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SOME EXAMPLES OF SMOOTH AND REGULAR RINGS(1)

BY
LESLIE G. ROBERTS

In this note I investigate smoothness and regularity of the ring $A = \mathbb{Z}[X, Y]/(aX^2+bXY+cY^2-1)$, $a, b, c \in \mathbb{Z}$, $\mathbb{Z}$ the ring of integers. These results do not seem to be well known, especially those dealing with regularity. At any rate, several of my colleagues knew that $\mathbb{Z}[X, Y]/(X^2+Y^2-1)$ was not smooth, but thought that it might be regular. I was surprised by the number of possibilities that can occur, as well as by the fact that A can be regular but not smooth. As expected the prime 2 plays a special role. Smoothness depends on $a, b, c \bmod 2$, and regularity depends on $a, b, c \bmod 4$. Note that A is of Krull dimension 2. I conclude by showing that the same techniques can be applied if there are more than two variables. This discussion is less specific than that in the two variable case.

Let R be a commutative Noetherian ring with unit. Then R is regular at P (or P is a regular prime of R) if R_P is a regular local ring (as defined on page 78 of [2]). The ring R is defined to be regular if R is regular at all prime ideals $P \subset R$. The definition of smoothness is that given in 28.D page 200 of [2]. The groundring will be $\mathbb{Z}$ unless stated otherwise, so I will usually write "smooth" instead of "smooth over $\mathbb{Z}$". The ring R is smooth at P if the local ring R_P is smooth. If R is smooth then R is smooth at all primes. Conversely if R is smooth at all primes then R is smooth, at least under relatively mild finiteness conditions that are satisfied if $R = A$. (Theorem 5.11 of [4]).

Let $J = (2aX+bY, bX+2cY)A$. By the Jacobian criterion for smoothness A is not smooth at P if and only if $J \subseteq P$. Corollary 8.4 of [4] is a more direct reference for this than [2] Theorem 64.

My motivation for trying to understand smoothness and regularity comes from the study of algebraic K-theory. In [3] Quillen has defined two sets of functors K_i, $K_i'(i \in \mathbb{Z},\ i \geq 0)$ from rings to abelian groups. There is a natural transformation $K_i \to K_i'$. If R is regular the resulting homomorphism $K_i(R) \to K_i'(R)$ is an isomorphism. There are different computational methods available for K_i and for K_i'. If R is regular these methods can be combined, with the result that the groups $K_i(R)$ are better understood if R is regular. Because of the Jacobian criterion smoothness is more easily determined than regularity (at

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least, at first glance). Finally let R be an algebra of finite type over the regular ring S. If P is a prime ideal of R and R_P is smooth over S then R_P is regular. I have not been able to find a published reference for the last fact, but it is Corollary 8.5 of [4].

First we determine when A is smooth. Let $f=aX^2+bXY+cY^2$. By Euler's theorem $X(\partial f/\partial X)+Y(\partial f/\partial Y)=2f=2$, so $2\in J$. Let d be the determinant of the 2×2 matrix

$$D=\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

Then $d=4ac-b^2$ and $D\,\mathrm{adj}D=dI_2$ so there exist $c_1, c_2, d_1, d_2\in\mathbb{Z}$ such that $c_1(\partial f/\partial X)+c_2(\partial f/\partial Y)=dX$, $d_1(\partial f/\partial X)+d_2(\partial f/\partial Y)=dY$. But $(X, Y)A=A$ so $d\in J$. Thus if d is odd then $J=A$. Clearly d is odd if and only if b is odd. We saw above that $2\in J$. If b is even then $\partial f/\partial X\in 2A$ and $\partial f/\partial Y\in 2A$, so $J=2A$. If all of a, b, c are even then 2 is a unit in A, so $J=A$. If one of a or c is odd and b even then $A/2A=\mathbb{Z}/2\mathbb{Z}[X, Y]/(\bar{a}X^2+\bar{c}Y^2-1)\neq 0$ so $J\neq A$. This proves

THEOREM 1. *Let $A=\mathbb{Z}[X, Y]/(aX^2+bXY+cY^2-1)$. Then A is smooth over $\mathbb{Z}$ if and only if b is odd or a, b, c are all even.*

In order to discuss the regularity of A we need the following fact about regular local rings. Let R be a regular local ring and I an ideal of R. Then $S=R/I$ is a regular local ring if and only if I can be generated by a subset of a regular system of parameters ([5] Theorem 26, p. 303). Let P be a prime ideal of A, and Q the inverse image of P in $\mathbb{Z}[X, Y]$. Then $A_P=\mathbb{Z}[X, Y]_Q/(aX^2+bXY+cY^2-1)$. Clearly $aX^2+bXY+cY^2-1\in Q$ so it follows from the above that A_P is regular if and only if $aX^2+bXY+cY^2-1\notin Q^2\mathbb{Z}[X, Y]_Q$.

Suppose P is a non-regular prime ideal of A. By Theorem 1 (and the fact that smooth implies regular) we have $2\in P$ and b even (write $b=2B$). Without loss of generality we can assume a odd. There are two possibilities, c odd or c even. First we consider the case c odd. Then $A/2A=\mathbb{Z}/2\mathbb{Z}[X, Y]/(X^2+Y^2-1)=\mathbb{Z}/2\mathbb{Z}[X, Y]/(X+Y-1)^2$. Set $Z=X+Y-1$, so that $A=\mathbb{Z}[Y, Z]/(aZ^2+(a+c-2B)Y^2+2(B-a)Y+2aZ+2(B-a)YZ+(a-1))$. We have $2\in P$ and $Z\in P$ so the terms aZ^2, $2aZ$, and $2(B-a)YZ$ lie in Q^2. Now let P be any prime $P\subset A$ such that $2\in P$. Then P is not regular if and only if $(a+c-2B)Y^2+2(B-a)Y+(a-1)\in Q^2\mathbb{Z}[X, Y]_Q$. Note that $a+c$ is even and $a-1$ is even, so that $(a+c-2B)Y^2+2(B-a)Y+(a-1)=2F$, where $F=[(a+c)2-B]Y^2+(B-a)Y+(a-1)/2\in\mathbb{Z}[Y]$. Now I claim that $2\notin Q^2\mathbb{Z}[X, Y]_Q$. For $\mathbb{Z}[Y, Z]/2)=\mathbb{Z}/2\mathbb{Z}[Y, Z]$ is regular. Again by [5] p. 303 we have that 2 is part of a regular system of parameters of $Q\mathbb{Z}[X, Y]_Q$, and hence does not lie in $Q^2\mathbb{Z}[X, Y]_Q$. Thus $2F\in Q^2\mathbb{Z}[X, Y]_Q$ if and only if $F\in Q\mathbb{Z}[X, Y]_Q$ if and only if $F\in Q$. Thus the non-regular prime ideals of A are those which contain $(2, Z, F)$, i.e. the closed subscheme $\operatorname{Spec} A/(2, Z, F) = \operatorname{Spec} \mathbb{Z}/2\mathbb{Z}[Y]/(F) \subset \operatorname{Spec} A$.

Now set $c_1 = (a+c)/2 - B$, $c_2 = B - a$, $c_3 = (a-1)/2$, so that $F = c_1Y^2 + c_2Y + c_3$. There are several different possibilities, depending on the parity of c_1, c_2, c_3.

(1) c_1, c_2, c_3 all even. Here $F = 0 \pmod 2$ so that the non-regular prime ideals of A are the height one prime $(2, Z)A$ and those maximal ideals that contain $(2, Z)A$. In this case the non-regular primes are the same as the non-smooth primes. An example is $A = \mathbb{Z}[X, Y]/(X^2 + 2XY + 5Y^2 - 1)$. Here $a = 1$, $B = 1$, $c = 5$ so that $c_1 = 3 - 1$, $c_2 = 1 - 1$, and $c_3 = 0$ are all even.

(2) c_1, c_2 even, c_3 odd. Here $F = 1 \pmod 2$ so A is regular at every prime ideal. Hence A is regular. An example is $A = \mathbb{Z}[X, Y]/(3X^2 + 2XY + 3Y^2 - 1)$.

(3) c_1 even, c_2 odd or c_1 odd, c_2 even. Then $F = Y - \bar{c}_3$ or $F = (Y - \bar{c}_3)^2$ so there is one non-regular prime ideal of A, namely the maximal ideal $P = (2, Z, Y - \bar{c}_3)$, and $A/P = Z/2Z$. Examples are $A = \mathbb{Z}[X, Y]/(X^2 + 3Y^2 - 1)$ and $A = \mathbb{Z}[X, Y]/(X^2 + 2XY + 3Y^2 - 1)$.

(4) c_1, c_2, c_3 odd. Then $F = Y^2 + Y + 1 \bmod 2$, which is irreducible in $\mathbb{Z}/2\mathbb{Z}[Y]$. There is one non-regular prime $P = (2, Z, Y^2 + Y + 1)$ and $A/P =$ the field with 4 elements. An example is $A = \mathbb{Z}[X, Y]/(X^2 + Y^2 + 1)$.

(5) c_1, c_2 both odd, c_3 even. Then $F = Y^2 + Y \pmod 2$. There are two non-regular primes $P_1 = (2, Z, Y)$ and $P_2 = (2, Z, Y+1)$, with $A/P_i = \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$. An example is $A = \mathbb{Z}[X, Y]/(X^2 + Y^2 - 1)$.

Interchanging X and Y interchanges a and c. Thus conditions (1)–(5) must be symmetric in a and c. This can be easily checked.

Now let us consider the case c even. Let P be a prime ideal of A such that $2 \in P$. Again we need only consider b even $(b = 2B)$. Then $A/2A = \mathbb{Z}/2\mathbb{Z}[X, Y]/(X^2 + 1) = \mathbb{Z}/2\mathbb{Z}[X, Y]/(X+1)^2$. Set $Z = X + 1$ so that $A = \mathbb{Z}[Y, Z]/(aZ^2 - 2aZ + 2BZY - 2BY + cY^2 + a - 1)$. We have $2 \in P$ and $Z \in P$ so that the aZ^2, $-2aZ$, and $2BZY$ terms lie in Q^2 (Q as before). Thus P is not regular if and only if $cY^2 - 2BY + a - 1 \in Q^2\mathbb{Z}[X, Y]_Q$. Again $2 \notin Q^2\mathbb{Z}[X, Y]_Q$. If we set $c_1 = c/2$, $c_2 = -B$, and $c_3 = (a-1)/2$ then we get the same five cases that we had with c odd. All cases can occur. It is also possible to change c from odd to even by a change of variable.

The regularity of A is characterized by cases (1)–(5). The ring A is Cohen–Macauley so by Theorem 39 page 125 of [2] A is integrally closed if and only if all primes of height one are regular. Thus case (1) is not integrally closed, and cases (2)–(5) are integrally closed. These results can be summarized as follows (with c_1, c_2, c_3 as previously defined—with different formulae depending on the parity of c):

THEOREM 2. *Let* $A = \mathbb{Z}[X, Y]/(aX^2 + bXY + cY^2 - 1)$. *Then A is regular if and only if either* (i) *A is smooth as in Theorem* 1, *or* (ii) *b is even, at least one of*

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a or c is odd, and c_1, c_2 are even, c_3 odd. In case (ii) *A is regular but not smooth. Also A is integrally closed except in the case b even, at least one of a or c odd, and c_1, c_2, c_3 all even.*

The case of more than two variables can be discussed in a similar way. Let $A=\mathbb{Z}[X_1, X_2, \ldots, X_n]/(F-1)$ where $F=\sum_{1\le i,j\le n} a_{ij}X_iX_j$ is a homogeneous form of degree 2, $a_{ij}\in\mathbb{Z}$. Let $J=(\partial F/\partial X_i)A$. Again $2\in J$. The non-smooth primes of A are those that contain J, and as a set equal the closed subscheme Spec $A/J\subset$ Spec A. But $A/J=(A/2A)/(J/2)$. If we apply the Jacobian criterion for smoothness over $\mathbb{Z}/2\mathbb{Z}$ to the ring $A/2A=\mathbb{Z}/2\mathbb{Z}[X_1,\ldots,X_n]/(\bar{F}-1)$ (where the $\bar{.}$ means reduction of coefficients mod 2) we get the same closed subscheme. Thus the non-smooth primes of A are the same as the non-smooth (over $\mathbb{Z}/2\mathbb{Z}$) primes of $A/2A$ (under the inclusion Spec$(A/2A)\subset$ Spec A). Over $\mathbb{Z}/2\mathbb{Z}$ we can make a change of variable so that $\bar{F}=\sum_{i=1}^{m}(a_iY_i^2+Y_iZ_i+c_iZ_i^2)+\sum_{j=1}^{m'}W_j^2$ $(2m+m'\le n)$ ([1] Satz 2). Then $\bar{J}(=J/2J)=(Y_i, Z_i)\bar{A}$, so we see that $\bar{A}$ is smooth over $\mathbb{Z}/2\mathbb{Z}$ (and hence A is smooth over $\mathbb{Z}$) if $m'=0$, otherwise the non-smooth points are of codimension $2m$ in Spec $\bar{A}$ and codimension $2m+1$ in Spec A. All possibilities can occur.

Now consider the regular primes of A. All non-regular primes must contain J and hence 2. Let P be any prime of A that contains J, and let Q be the inverse image of P in $\mathbb{Z}[X_1,\ldots,X_n]$. An invertible matrix over $\mathbb{Z}/2\mathbb{Z}$ is the product of elementary matrices, hence can be lifted to an invertible matrix over $\mathbb{Z}$. Therefore the change of variable carried out above in $A/2A$ can be lifted to A and we can assume that we have variables Y_i, Z_i $(1\le i\le m)$, W_j $(1\le j\le m')$ and possibly other variables, so that

$$F=\sum_{i=1}^{m}(a_iY_i^2+b_iY_iZ_i+c_iZ_i^2)+\sum_{j=1}^{m'}d_jW_j^2+2G,$$

where b_i, d_j are odd. If $m'=0$ then A is smooth, thus regular. Hence assume $m'>0$. We have $\partial F/\partial Y_i=b_iZ_i+2(\cdots)$. Since b_i is odd and $2\in J$ we conclude that $Z_i\in J$. Similarly $Y_i\in J$ and $\sum_{i=1}^{m}(a_iY_i^2+b_iY_iZ_i+c_iZ_i^2)\in P^2$. The ring A is not regular at P if and only if $F-1\in Q^2\mathbb{Z}[Y_i, Z_i, W_i,\ldots]_Q$ if and only if $T=\sum_{j=1}^{m'}d_jW_j^2+2G-1\in Q^2\mathbb{Z}[Y_i, Z_i, W_j,\ldots]_Q$. Now set $Z=(\sum_{j=1}^{m'}W_j)-1$ and replace one of the variables W_j by Z. Then $T=Z^2+2H$. Since 2 and T lie in Q we have $Z^2\in Q$. But Q is prime, so $Z\in Q$ and thus $Z^2\in Q^2$. We now have P non-regular if and only if $2H\in Q^2\mathbb{Z}[Y_i, Z_i, W_j,\ldots]_Q$. As in the two variable case we can cancel off a 2 and conclude that P is non-regular if and only if $H\in Q$. This is one extra condition to be satisfied, so the set of non-regular primes is either empty, or has codimension exceeding that of the non-smooth primes by at most 1. These results can be summarized as

THEOREM 3. *Let $A=\mathbb{Z}[X_1,\ldots,X_n]/(F-1)$, where F is a homogeneous form of degree 2 with coefficients in $\mathbb{Z}$. Then the non-smooth primes of A form a*

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closed subset of Spec A which is either empty or of odd codimension. Every allowable codimension can occur. The non-regular primes of A form a closed subset of the non-smooth primes. This subset is either empty or of codimension at most one in the set of non-smooth primes.

If F is in "diagonal" form H is easily calculated. Let $F=\sum_{i=1}^{n} a_i X_i^2+\sum_{j=1}^{m} b_j Y_j^2$ where the a_i are odd and the b_j are even. Here $J=(2)$ and we can assume $n>0$, otherwise 2 is a unit and A is smooth. Let $Z=(\sum_{i=1}^{n} X_i)-1$, so that $X_1=Z-(\sum_{i=2}^{n} X_i)+1$. Then $F-1$ becomes

$$a_1Z^2+\sum_{i=2}^{n}(a_1+a_i)X_i^2-2a_1(\sum_{i=2}^{n}X_i)+2a_1(\sum_{1<i<j}X_iX_j)+2Z(\cdots)+\sum_{j=1}^{m}b_jY_j^2 + a_1-1.$$

Thus

$$H=\sum_{i=2}^{n}[(a_1+a_i)/2]X_i^2-a_1(\sum_{i=2}^{n}X_i)+a_1(\sum_{1<i<j}X_iX_j)+\sum_{j=1}^{m}(b_j/2)Y_j^2+(a_1-1)/2$$

(the $2Z(\cdots)$ term can be omitted since $Z\in Q$). Then A is regular at P $(2\in P)$ if and only if $H\notin Q$. For A to be regular everywhere we require that all coefficients be 0 (mod 2) except $(a_1-1)/2$ which should be non-zero. This happens if and only if $n=1$, the b_j are all divisible by 4, and $a_1=3 \bmod 4$. In order for A to be non-regular at the set of non-smooth points (Spec $A/2A$) we require that $H=0 \bmod 2$, and this happens if and only if $n=1$, the b_j are all divisible by 4, and $a_1\equiv 1 \bmod 4$. (The latter is the only non-integrally closed case). Otherwise A is non-regular in codimension 2. These results can be summarized as

THEOREM 4. *Let* $A=\mathbb{Z}[X_i, Y_j]/(\sum_{i=1}^{n} a_iX_i^2+\sum_{j=1}^{m} b_jY_j^2-1)(1\le i\le n,\ 1\le j\le m,$ *a_i odd, b_j even). Then the non-smooth primes of A are those that contain 2, and A is smooth if and only if $n=0$. The ring A is regular if and only if $n=0$ or $n=1$, $a_1\equiv 3 \bmod 4$ and the b_j are all divisible by 4. The ring A is non-regular at all non-smooth primes if and only if $n=1$, $a_1\equiv 1 \bmod 4$, and the b_j are all divisible by 4. Otherwise A is non-regular on Spec $A/(2,H)$, a subset of codimension 2, where H is given above. The ring A is integrally closed except in the case $n=1$, $a_1\equiv 1 \bmod 4$, and the b_j are all divisible by 4.*

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DEPARTMENT OF MATHEMATICS
QUEEN'S UNIVERSITY
KINGSTON, ONTARIO, K7L 3N6