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COMPARISON THEOREMS OF HILLE-WINTNER TYPE FOR THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS

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Integral comparison theorems of Hille-Wintner type of second order linear equations are shown to be valid for the third order linear equation $y''' + q(t)y = 0$.

1. Introduction

The Sturm comparison theorem and its generalizations play an important role in the study of the oscillatory character of the second order linear equation

$$y'' + q(t)y = 0 , \quad q \in C[a, +\infty) . \tag{1}$$

One of the simpler forms of the theorem states that if equation (1) is disconjugate on $[a, +\infty)$ (that is, no solution of (1) has more than one zero on $[a, +\infty)$), and if $q_1 \in C[a, +\infty)$ with $q_1(t) \le q(t)$ on $[a, +\infty)$, then the equation

$$y'' + q_1(t)y = 0 \tag{2}$$

is also disconjugate on $[a, +\infty)$. This result may be extended to comparisons of an integral type, one of which is the so-called Hille-Wintner comparison theorem.

THEOREM 1 [8], [15]. *Let* $Q(t) \equiv \int_t^\infty q(s)ds$ *and* $Q_1(t) \equiv \int_t^\infty q_1(s)ds$

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exist with $0 \le Q_1(t) \le Q(t)$ *on* $[a, +\infty)$ *and assume equation* (1) *is disconjugate on* $[a, +\infty)$. *Then so also is equation* (2).

In this paper we shall be interested in extending an analogue of Theorem 1 to the third order linear equations

$$Ly \equiv y''' + q(t)y = 0 \tag{3}$$

and

$$L_1 y \equiv y''' + q_1(t)y = 0 \ . \tag{4}$$

We recall that equation (3) is said to be disconjugate on an interval $I \subset [a, +\infty)$ in case no nontrivial solution has more than two zeros on I. Disconjugacy and its connection with oscillation and nonoscillation have been studied by many authors (*cf.* Barrett [*1*], Hanan [*7*], Lazer [*11*], Etgen and Shih [*3*], [*4*], [*5*], Jones [*9*], [*10*], and the references therein). In particular, it has been shown in [*13*] (see also [*7*, Theorem 3.11]) that if $\hat{q}(t) \le q_1(t) \le q(t)$ and if $Ly = y''' + q(t)y = 0$ and $\hat{L}y = y''' + \hat{q}(t)y = 0$ are both disconjugate on $[a, +\infty)$, then so is $L_1 y = y''' + q_1(t)y = 0$. This may be thought of as one analogue of the Sturm Comparison Theorem in the study of the oscillatory character of $Ly = 0$. In order to compare our results with other known criteria for disconjugacy, we recall that if $q(t) \ge 0$ on $[a, +\infty)$ and if $Ly = 0$ is disconjugate on $[a, +\infty)$, then ([*7*]),

$$\int^{\infty} tq(t)dt < +\infty \ . \tag{5}$$

Further, if

$$\int^{\infty} t^2 q(t)dt < +\infty \ , \tag{6}$$

then $Ly = 0$ is disconjugate on $[t_0, +\infty)$, some $t_0 \ge a$, ([*6*]). On the other hand, if for some δ, $0 < \delta < 1$, we have

$$\int^{\infty} t^{1+\delta} q(t)dt = +\infty \ , \tag{7}$$

then $Ly = 0$ is oscillatory (that is, $Ly = 0$ has a solution which changes sign on each half-line $[t_0, +\infty)$) ([*11*]). Finally, comparison with the Euler equation shows that $Ly = 0$ is (eventually) disconjugate if

$$\limsup_{t\to\infty} t^3 q(t) < 2/3\sqrt{3} \tag{8}$$

and is oscillatory if

$$\liminf_{t\to\infty} t^3 q(t) > 2/3\sqrt{3} . \tag{9}$$

Additional criteria may be found in the references cited above and in the book of Swanson [14]. We remark also that $Ly = 0$ is disconjugate on an interval I (finite or infinite) in case there exist $\alpha, \beta \in C^2(I)$ with $\alpha < \beta$ on I and $\alpha'' + f(t, \alpha, \alpha') \geq 0 \geq \beta'' + f(t, \beta, \beta')$, $t \in I$, where $f(t, r, r') = 3rr' + r^3 + p(t)$, (that is, α, β are lower and upper solutions of the Riccati equation for $Ly = 0$) (see [2]).

2. Statement and proof of the results

THEOREM 2. *Assume* $Ly = 0$ *is disconjugate on* $[a, +\infty)$, *and let* $q, q_1 \in C[a, +\infty)$ *satisfy*

$$q(t) \geq 0 , \quad q_1(t) \geq 0 , \quad t \geq a . \tag{10}$$

Assume further that

$$Q(t) \equiv \int_t^\infty q(s)ds \quad \text{and} \quad Q_1(t) \equiv \int_t^\infty q_1(s)ds \tag{11}$$

exist and satisfy

$$Q_1(t) \leq Q(t) , \quad t \geq a . \tag{12}$$

Then $L_1 y = y''' + q_1(t)y = 0$ *is disconjugate on* $[t_0, +\infty)$ *for some* $t_0 \geq a$.

If we relax the requirement that q_1 be nonnegative, we may establish

COROLLARY 3. *Assume* $Ly = 0$ *is disconjugate on* $[a, +\infty)$, $q \geq 0$, *and assume that*

$$Q_1^+(t) \leq Q(t) , \quad t \geq a , \tag{13}$$

and

(14) $$Q_1^-(t) \le Q(t) \ , \quad t \ge a \ ,$$

where

$$Q_1^+(t) \equiv \int_t^\infty q_1^+(s)ds \quad \text{and} \quad Q_1^-(t) \equiv \int_t^\infty q_1^-(s)ds$$

and where $q_1^+(t) \equiv \max[0, q_1(t)]$, $q_1^-(t) \equiv \max[0, -q_1(t)]$. *Then* $L_1y = 0$ *is disconjugate on* $[t_0, +\infty)$ *for some* $t_0 \ge a$.

It is not difficult to see that Theorem 2 and Corollary 3 are sharp by considering the Euler equation

(15) $$y''' + \alpha t^{-3}y = 0 \ , \quad \alpha \text{ real constant,}$$

which is disconjugate on $[a, +\infty)$ $(a > 0)$ if and only if $|\alpha| \le 2/3\sqrt{3}$. We shall give below an example whose disconjugate behaviour may not be inferred by any criteria known to the author.

For the case of a finite interval $I = [a, b]$, we have the following analogue of Theorem 2. This result is also related to the so-called Levin comparison theorems for the second-order equation (1) and (2) (see [*12*] and [*14*]). We recall that the adjoint equation of (3) is

(16) $$L^*y = y''' - q(t)y = 0 \ .$$

THEOREM 4. *Assume* $q \ge 0$ *on* $[a, b]$ *and that the following conditions hold:*

(17) $$\left\{\begin{array}{l} \displaystyle\int_t^b q_1^+(s)ds \le \int_t^b q(s)ds \ , \\ \displaystyle\int_t^b q_1^-(s)ds \le \int_t^b q(s)ds \ . \end{array}\right. \quad a \le t \le b \ ,$$

Further, let $L^*y = 0$ *have a solution* $y = y(t)$ *satisfying*

(18) $$y > 0 \ , \quad y' > 0 \ , \quad y'' < 0 \ , \quad a \le t \le b \ .$$

Then $L_1y = 0$ *is disconjugate on* $[a, b]$.

Before proving the above results, we shall need to establish some properties of the nonoscillatory solutions of the adjoint equation (16) under the assumption $q \ge 0$, $q \not\equiv 0$. Recall that $L^*y = 0$ is disconjugate on an interval I if and only if $Ly = 0$ is disconjugate on I (*cf.* [7]). If $I = [a, +\infty)$ and $y = y(t) \not\equiv 0$ is a nonoscillatory solution of $L^*y = 0$, we may suppose that $y(t) > 0$ for $t \geq t_0$. Since $y''' = q(t)y \geq 0$, $t \geq t_0$, it follows that y'' is increasing on $[t_0, +\infty)$ and hence y' can change from negative to positive values at most once in $[t_0, +\infty)$. Therefore, either $y'(t) \leq 0$ for all $t \geq t_0$ or there exists $t_1 \geq t_0$ with $y'(t) > 0$ on $[t_1, +\infty)$. Suppose then that $y'(t) \leq 0$ for all $t \geq t_0$. Now since $y''(t)$ is increasing, we must also have either $y''(t) \leq 0$ or $y''(t) > 0$ eventually. But if $y''(t) \leq 0$, then $y'(t)$ is decreasing and hence $y'(t) \leq \delta < 0$ for all large t and some $\delta < 0$, which contradicts the fact that $y(t) > 0$ on $[t_0, +\infty)$. On the other hand, if $y''(t) > 0$ eventually, then it follows that $y''(t) \geq \eta > 0$ for all large t and some $\eta > 0$, which implies that $y'(t) \to +\infty$, a contradiction. We may conclude, therefore, that we must have $y'(t) > 0$ on $[t_1, +\infty)$ for some $t_1 \geq t_0$. We summarize the above remarks in

LEMMA 5. *If $q(t) \geq 0$ and if $y = y(t) \not\equiv 0$ is a nonoscillatory solution of $L^*y = y''' - q(t)y = 0$, then there exists $t_1 \geq a$ such that*

(19) $$y(t)y'(t) > 0 \quad \text{on} \quad [t_1, +\infty) .$$

The following result gives a sufficient condition for disconjugacy of $L^*y = 0$ on an arbitrary interval $I = [a, b]$ or $[a, +\infty)$. In the infinite interval case, it is actually a special case of a result of Lazer [*11*, Theorem 2.1]. However, the proof given below is different.

LEMMA 6. *Let $q \geq 0$, $q \not\equiv 0$, on I and assume there exists a solution of $L^*y = 0$ satisfying*

(20) $$y(t) > 0 , \quad y'(t) > 0 , \quad y''(t) < 0 , \quad t \in I .$$

Then $L^*y = 0$ is disconjugate on I.

Proof. Given the solution y of $L^*y = 0$ satisfying (20), let z be the solution of $L^*y = 0$ with

(21) $$z(a) = 0 , \quad z'(a) = 1 , \quad z''(a) = 1 ,$$

where $I = [a, b]$ or $[a, \infty)$. Then since $z''' = qz \geq 0$ on I, it

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follows that $z > 0$, $z' > 0$, $z'' > 0$, $t > a$. Therefore, the function $W(t) = yz' - zy'$ is a solution of $Ly = 0$ satisfying $W(a) = y(a)z'(a) > 0$ and $W' = yz'' - zy'' > 0$ on I so that $W > 0$, $W' > 0$ on I . Therefore, $\alpha(t) \equiv 0 < \beta(t) \equiv W'/W$ are lower and upper solutions of the Riccati equation for $Ly = 0$ and hence $Ly = 0$ is disconjugate on I ([2]), that is, $L^*y = 0$ is disconjugate on I .

The converse of Lemma 6 is true, under an additional assumption, for the infinite interval case. This is a special case of a result of Lazer [11, Theorem 2.2] to which we refer for the proof:

LEMMA 7. *Let* $q(t) \geq 0$ *and* $q(t) \not\equiv 0$, $I = [a, +\infty)$, *and assume*

$$\int_a^\infty t^4 q(t)dt = +\infty . \tag{22}$$

Then $L^*y = 0$ *is disconjugate on* $[t_0, +\infty)$ *for some* $t_0 \geq a$ *if and only if there exists a nonoscillatory solution* $y = y(t)$ *of* $L^*y = 0$ *satisfying*

$$y(t) > 0 , \quad y'(t) > 0 , \quad y''(t) < 0 \quad on \quad [t_1, +\infty) , \tag{23}$$

for some $t_1 \geq a$.

Whether condition (22) is necessary for the existence of a solution of L^*y satisfying (23) under the assumption that $L^*y = 0$ is disconjugate appears to still be an open question ([11]).

The proof of Theorem 2 will be given by considering an appropriate two dimensional nonlinear Riccati system. To that end, we make the following change of variable in the equation $L^*y = y''' - q(t)y = 0$:

$$u_1 = y'/y , \quad u_2 = y''/y , \tag{24}$$

to obtain the system

$$\begin{cases} u_1' = u_2 - u_1^2 , \\ u_2' = -u_1u_2 + q . \end{cases} \tag{25}$$

Thus, if y is a nonoscillatory solution of $L^*y = 0$ with, say $y(t) > 0$, $y'(t) > 0$ for $t \geq t_0$, then u_1, u_2 defined by (24) satisfy (25) which becomes, after an integration from t to T ,

$t_0 \leq t < T < +\infty$,

$$(26) \qquad \begin{cases} u_1(t) = u_1(T) + \int_t^T \left(u_1^2 - u_2\right) ds , \\ u_2(t) = u_2(T) + \int_t^T \left(u_1 u_2\right) ds - \int_t^T q ds . \end{cases}$$

We may now give the proof of Theorem 2.

Proof of Theorem 2. The proof of the theorem will be separated into two cases, according to whether condition (22) does or does not hold. We shall also assume that $q_1 \not\equiv 0$ for all large t (otherwise there is nothing to prove).

Case (i): $\int^\infty t^4 q(t) dt = +\infty$.

Suppose that q, q_1 , are as in the hypotheses of Theorem 2. Since disconjugacy of $Ly = 0$ is equivalent to disconjugacy of $L^* y = 0$, let $y = y(t) \not\equiv 0$ be a nonoscillatory solution of $L^* y = 0$ satisfying (23) on $[t_0, +\infty)$, $t_0 \geq a$. Defining u_1, u_2 as in (24), we see that $u_1 > 0$, $u_2 < 0$ on $[t_0, \infty)$ and since $y'(t)$ is decreasing and $y(t)$ is increasing on $[t_0, \infty)$, it follows that $\lim_{t\to\infty} u_1(t) = 0$. Likewise, $\lim_{t\to\infty} u_2(t) = 0$. Hence, letting $T \to \infty$ in system (26), we see that u_1, u_2 satisfy

$$(27) \qquad \begin{cases} u_1(t) = \int_t^\infty g_1\left(u_1, u_2\right) ds , \quad t \geq t_0 , \\ u_2(t) = \int_t^\infty g_2\left(u_1, u_2\right) ds - Q(t) , \quad t \geq t_0 , \end{cases}$$

where $Q(t) = \int_t^\infty q(s) ds$, and

$$(28) \qquad \begin{cases} g_1\left(u_1, u_2\right) = u_1^2 - u_2 , \\ g_2\left(u_1, u_2\right) = u_1 u_2 . \end{cases}$$

Notice that if $u_1 > 0$, $u_2 < 0$, then g_1 is increasing in u_1 and decreasing in u_2 and g_2 is decreasing in u_1 and increasing in u_2 . Consider now the system corresponding to $L_1y = y''' + q_1y = 0$ and its adjoint $L_1^*y = y''' - q_1y = 0$:

$$\text{(29)} \qquad \begin{cases} v_1(t) = \int_t^\infty g_1(v_1, v_2)ds \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \ge t_0 \ . \\ v_2(t) = \int_t^\infty g_2(v_1, v_2)ds - Q_1(t) \ , \end{cases}$$

We show first that (29) has a solution defined on $[t_0, +\infty)$ which is obtainable by successive approximations. To see this, define the sequences $\{v_{1n}(t)\}$, $\{v_{2n}(t)\}$ by

$$\text{(30)} \qquad \begin{cases} v_{10}(t) \equiv u_1(t) \ , \quad t \ge t_0 \ , \\ v_{1n}(t) = \int_t^\infty g_1(v_{1n-1}, v_{2n-1})ds \ , \quad n \ge 1 \ , \quad t \ge t_0 \ , \end{cases}$$

and

$$\text{(31)} \qquad \begin{cases} v_{20}(t) \equiv u_2(t) \ , \quad t \ge t_0 \ , \\ v_{2n}(t) = \int_t^\infty g_2(v_{1n-1}, v_{2n-1})ds - Q_1(t) \ , \quad n \ge 1 \ , \quad t \ge t_0 \ . \end{cases}$$

By induction, using the monotoneity of g_1 and g_2 , it follows that $\{v_{1n}\}$, $\{v_{2n}\}$ are well-defined for all $t \ge t_0$ and satisfy

$$\text{(32)} \quad v_{1n+1}(t) \le v_{1n}(t) \le u_1(t) \ , \quad v_{2n+1}(t) \ge v_{2n}(t) \ge u_2(t) \ , \quad t \ge t_0 \ .$$

Furthermore, since $g_1(u_1(t), u_2(t)) > 0 > g_2(u_1(t), u_2(t))$, it follows (using $Q_1(t) \ge 0$) that

$$\text{(33)} \qquad v_{1n}(t) > 0 > v_{2n}(t) \ , \quad t \ge t_0 \ , \quad n = 1, 2, \ldots \ .$$

Define $\hat{v}_1(t)$, $\hat{v}_2(t)$ by

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$$\hat{v}_1(t) \equiv \lim_{n\to\infty} v_{1n}(t) \ , \quad \hat{v}_2(t) \equiv \lim_{n\to\infty} v_{2n}(t) \ , \quad t \geq t_0 \ .$$

It follows by the Monotone Convergence Theorem and Dini's Theorem that $\hat{v}_1(t)$, $\hat{v}_2(t)$ solve system (29) on $[t_0, +\infty)$, and that $\{v_{1n}\}$, $\{v_{2n}\}$ converge uniformly on compact intervals to $\hat{v}_1$, $\hat{v}_2$. Further, since $q_1(t) \not\equiv 0$, it follows that $\hat{v}_2(t) < 0$, $t > t_0$. We may now define

$$z(t) = \exp \int_{t_0}^{t} \hat{v}_1(s)ds \ , \quad t \geq t_0 \ ,$$

and it follows that $z(t)$ is a solution of $y''' - q_1 y = 0$ on $[t_0, +\infty)$ and satisfies $z(t) > 0$, $z'(t) = \hat{v}_1(t)z(t) > 0$, and

$$z''(t) = \left(\hat{v}_1^2 + \hat{v}_1'\right)z = \hat{v}_2(t)z(t) < 0 \ , \quad t \geq t_0 \ .$$

Therefore, by Lemma 6, $L_1^* y = y''' - q_1 y = 0$ is disconjugate on $[t_0, +\infty)$ and hence so also is $L_1 y = y''' + q_1 y = 0$. This completes the proof for case (i).

Case (ii): $\displaystyle\int^{\infty} t^4 q(t)dt < +\infty$.

In this case, we show that under the assumption that (12) holds (that is, $Q_1(t) \leq Q(t)$), it follows that

$$\int^{\infty} t^4 q_1(t)dt < +\infty$$

so that condition (6) holds for $L_1 y = 0$ and therefore $L_1 y = 0$ is disconjugate ([5]). (The author is indebted to Professor G. Butler for the following observations.) Define $\mu(t) = t^4$, $t \geq a$. Then an integration by parts yields:

$$(34) \qquad \int_a^t s^4 q(s)ds = \int_a^t \mu(s)q(s)ds$$

$$= \int_a^t \mu'(s)Q(s)ds - \mu(t)Q(t) + \mu(a)Q(a) \ .$$

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Similarly,

$$(35) \qquad \int_a^t s^4 q_1(s)ds = \int_a^t \mu'(s)Q_1(s)ds - \mu(t)Q_1(t) + \mu(a)Q_1(a) .$$

Suppose, if possible, that $\int_a^\infty t^4 q_1(t)dt = +\infty$. Then from (35) it follows that

$$(36) \qquad \lim_{t\to\infty} \int_a^t \mu'(s)Q_1(s)ds = +\infty$$

so that if we let

$$\phi(t) \equiv \int_a^t \mu'(s)Q(s)ds ,$$

then $\lim_{t\to\infty} \phi(t) = +\infty$ $\big($condition (12)$\big)$. Now from (34), since $\lim_{t\to\infty} [\phi(t)-\mu(t)Q(t)]$ is finite, there exists $c > 0$ and $t_0 \geq a$ so that

$$(37) \qquad \mu(t)Q(t) \geq \phi(t) + \mu(a)Q(a) - c , \quad t \geq t_0 ,$$

which implies

$$(38) \qquad \phi'(t) \geq \big(\mu'(t)/\mu(t)\big)\big(\phi(t)+c_1\big) , \quad t \geq t_0 ,$$

where $c_1 = \mu(a)Q(a) - c$. Let $t_1 \geq t_0$ such that $\phi(t) \geq 2|c_1|$ for $t \geq t_1$. Then integrating (38) from t_1 to $t \geq t_1$ we get

$$(39) \qquad \big(\phi(t)+c_1\big)/\big(\phi(t_1)+c_1\big) \geq \mu(t)/\mu(t_1) , \quad t \geq t_1 ,$$

and so from (37) we have

$$(40) \qquad \mu(t)Q(t) \geq \big(\phi(t_1)+c_1\big)\big(\mu(t)/\mu(t_1)\big) , \quad t \geq t_1 ,$$

and hence

$$(41) \qquad Q(t) \geq \big(\phi(t_1)+c_1\big)/\mu(t_1) > 0 , \quad t \geq t_1 ,$$

contradicting the integrability of q . Therefore, $\int_a^\infty t^4 q_1(t)dt < +\infty$ and $L_1 y = 0$ is disconjugate. This completes the proof of Theorem 2.

Proof of Corollary 3. The proof of Corollary 3 follows immediately from Theorem 2. If the hypotheses of Corollary 3 hold, then it follows by Theorem 2 that both $y''' + q_1^+ y = 0$ and $y''' - q_1^- y = 0$ are disconjugate on $[t_0, +\infty)$. Since $-q_1^-(t) \le q_1(t) \le q_1^+(t)$, it follows ([7], Theorem 3.11, [13]) that $y''' + q_1(t)y = 0$ is disconjugate on $[t_0, +\infty)$.

Proof of Theorem 4. The proof of Theorem 4 is similar to the proof of Theorem 2 and Corollary 3. We consider system (26) with $t_0 = a$ and $T = b$ and with u_1, u_2 defined as in (24), where y satisfies (18). Assuming $q_1 \ge 0$, then the corresponding system for $L_1^* y = 0$ is

$$(42)\qquad \begin{cases} v_1(t) = \displaystyle\int_t^b g_1(v_1, v_2)ds + v_1(b) , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a \le t \le b , \\ v_2(t) = \displaystyle\int_t^b g_2(v_1, v_2)ds - \hat{Q}_1(t) + v_2(b) , \end{cases}$$

where $\hat{Q}_1(t) = \int_t^b q_1(s)ds$. We may now define the successive approximations $\{v_{1n}\}$, $\{v_{2n}\}$ for $a \le t \le b$ by

$$v_{10}(t) = u_1(t) , \quad v_{20}(t) = u_2(t) ,$$

and

$$(43)\qquad v_{1n}(t) = \int_t^b g_1(v_{1n-1}, v_{2n-1})ds + u_1(b) ,$$

$$v_{2n}(t) = \int_t^b g_2(v_{1n-1}, v_{2n-1})ds - \hat{Q}_1(t) + u_2(b) .$$

The proof now proceeds as in Theorem 2 and we conclude that there exists a solution z of $L_1 y = 0$ satisfying (20) so that $L_1 y = 0$ is disconjugate on $[a, b]$ by Lemma 6. If now $q_1 \ge 0$ is not assumed but condition (17) holds, then we argue as in Corollary 3 to show that $L_1 y = 0$ is disconjugate on $[a, b]$. This completes the proof of Theorem 4.

EXAMPLE 1. A special case of a result of Lazer [11, Theorem 3.5] implies that equation (4) is disconjugate on $[t_0, +\infty)$ for some $t_0 \ge a$ in case $\int_a^\infty q_1(t)dt < +\infty$, $q_1 \ge 0$ and $q_1 \not\equiv 0$ in any subinterval and provided the second order equation

$$y'' + \tfrac{3}{2}\left(\int_t^\infty q_1(t)dt\right)y = 0 \tag{44}$$

is nonoscillatory. In particular, (44) is nonoscillatory (by comparison with $y'' + \frac{1}{4}t^{-2}y = 0$) in case

$$\limsup_{t\to+\infty} t^2 \int_t^\infty q_1(t)dt < \tfrac{1}{4} \cdot \tfrac{2}{3} = \tfrac{1}{6} . \tag{45}$$

Applying Theorem 2 with $q(t) = (2/3\sqrt{3})t^{-3}$ we conclude that $L_1y = 0$ is disconjugate on $[t_0, +\infty)$ for some $t_0 \ge a$ in case $q_1 \ge 0$ and

$$\int_t^\infty q_1(t)dt \le (1/3\sqrt{3})t^{-2} , \quad t \ge t_0 , \tag{46}$$

which improves (45).

Thus, if $q_1(t) = k\left(1 + \sin t^\delta\right)t^{-3}$, $\delta > 0$, then

$$\int_t^\infty q_1(s)ds = (k/2)t^{-2} + O\left(t^{-2-\delta}\right) , \quad t \to \infty ,$$

so that if $\frac{1}{3} < k < 2/3\sqrt{3}$, then (46) holds for large t and hence $L_1y = 0$ is disconjugate on $[t_0, +\infty)$ for some $t_0 \ge a$ by Theorem 2. However, equation (44) is oscillatory since $\frac{3}{2}\int_t^\infty q_1(s)ds > \frac{1}{4}t^{-2}$ for large t . Thus, the criterion of Lazer is not applicable to this example nor do the conditions (6) or (8) hold.

EXAMPLE 2. If $q(t) = (2/3\sqrt{3})t^{-3}$, then $L^*y = y''' - qy = 0$ has a solution $y = y(t) \not\equiv 0$ satisfying (2) on $I = [a, b]$ for all $0 < a < b < +\infty$, (that is, $y(t) = t^\lambda$, where $0 < \lambda < 1$). Therefore, if $q_1 \in C[a, b]$ and if

$$(47)\qquad \begin{cases} \displaystyle\int_t^b q_1^+(s)ds \le \left(t^{-2}-b^{-2}\right)/3\sqrt{3} \ , \\ \\ \displaystyle\int_t^b q_1^-(s)ds \le \left(t^{-2}-b^{-2}\right)/3\sqrt{3} \ , \end{cases} \qquad a \le t \le b \ ,$$

then it follows by Theorem 4 that $L_1y = 0$ is disconjugate on $[a, b]$.

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