(H, d\lambda)$  for the range  $1 < p \leq 2 \leq q < \infty$ .

**THEOREM 3.5.** *Let  $1 < p \leq 2 \leq q < \infty$  and let  $H$  be a commutative hypergroup. Suppose that  $T$  is a Fourier multiplier with symbol  $h$ . Then,*

$$\|T\|_{L^p(H, d\lambda) \rightarrow L^q(H, d\lambda)} \lesssim \sup_{s>0} s \left[ \int_{\{\chi \in \widehat{H} : |h(\chi)| \geq s\}} d\pi(\chi) \right]^{1/p-1/q}. \tag{3-8}$$

**PROOF.** Since for  $p = q = 2$  the inequality (3-8) always holds, we assume that  $1 < p \leq 2 \leq q < \infty$  with  $p$  and  $q$  both not equal to 2. Let us first assume that  $p \leq q'$ , where  $1/q + 1/q' = 1$ . Since  $q' \leq 2$ , dualising the Hausdorff–Young inequality gives that

$$\|Tf\|_{L^q(H, d\lambda)} \leq \|\widehat{Tf}\|_{L^{q'}(\widehat{H}, d\pi)} = \|h\widehat{f}\|_{L^{q'}(\widehat{H}, d\pi)}.$$

The case  $q' \leq (p')' = p$  can be reduced to the case  $p \leq q'$  as follows. Using the duality of  $L^p$ -spaces, we have  $\|T\|_{L^p(H, d\lambda) \rightarrow L^q(H, d\lambda)} = \|T^*\|_{L^q(H, d\lambda) \rightarrow L^{p'}(H, d\lambda)}$ . The symbol of adjoint operator  $T^*$  is equal to  $\bar{h}$  and obviously we have  $|\bar{h}| = |h|$  (see [2, Theorem 4.2]). Now, we are in a position to apply Theorem 3.4. Set  $1/p - 1/q = 1/r$ . Now, by applying Theorem 3.4 with  $\psi = |h|^r$  with  $b = q'$ ,

$$\|h\widehat{f}\|_{L^{q'}(H, d\lambda)} \lesssim \left( \sup_{s>0} s \int_{\substack{\chi \in \widehat{H} \\ |h(\chi)|^r > s}} d\pi(\chi) \right)^{1/r} \|f\|_{L^p(H, d\lambda)}$$

for all  $f \in L^p(H, d\lambda)$ , in view of  $1/p - 1/q = 1/q' - 1/p' = 1/r$ . Thus, for  $1 < p \leq 2 \leq q < \infty$ ,

$$\|Tf\|_{L^q(H, d\lambda)} \lesssim \left( \sup_{s>0} s \int_{\substack{\chi \in \widehat{H} \\ |h(\chi)|^r > s}} d\pi(\chi) \right)^{1/r} \|f\|_{L^p(H, d\lambda)}.$$

Further, the proof follows from the following easy identities:

$$\begin{aligned} \left( \sup_{s>0} s \int_{\substack{\chi \in \widehat{H} \\ |h(\chi)|^r > s}} d\pi(\chi) \right)^{1/r} &= \left( \sup_{s>0} s \int_{\substack{\chi \in \widehat{H} \\ |h(\chi)| > s^{1/r}}} d\pi(\chi) \right)^{1/r} = \left( \sup_{s>0} s^r \int_{\substack{\chi \in \widehat{H} \\ |h(\chi)| > s}} d\pi(\chi) \right)^{1/r} \\ &= \sup_{s>0} s \left( \int_{\substack{\chi \in \widehat{H} \\ |h(\chi)| > s}} d\pi(\chi) \right)^{1/r}, \end{aligned}$$

proving Theorem 3.5. □

### 4. $L^p$ – $L^q$ multipliers on Chébli–Trimèche hypergroups

**4.1. Interlude on Chébli–Trimèche hypergroups.** An important and motivating example of a hypergroup is the algebra of finite radial measures on a noncompact rank-one symmetric space under the usual convolution; as radial measures on a noncompact symmetric space can be viewed as measures on  $\mathbb{R}_+ := [0, \infty)$ , this endows  $\mathbb{R}_+$  with a hypergroup structure with involution being the identity map. The hypergroup structures arising in this way on  $\mathbb{R}_+$  are a particular case of a general class of hypergroup structures on  $\mathbb{R}_+$  arising from the Strum–Liouville boundary value problems where the solutions coincide with the characters of the hypergroup in question [13, Section 3.5].

In this paper, we are interested in a special class of ‘one-dimensional hypergroups’ [44] on  $\mathbb{R}_+$  called *Chébli–Trimèche hypergroups* with the convolution structure related to the following second-order differential operator:

$$L = L_{A,x} := -\frac{d^2}{dx^2} - \frac{A'(x)}{A(x)} \frac{d}{dx}, \tag{4-1}$$

where the function  $A$ , called a *Chébli–Trimèche function*, is continuous on  $\mathbb{R}_+$ , twice continuously differentiable on  $\mathbb{R}_+^* := (0, \infty)$  and satisfies the following properties:

- (i)  $A(0) = 0$  and  $A$  is positive on  $\mathbb{R}_+^*$ ;
- (ii)  $A$  is an increasing function and  $A(x) \rightarrow \infty$  as  $x \rightarrow \infty$ ;
- (iii)  $A'/A$  is a decreasing  $C^\infty$ -function on  $\mathbb{R}_+^*$ ; hence  $\rho := \frac{1}{2} \lim_{x \rightarrow \infty} A'(x)/A(x) \geq 0$  exists;
- (iv)  $A'(x)/A(x) = (2\alpha + 1)/x + B(x)$  on a neighbourhood of 0, where  $\alpha > -\frac{1}{2}$  and  $B$  is an odd  $C^\infty$ -function on  $\mathbb{R}$ .

Now, we define the main object of this section, namely, the Chébli–Trimèche hypergroups. For a detailed study of these hypergroups, one can refer to [12, 19, 41].

**DEFINITION 4.1.** [13, 14, 44] A hypergroup  $(\mathbb{R}_+, *)$  is called a Chébli–Trimèche hypergroup if there exists a Chébli–Trimèche function  $A$  such that for any real-valued  $C^\infty$ -function  $f$  on  $\mathbb{R}_+$ , that is, the restriction of an even nonnegative  $C^\infty$ -function on  $\mathbb{R}$ , the generalised translation  $u(x, y) = T_x f(y) := \int_0^\infty f(t) d(\delta_x * \delta_y)(t)$ ,  $y \in \mathbb{R}_+$  is the solution of the following Cauchy problem:

$$\begin{aligned} (L_{A,x} - L_{A,y})u(x, y) &= 0, \\ u(x, 0) = f(x), \quad u_y(x, 0) &= 0, \quad x > 0. \end{aligned}$$

The Chébli–Trimèche hypergroup associated with the Chébli–Trimèche function  $A$  is denoted by  $(\mathbb{R}_+, *(A))$ . The growth of  $(\mathbb{R}_+, *(A))$  is determined by the number  $\rho := \frac{1}{2} \lim_{x \rightarrow \infty} A'(x)/A(x)$ . We say that  $(\mathbb{R}_+, *(A))$  is of *exponential growth* if and only if  $\rho > 0$ . Otherwise, we say that the hypergroup is of *subexponential growth*, which also includes polynomial growth.

The class of Chébli–Trimèche hypergroups contains many important classes of hypergroups. We discuss three of them here.

- (i) If  $A(x) := x^{2\alpha+1}$  with  $2\alpha \in \mathbb{N}$  and  $\alpha > -\frac{1}{2}$ , then  $L_{A,x}$  is the radial part of the Laplace operator on the Euclidean space and  $(\mathbb{R}_+, *(A))$  is a Bessel–Kingman hypergroup.
- (ii) If  $A(x) := (\sinh x)^{2\alpha+1} (\cosh x)^{2\beta+1}$  with  $2\alpha, 2\beta \in \mathbb{N}$ ,  $\alpha \geq \beta \geq -\frac{1}{2}$  and  $\alpha \neq -\frac{1}{2}$ , then  $L_{A,x}$  is the radial part of the Laplace–Beltrami operator on a noncompact Riemannian symmetric space of rank one (also of Damek–Ricci spaces) and  $(\mathbb{R}_+, *(A))$  is a Jacobi hypergroup [7, 12].
- (iii) If  $A$  is the density function on the simply connected harmonic manifold  $X$  of purely exponential volume growth, then  $L_{A,x}$  is the radial part of the Laplace–Beltrami operator on  $X$  and  $(\mathbb{R}_+, *(A))$  is the ‘radial hypergroup’ of  $X$  (see [11]).

The Chébli–Trimèche hypergroup  $(\mathbb{R}_+, *(A))$  is a noncompact commutative hypergroup with identity element 0 and involution as the identity map. The Haar measure  $m$  on  $(\mathbb{R}_+, *(A))$  is given by  $m(x) := A(x) dx$ , where  $dx$  is the usual Lebesgue measure

on  $\mathbb{R}_+$ . For any  $x, y \in \mathbb{R}_+$ , the probability measure  $\delta_x * \delta_y$  is absolutely continuous with respect to  $m$  and  $\text{supp}(\delta_x * \delta_y) \subset [|x - y|, x + y]$ . For  $1 \leq p \leq \infty$ , the Lebesgue space  $L^p(\mathbb{R}_+, m)$  on  $(\mathbb{R}_+, *(A))$  is defined as usual; we denote by  $\|f\|_{p,A}$  the  $L^p$ -norm of  $f \in L^p(\mathbb{R}_+, A dx)$ .

For the Chébli–Trimèche hypergroup  $(\mathbb{R}_+, *(A))$ , the multiplicative functions on  $(\mathbb{R}_+, *(A))$  are given by the eigenfunctions of the operator  $L := L_{A,x}$  defined in (4-1). For any  $\lambda \in \mathbb{C}$ , the equation

$$Lu = (\lambda^2 + \rho^2)u \tag{4-2}$$

has a unique solution  $\phi_\lambda$  on  $\mathbb{R}_+^*$  that extends continuously to 0 and satisfies  $\phi_\lambda(0) = 1$ . We point out here that since the coefficient  $A'/A$  of  $L$  is singular at  $x = 0$  as  $A(0) = 0$ , the existence of a solution continuous at 0 is not immediate. Since (4-2) reads the same for  $\lambda$  and  $-\lambda$ , by uniqueness, we have  $\phi_\lambda = \phi_{-\lambda}$ . The multiplicative functions, that is, semicharacters, on  $(\mathbb{R}_+, *(A))$  are then exactly the functions  $\phi_\lambda$ ,  $\lambda \in \mathbb{C}$ . However, a semicharacter  $\phi_\lambda$  is bounded, and therefore becomes a character, if and only if  $|\text{Im}\lambda| \leq \rho$ . Since the involution of  $(\mathbb{R}_+, *(A))$  is just the identity map, it follows that the characters  $\phi_\lambda$  of the hypergroup  $(\mathbb{R}_+, *(A))$  are real-valued, which is the case for every  $\lambda \in \mathbb{R} \cup i\mathbb{R}$ . Therefore, in view of the above discussion, the dual space  $\widehat{\mathbb{R}}_+$  of  $(\mathbb{R}_+, *(A))$  is described by  $\{\phi_\lambda : \lambda \in [0, \infty) \cup [0, i\rho]\}$ , which can be identified with the parameter set  $\mathbb{R}_+ \cup [0, i\rho]$ .

We define the Fourier transform  $\widehat{f}$  of  $f \in L^1(\mathbb{R}_+, A dx)$  at a point  $\lambda \in \widehat{\mathbb{R}}_+$  by

$$\widehat{f}(\lambda) := \int_0^\infty f(x) \phi_\lambda(x) A(x) dx.$$

It is worth noting that the aforementioned Fourier transform turns out to be a Jacobi transform [28] when the hypergroup arises from convolution of radial measures on a rank-one symmetric space or, equivalently,  $A(x) := (\sinh x)^{2\alpha+1} (\cosh x)^{2\beta+1}$  with  $\alpha \geq \beta \geq -\frac{1}{2}$  and  $\alpha \neq -\frac{1}{2}$ . The following theorem is the Levitan–Plancherel theorem for  $(\mathbb{R}_+, *(A))$ .

**THEOREM 4.2** [12, Theorem 2.1]. *There exists a unique nonnegative measure  $\pi$  on  $\widehat{\mathbb{R}}_+$  with support  $[0, \infty)$  such that the Fourier transform induces an isometric isomorphism from  $L^2(\mathbb{R}_+, A dx)$  onto  $L^2(\widehat{\mathbb{R}}_+, \pi)$  and for any  $f \in L^1(\mathbb{R}_+, A dx) \cap L^2(\mathbb{R}_+, A dx)$ ,*

$$\int_0^\infty |f(x)|^2 A(x) dx = \int_{\widehat{\mathbb{R}}_+} |\widehat{f}(\lambda)|^2 d\pi(\lambda).$$

In addition to the aforementioned theorem, we also have the following identity known as Parseval’s identity. For  $f_1, f_2 \in L^2(\mathbb{R}_+, A dx)$ ,

$$\int_0^\infty f_1(x) f_2(x) A(x) dx = \int_{\widehat{\mathbb{R}}_+} \widehat{f}_1(\lambda) \widehat{f}_2(\lambda) d\pi(\lambda).$$

*Condition (P):* We say that a function  $f$  satisfies *Condition (P)* if for some  $a > 0$ ,  $f$  can be expressed as

$$f(x) = \frac{a^2 - \frac{1}{4}}{x^2} + \zeta(x)$$

for all large  $x$ , where

$$\int_{x_0}^{\infty} x^{\gamma(a)} |\zeta(x)| dx < \infty$$

for some  $x_0 > 0$  and  $\zeta(x)$  is bounded for  $x > x_0$ ; here  $\gamma(a) = a + \frac{1}{2}$  if  $a \geq \frac{1}{2}$  and  $\gamma(a) = 1$  otherwise.

Now consider the function  $G$  defined by

$$G(x) := \frac{1}{4} \left( \frac{A'(x)}{A(x)} \right)^2 + \frac{1}{2} \frac{d}{dx} \left( \frac{A'(x)}{A(x)} \right) - \rho^2.$$

It is possible to determine the Plancherel measure  $\pi$  explicitly. In fact, Bloom and Xu [13] determined it by placing an extra growth condition on  $A$  described as follows.

**THEOREM 4.3** [13, Proposition 3.17]. *If the function  $G$  defined previously satisfies Condition (P) with any of the following conditions:*

- (i)  $a > \frac{1}{2}$ ;
- (ii)  $a \neq |\alpha|$ , where  $\alpha$  is the constant appearing in the definition of Chébli–Trimèche function;
- (iii)  $a = \alpha \leq \frac{1}{2}$  and  $\int_0^{\infty} x^{1/2-\alpha} \zeta(x) \phi_0(x) A(x)^{1/2} dx \neq -2\alpha \sqrt{M_A}$  or  $\int_0^{\infty} x^{1/2+\alpha} \zeta(x) \phi_0(x) A(x)^{1/2} dx = 0$  where  $M_A = \lim_{x \rightarrow 0^+} x^{-2\alpha-1} A(x)$  and  $\zeta(x) = G(x) + (\frac{1}{4} - a^2)/x^2$ ,

then the Plancherel measure  $\pi$  is absolutely continuous with respect to the Lebesgue measure and has density  $|c(\lambda)|^2$ , where the function  $c$  satisfies the following: there exist positive constants  $C_1, C_2$  and  $K$  such that for any  $\lambda \in \mathbb{C}$  with  $\text{Im}(\lambda) \leq 0$ ,

$$C_1 |\lambda|^{a+1/2} \leq |c(\lambda)|^{-1} \leq C_2 |\lambda|^{a+1/2} \quad \text{for } |\lambda| \leq K,$$

$$C_1 |\lambda|^{a+1/2} \leq |c(\lambda)|^{-1} \leq C_2 |\lambda|^{a+1/2} \quad \text{for } |\lambda| > K.$$

In the following, we always assume that the function  $A$  (and so  $G$ ) satisfies Condition (P) of Theorem 4.3 for  $\alpha \geq 0$ . Therefore,  $d\pi = C_0 |c(\lambda)|^{-2} d\lambda$ , where  $C_0$  is a positive constant and  $c$  is a certain complex function on  $\mathbb{C} \setminus \{0\}$ . We also assume that if  $\rho = 0$ ,  $A$  also satisfies  $A(x) = O(x^{2\alpha+1})$ , as  $x \rightarrow \infty$ .

Note that the space of infinitely differentiable and compactly supported functions on  $\mathbb{R}_+$ , denoted by  $C_c^\infty(\mathbb{R}_+)$ , is the space of all even  $f \in C_c^\infty(\mathbb{R})$  restricted to  $\mathbb{R}_+$ .

We have the following Fourier inversion formula for  $f \in C_c^\infty(\mathbb{R}_+)$  :

$$f(x) = C_0 \int_0^\infty \widehat{f}(\lambda)\phi_\lambda(x) |c(\lambda)|^{-2} d\lambda.$$

For any  $\lambda \in \mathbb{C}$ ,  $\phi_\lambda$  is an even  $C^\infty$ -function and  $\lambda \mapsto \phi_\lambda(x)$  is analytic. Also,  $|\phi_\lambda(x)| \leq 1$  for all  $x \in \mathbb{R}_+$ ,  $\lambda \in \mathbb{C}$  with  $|\operatorname{Im} \lambda| \leq \rho$ . For each  $\lambda \in \mathbb{C}$ ,  $\phi_\lambda$  has a Laplace representation

$$\phi_\lambda(x) = \int_{-x}^x e^{(i\lambda-\rho)t} d\nu_x(t) \quad \text{for } x \in \mathbb{R}_+,$$

where  $\nu_x$  is a probability measure on  $\mathbb{R}$  supported in  $[-x, x]$ . Further, if  $\lambda \in \mathbb{C}$ , then  $|\phi_\lambda| \leq C(1+x)e^{(\operatorname{Im} \lambda-\rho)x}$ .

**4.2.  $L^p$ – $L^q$  boundedness of multipliers on Chébli–Trimèche hypergroups.** The following theorem is an analogue of the Paley inequality for Chébli–Trimèche hypergroups on half line.

**THEOREM 4.4.** *Suppose that  $\psi$  is a positive function on  $\mathbb{R}_+$  satisfying the condition*

$$M_\psi := \sup_{t>0} t \int_{\substack{\lambda \in \mathbb{R}_+ \\ \psi(\lambda) \geq t}} |c(\lambda)|^{-2} d\lambda < \infty.$$

Then for  $f \in L^p(\mathbb{R}_+, A dx)$ ,  $1 < p \leq 2$ ,

$$\left( \int_0^\infty |\widehat{f}(\lambda)|^p \psi(\lambda)^{2-p} |c(\lambda)|^{-2} d\lambda \right)^{1/p} \lesssim M_\psi^{(2-p)/p} \|f\|_{L^p(\mathbb{R}_+, A dx)}.$$

The following Hausdorff–Young–Paley inequality holds for the Chébli–Trimèche hypergroups.

**THEOREM 4.5.** *Let  $1 < p \leq 2$  and let  $1 < p \leq b \leq p' < \infty$ , where  $p' = p/(p-1)$ . If  $\psi(\lambda)$  is a positive function on  $\mathbb{R}_+$  such that*

$$M_\psi := \sup_{t>0} t \int_{\substack{\lambda \in \mathbb{R}_+ \\ \psi(\lambda) \geq t}} |c(\lambda)|^{-2} d\lambda$$

is finite, then for every  $f \in L^p(\mathbb{R}_+, A dx)$ ,

$$\left( \int_{\mathbb{R}_+} (|\widehat{f}(\lambda)|\psi(\lambda)^{1/b-1/p'})^b |c(\lambda)|^{-2} d\lambda \right)^{1/b} \lesssim M_\psi^{1/b-1/p'} \|f\|_{L^p(\mathbb{R}_+, A dx)}.$$

Next, we establish the following  $L^p$ – $L^q$  boundedness result for Fourier multipliers on Chébli–Trimèche hypergroups.

**THEOREM 4.6.** *Let  $1 < p \leq 2 \leq q < \infty$  and let  $T$  be a Fourier multiplier with symbol  $h$ . Then,*

$$\|T\|_{L^p(\mathbb{R}_+, A dx) \rightarrow L^q(\mathbb{R}_+, A dx)} \lesssim \sup_{s>0} s \left[ \int_{\{\lambda \in \mathbb{R}_+ : |h(\lambda)| \geq s\}} |c(\lambda)|^{-2} d\lambda \right]^{1/p-1/q}.$$

**REMARK 1.** It is well known that it is necessary for a Fourier multiplier  $T_h$  to be bounded from  $L^p(\mathbb{R}_+, A dx)$  to  $L^p(\mathbb{R}_+, A dx)$  so that the symbol  $h$  must be holomorphic in a strip in the complex plane (see [15]). It is evident from our theorem that it is no longer a necessary condition for  $T_h$  to be bounded from  $L^p$  to  $L^q$ , for  $1 < p \leq 2 \leq q < \infty$ .

**4.3. Spectral multipliers of the generalised radial Laplacian.** Now, we apply Theorem 3.5 to prove the  $L^p$ - $L^q$  boundedness of spectral multipliers for the operator  $L := L_{A,x}$ . If  $\varphi \in L^\infty(\mathbb{R}_+, A dx)$ , the spectral multiplier  $\varphi(L)$ , defined by  $\varphi$  coincides with the Chébli–Trimèche Fourier multiplier  $T_h$  with  $h(\lambda) = \varphi(\lambda^2 + \rho^2)$  for  $\lambda \in \mathbb{R}_+$ . The  $L^p$ -boundedness of spectral multipliers has been proved by several authors in many different settings, for example, Bessel transforms [10], Dunkl harmonic oscillators [42] and multidimensional Hankel transforms [24]. The  $L^p$ - $L^q$  boundedness of spectral multipliers for compact Lie groups, Heisenberg groups and graded Lie groups has been proved by Akylzhanov and the second author [4]. Chatzakou and the first author recently studied  $L^p$ - $L^q$  boundedness of a spectral multiplier for the anharmonic oscillator [17, 18] (see also [16]). Now, we state the main result of this subsection.

**THEOREM 4.7.** *Let  $1 < p \leq 2 \leq q < \infty$  and let  $\varphi$  be a monotonically decreasing continuous function on  $[\rho^2, \infty)$  such that  $\lim_{u \rightarrow \infty} \varphi(u) = 0$ . Then,*

$$\|\varphi(L)\|_{op} \lesssim \sup_{u > \rho^2} \varphi(u) \begin{cases} (u - \rho^2)^{(\alpha+1)(1/p-1/q)} & \text{if } (u - \rho^2)^{1/2} \leq K, \\ [K^{2\alpha+2} - K^{2\alpha+2} + (u - \rho^2)^{(\alpha+1)}]^{1/p-1/q} & \text{if } (u - \rho^2)^{1/2} > K, \end{cases}$$

where  $K$  is a constant appearing in the estimate of the  $c$ -function and  $\|\cdot\|_{op}$  denotes the operator norm from  $L^p(\mathbb{R}_+, A dx)$  to  $L^q(\mathbb{R}_+, A dx)$ .

**PROOF.** Since  $\varphi(L)$  is a Fourier multiplier with the symbol  $\varphi(\lambda^2 + \rho^2)$ , as an application of Theorem 4.6,

$$\begin{aligned} \|\varphi(L)\|_{op} &\lesssim \sup_{s > 0} s \left[ \int_{\{\lambda \in \mathbb{R}_+ : \varphi(\lambda^2 + \rho^2) \geq s\}} |c(\lambda)|^{-2} d\lambda \right]^{1/p-1/q} \\ &= \sup_{0 < s < \varphi(\rho^2)} s \left[ \int_{\{\lambda \in \mathbb{R}_+ : \varphi(\lambda^2 + \rho^2) \geq s\}} |c(\lambda)|^{-2} d\lambda \right]^{1/p-1/q}, \end{aligned}$$

since  $\varphi \leq \varphi(\rho^2)$ . Now, as  $s \in (0, \varphi(\rho^2)]$ , we can write  $s = \varphi(u)$  for some  $u \in [\rho^2, \infty)$  and, therefore,

$$\|\varphi(L)\|_{op} \lesssim \sup_{\varphi(u) < \varphi(\rho^2)} \varphi(u) \left[ \int_{\{\lambda \in \mathbb{R}_+ : \varphi(\lambda^2 + \rho^2) \geq \varphi(u)\}} |c(\lambda)|^{-2} d\lambda \right]^{1/p-1/q}.$$



Since  $\varphi$  is monotonically decreasing,

$$\|\varphi(L)\|_{\text{op}} \lesssim \sup_{u>\rho^2} \varphi(u) \left[ \int_{\{\lambda \in \mathbb{R}_+ : \lambda \leq (u-\rho^2)^{1/2}\}} |c(\lambda)|^{-2} d\lambda \right]^{1/p-1/q}.$$

Now, we use the estimate of the  $c$ -function (see Theorem 4.3) to get

$$\begin{aligned} \|\varphi(L)\|_{\text{op}} &\lesssim \sup_{u>\rho^2} \varphi(u) \begin{cases} \left[ \int_0^{(u-\rho^2)^{1/2}} \lambda^{2a+1} d\lambda \right]^{1/p-1/q} & \text{if } (u-\rho^2)^{1/2} \leq K, \\ \left[ \int_0^K \lambda^{2a+1} d\lambda + \int_K^{(u-\rho^2)^{1/2}} \lambda^{2\alpha+1} d\lambda \right]^{1/p-1/q} & \text{if } (u-\rho^2)^{1/2} > K \end{cases} \\ &= \sup_{u>\rho^2} \varphi(u) \begin{cases} (u-\rho^2)^{(a+1)(1/p-1/q)} & \text{if } (u-\rho^2)^{1/2} \leq K, \\ [K^{2a+2} - K^{2\alpha+2} + (u-\rho^2)^{\alpha+1}]^{1/p-1/q} & \text{if } (u-\rho^2)^{1/2} > K, \end{cases} \end{aligned}$$

proving Theorem 4.7. □

**4.3.1. Heat equation.** Let us consider the  $L$ -heat equation

$$\partial_t u + Lu = 0, \quad u(0) = u_0 \in L^p(\mathbb{R}_+, A dx), \quad 1 < p \leq 2. \tag{4-3}$$

One can verify that for each  $t > 0$ ,  $u(t, x) = e^{-tL}u_0$  is a solution of initial value problem (4-3). To apply Theorem 4.7, we consider the function  $\varphi(u) = e^{-tu}$ , which satisfies the condition of Theorem 4.7 and therefore,

$$\|e^{-tL}\|_{\text{op}} \leq \sup_{u>\rho^2} e^{-tu} \begin{cases} (u-\rho^2)^{(a+1)(1/p-1/q)} & \text{if } (u-\rho^2)^{1/2} \leq K, \\ [K^{2a+2} - K^{2\alpha+2} + (u-\rho^2)^{\alpha+1}]^{1/p-1/q} & \text{if } (u-\rho^2)^{1/2} > K, \end{cases}$$

by setting  $s = (u - \rho^2)^{1/2}$ ,

$$\|e^{-tL}\|_{\text{op}} \leq e^{-t\rho^2} \sup_{s>0} e^{-ts^2} \begin{cases} s^{2(a+1)(1/p-1/q)} & \text{if } s \leq K, \\ [K^{2a+2} - K^{2\alpha+2} + s^{2(\alpha+1)}]^{1/p-1/q} & \text{if } s > K. \end{cases} \tag{4-4}$$

Now, we first calculate  $e^{-t\rho^2} \sup_{s>0} e^{-ts^2} s^{2(a+1)/r}$ , where  $1/r = 1/p - 1/q$ . Consider the function

$$\phi(s) = e^{-ts^2} s^{2(a+1)/r}.$$

Then,

$$\phi'(s) = e^{-ts^2} \left( -2s^2t + \frac{2(a+1)}{r} \right) s^{2(a+1)/r-1}.$$

So, the derivative  $\phi'$  is zero only at  $s_0 = \sqrt{(a+1)/rt}$  and changes sign from positive to negative at  $s_0$ . Thus,  $s_0$  is the point of maximum of  $\phi$  if  $s_0 \leq K$ , that is,  $\sqrt{(a+1)/rt} \leq K$  implies that  $t \geq (a+1)/rK^2$ . Thus,

$$e^{-t\rho^2} \sup_{s>0} e^{-ts^2} s^{2(a+1)/r} = e^{-t\rho^2} e^{-(a+1)/r} \left( \frac{a+1}{rt} \right)^{(a+1)/r} \leq C_{a,r} e^{-t\rho^2} t^{-(a+1)/r}. \tag{4-5}$$

Now, for  $s > K$ , consider the function  $\psi(s) = e^{-ts^2} [K^{2a+2} - K^{2\alpha+2} + s^{2(\alpha+1)}]^{1/r}$ . Then it is easy to see that

$$\psi'(s) = 2e^{ts^2} s [K^{2a+2} - K^{2\alpha+2} + s^{2(\alpha+1)}]^{1/r-1} \left( -t[K^{2a+2} - K^{2\alpha+2} + s^{2(\alpha+1)}] + \frac{\alpha+1}{r} s^{2\alpha} \right).$$

Now, we see that  $\psi'(s_0) = 0$  at the point  $s_0 > K$  if  $s_0$  satisfies the equation

$$t[K^{2a+2} - K^{2\alpha+2} + s^{2(\alpha+1)}] = \frac{\alpha+1}{r} s^{2\alpha},$$

that is,  $t[K^{2a+2} - K^{2\alpha+2} + s_0^{2(\alpha+1)}] = (\alpha+1)/rs_0^{2\alpha}$ . Therefore,  $\psi'$  changes sign from positive to negative at  $s_0$  and so it is a point of maximum. Thus,

$$\begin{aligned} e^{-t\rho^2} \sup_{s>0} e^{-ts^2} [K^{2a+2} - K^{2\alpha+2} + s^{2(\alpha+1)}]^{1/r} &= e^{-t\rho^2} e^{-ts_0^2} \left( \frac{\alpha+1}{rt} s_0^{2\alpha} \right)^{1/r} \leq C_{\alpha,a,r,K} e^{-t\rho^2} t^{-1/r} e^{-ts_0^2} s_0^{2\alpha/r} \\ &\lesssim t^{-1/r} \sup_{s>K} e^{-ts^2} s^{2(\max\{\alpha,a+1\})/r}. \end{aligned}$$

Now, we compute  $\sup_{s>K} e^{-ts^2} s^{2(\max\{\alpha,a+1\})/r}$ . For this purpose, we set  $\eta(s) = e^{-ts^2} s^{2(\max\{\alpha,a+1\})/r}$ . Again, the derivative  $\eta'(s) = 0$  for  $s = \sqrt{\max\{\alpha, a+1\}/rt} > K$ , that is,  $t < \max\{\alpha, a+1\}/K^2r$  and changes its sign from positive to negative at  $s = \sqrt{\max\{\alpha, a+1\}/rt}$ . Therefore,

$$\begin{aligned} e^{-t\rho^2} \sup_{s>0} e^{-ts^2} [K^{2a+2} - K^{2\alpha+2} + s^{2(\alpha+1)}]^{1/r} &\lesssim t^{-1/r} \sup_{s>K} e^{-ts^2} s^{2\max\{\alpha,a+1\}/r} \\ &\lesssim e^{-t\max\{\alpha,a+1\}/rt} t^{-1/r} \left( \sqrt{\frac{\max\{\alpha, a+1\}}{rt}} \right)^{2\max\{\alpha,a+1\}/r} \lesssim t^{-(\max\{\alpha,a+1\}+1)/r}. \end{aligned} \tag{4-6}$$

Next, by putting the estimates from (4-5) and (4-6) in (4-4),

$$\|e^{-tL}\|_{\text{op}} \lesssim \begin{cases} t^{-(\max\{\alpha,a+1\}+1)(1/p-1/q)} & \text{if } 0 < t < \frac{\max\{\alpha, a+1\}}{K^2} \left( \frac{1}{p} - \frac{1}{q} \right) \\ e^{-t\rho^2} t^{-(a+1)(1/p-1/q)} & \text{if } t \geq \frac{a+1}{K^2} \left( \frac{1}{p} - \frac{1}{q} \right). \end{cases}$$

Therefore,

$$\begin{aligned} \|u(t, \cdot)\|_{L^q(\mathbb{R}_+, A dx)} &\leq \|e^{-tL} u_0\|_{L^q(\mathbb{R}_+, A dx)} \leq \|e^{-tL}\|_{\text{op}} \|u_0\|_{L^p(\mathbb{R}_+, A dx)} \\ &\lesssim \|u_0\|_{L^p(\mathbb{R}_+, A dx)} \begin{cases} t^{-(\max\{\alpha,a+1\}+1)(1/p-1/q)} & \text{if } 0 < t < \frac{\max\{\alpha, a+1\}}{K^2} \left( \frac{1}{p} - \frac{1}{q} \right) \\ e^{-t\rho^2} t^{-(a+1)(1/p-1/q)} & \text{if } t \geq \frac{a+1}{K^2} \left( \frac{1}{p} - \frac{1}{q} \right). \end{cases} \end{aligned}$$

**4.3.2. Sobolev type embeddings.** Let  $1 < p \leq 2 \leq q < \infty$ . As an application of Theorem 4.7, we obtain a Sobolev-type embedding theorem for operator  $L$ . Consider the function  $\varphi(u) = (1 + u)^{-b}$ ,  $u > \rho^2$ , which  $\varphi$  satisfies the condition of Theorem 4.7. Therefore, from Theorem 4.7,

$$\|(1 + L)^{-b}\|_{\text{op}} \lesssim \sup_{u > \rho^2} (1 + u)^{-b} \begin{cases} (u - \rho^2)^{(a+1)(1/p-1/q)} & \text{if } (u - \rho^2)^{1/2} \leq K, \\ [K^{2a+2} - K^{2\alpha+2} + (u - \rho^2)^{(a+1)}]^{1/p-1/q} & \text{if } (u - \rho^2)^{1/2} > K, \end{cases}$$

which is finite provided that  $b \geq (\alpha + 1)(1/p - 1/q)$ .

Therefore,

$$\|f\|_{L^q(\mathbb{R}_+, A \, dx)} \leq C \|(1 + L)^b f\|_{L^q(\mathbb{R}_+, A \, dx)}$$

provided that  $b \geq (\alpha + 1)(1/p - 1/q)$ ,  $1 < p \leq 2 \leq q < \infty$ .

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