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THE RESIDUAL FINITENESS OF POLYGONAL PRODUCTS—TWO COUNTEREXAMPLES

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ABSTRACT. We show that, even under very favourable hypotheses, a polygonal product of finitely generated torsion free nilpotent groups amalgamating infinite cyclic subgroups is, in general, *not* residually finite, thus answering negatively a question of C. Y. Tang. A second example shows similar kinds of limitations apply even when the factors of the product are free abelian groups.

Introduction. Over the last decade there has been renewed interest in so-called *polygonal products* of groups. (The definition is given below.) Their first explicit occurence may well have been in [4] where Higman gave an example of a finitely generated infinite simple group as a suitable factor group of such a polygonal product. The term *polygonal product* was introduced in [5] where the subgroup structure of such products was studied. This was used in [3] to investigate the subgroup structure of the Picard group. In [8] the question of SQ-universality was (amongst other things) considered whilst, in [2], [6] and [7], residual properties of polygonal products were investigated. Three results which set the present ones in context are stated below.

Definitions and notation. Let $\mathcal{P}$ be a polygon. To each vertex, v, of $\mathcal{P}$ assign a "vertex group" A_v and to each edge, e, of $\mathcal{P}$ assign an "edge group" A_e *and* a pair λ_e, μ_e of monomorphisms embedding A_e as a subgroup of each of the two vertex groups at the ends of e. The *polygonal product* of this system of groups is the group P with generators and relations those of the vertex groups together with the extra relations obtained by identifying $a_e\lambda_e$ and $a_e\mu_e$ for each a_e in A_e. Note that, if the polygon is a triangle the group P may not contain the vertex groups isomorphically. As a consequence we shall consider here only the case where the polygon has four edges (the polygonal product then being most naturally called a *square product*), the cases of polygons with more than four sides then being easily dealt with. We shall denote the vertex groups by A_i, $1 \leq i \leq 4$ and the subgroup $A_i \cap A_{i+1}$ determined by the edge joining A_i and A_{i+1} in $\mathcal{P}$ by K_i. *Throughout we shall assume that each $K_i \cap K_{i+1}$ is trivial.* Then P is a generalised free product (of generalised free products), $P = (A_1 *_{K_1} A_2) *_{K_2 * K_4} (A_3 *_{K_3} A_4)$—so that P contains all the A_i and their intersections isomorphically.

Three results we wish to record are:

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THEOREM 1 [6]. *Let each A_i be polycyclic-by-finite and suppose that each K_i lies in the centre of A_i and A_{i+1}. Then P is residually finite. (In fact P has the stronger property of being Π_c.)*

THEOREM 2 [7]. *Let each A_i be finitely generated, torsion free and nilpotent and let each K_i be maximal cycle (that is, K_i is contained in no larger cyclic subgroup of P). If in addition,* each subgroup $\langle K_{i-1}, K_i\rangle$ *of A_i is* isolated in A_i, *then P is residually finite.*

[Recall that the subgroup K of the group A is *isolated* in A if and only if ($g \in A$ and $g^n \in K$ for some $n \in \mathbb{Z}$) $\Rightarrow g \in K$.]

Theorem 1 contains the result in [2] where each group A_i was chosen to be finitely generated free abelian and each K_i to be cyclic. Theorem 2 was obtained as an antidote to the following example in [2]:

EXAMPLE 3. Suppose each A_i is finitely generated, torsion free and nilpotent and each K_i is infinite cyclic. Then P need not be residually finite.

The first of our examples, answering a question put to us by Prof. C. Y. Tang, shows that the (rather "untidy") condition in roman type in Theorem 2 cannot simply be ignored (as I at first believed it could!) The second shows that any reasonable attempt to strengthen the conclusion of Theorem 1 also seems doomed to failure.

The examples. (1) *A polygonal product of finitely generated torsion free nilpotent groups which is not residually finite.* For $1 \le i \le 4$, let $A_i = \langle a_i, b_i : [a_i, b_i, a_i] = [a_i, b_i, b_i] = 1\rangle$, the 2-generator free nilpotent group of class 2. Taking suffices modulo 4, set $K_i = \langle k_i\rangle = \langle h_{i+1}\rangle$ where $h_i = a_i^2[a_i, b_i]^7$ and $k_i = a_i^3[a_i, b_i]^7$. It is clear that each K_i is a maximal cyclic subgroup in the sense defined above. We claim that the resulting group P is not residually finite.

To prove this let P/M be any finite homomorphic image of P. Set $N = M \cap M\vartheta \cap M\vartheta^2 \cap M\vartheta^3$ where ϑ is the automorphism of P defined by $a_i \longrightarrow a_{i+1}$, $b_i \longrightarrow b_{i+1}$ for $1 \le i \le 4$. Write $\bar{P}$ for P/N. Clearly, in $\bar{P}$, the four elements $[\bar{a}_i, \bar{b}_i]$ ($1 \le i \le 4$) have the same (finite) order. We claim that this order is *not* a multiple of 7. For suppose that $|[\bar{a}_i, \bar{b}_i]| = 7n$ exactly. We then have, for $1 \le i \le 4$, $\bar{a}_i^{3n}[\bar{a}_i, \bar{b}_i]^{7n} = \bar{a}_{i+1}{}^{2n}[\bar{a}_{i+1}, \bar{b}_{i+1}]^{7n}$ and hence $\bar{a}_i^{3n} = \bar{a}_{i+1}^{2n}$. But this leads immediately to $\bar{a}_i^{81n} = \bar{a}_i^{16n}$, that is, $\bar{a}_i^{65n} = \bar{1}$ in $\bar{P}$. From this we conclude, since $[\bar{a}_i, \bar{b}_i]$ is central in $\bar{A}_i$, that $[\bar{a}_i, \bar{b}_i]^{65n} = [\bar{a}_i^{65n}, \bar{b}_i] = \bar{1}$. Together with $[\bar{a}_i, \bar{b}_i]^{7n} = \bar{1}$ this yields $[\bar{a}_i, \bar{b}_i]^n = \bar{1}$, a manifest contradiction. The above implies that, in $\bar{P}$, $[\bar{a}_i, \bar{b}_i] \in \langle [\bar{a}_i, \bar{b}_i]^7\rangle$, so that $[\bar{a}_i, \bar{b}_i] = [\bar{a}_i, \bar{b}_i]^{7m}$, for some integer m.

Now let α_i denote $[a_i, b_i]$ and consider the element $X = [\alpha_1^{27}\alpha_2^{18}\alpha_3^{12}\alpha_4^8, \alpha_1]$ of P. Since, in A_i, $\langle h_i, k_i\rangle = \langle a_i\rangle \times \langle [a_i, b_i]^7\rangle$ we see that none of α_1^{27}, α_2^{18}, α_3^{12}, α_4^8, α_1 belongs to $\langle h_1, k_1\rangle$, $\langle h_2, k_2\rangle$, $\langle h_3, k_3\rangle$, $\langle h_4, k_4\rangle$, $\langle h_1, k_1\rangle$ respectively. We can therefore deduce that $X \ne 1$ in P. On the other hand, for each i, $[a_i, b_i]^7 = h_i^3 k_i^{-2}$ so that in $\bar{P}$ we have $\bar{\alpha}_i = [\bar{a}_i, \bar{b}_i]^{7m} = \bar{h}_i^{3m}\bar{k}_i^{-2m}$ and $\bar{\alpha}_1^{27m}\bar{\alpha}_2^{18m}\bar{\alpha}_3^{12m}\bar{\alpha}_4^{8m} = \bar{h}_1^{81m}\bar{k}_1^{-54m}\bar{h}_2^{54m}\bar{k}_2^{-36m}\bar{h}_3^{36m}\bar{k}_3^{-24m}\bar{h}_4^{24m}\bar{k}_4^{-16m} = \bar{h}_1^{65m}$—since $k_i = h_{i+1}$ for each i. But then $\bar{X} = [\bar{h}_1^{65m}, \bar{\alpha}_1] = \bar{1}$ since $\bar{\alpha}_1$ is central in $\bar{A}_1$.

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(2) *A polygonal product of finitely generated free abelian groups which is not residually-finite-p for any prime p.*

This time, for $1 \leq i \leq n$, let $A_i = \langle a_i, b_i : [a_i, b_i] = 1\rangle$, the free abelian group of rank 2. Taking suffices modulo 4, set $K_i = \langle k_i \rangle = \langle h_{i+1} \rangle$ where $h_i = a_i^2 b_i^3$ and $k_i = a_i^4 b_i^3$. Again each K_i is a maximal cyclic subgroup of A_i. We claim that the resulting group P is not residually-finite-p for any prime p.

To prove this, consider the element $X = [b_1, b_1^8 b_2^4 b_3^2 b_4, a_1 a_2 a_3 a_4]$ of P. It is not difficult to check that $X \neq 1$ in P. Suppose first that P is residually-finite-p for some *odd* prime p. Then, as above, there is a normal subgroup N this time of p-power index in P such that, in $\bar{P} = P/N$, $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4$ all have the same order p^n, say. Consequently, for each i, $\bar{a}_i \in \langle \bar{a}_i^2 \rangle$, that is, $\bar{a}_1 = \bar{a}_1^{2m}$ for some integer m. But $a_i^2 = h_i^{-1} k_i$. Hence $\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4 = \bar{a}_1^{2m} \bar{a}_2^{2m} \bar{a}_3^{2m} \bar{a}_4^{2m} = \bar{h}_1^{-m} \bar{k}_1^m \bar{h}_2^{-m} \bar{k}_2^m \bar{h}_3^{-m} \bar{k}_3^m \bar{h}_4^{-m} \bar{k}_4^m = \bar{1}$ in $\bar{P}$, since $k_i = h_{i+1}$.

Now suppose that P is residually a finite 2-group so that there is a normal subgroup of N of 2-power index in P such that, in $\bar{P} = P/N$, $\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4$ all have the same order 2^u, say. Consequently, for each i, $\bar{b}_i \in \langle \bar{b}_i^3 \rangle$, $-\bar{b}_1 = \bar{b}_1^{3v}$, say, for some integer v. But $b_i^3 = h_i^2 k_i^{-1}$. Hence $\bar{b}_1^8 \bar{b}_2^4 \bar{b}_3^2 \bar{b}_4 = \bar{b}_1^{24v} \bar{b}_2^{12v} \bar{b}_3^{6v} \bar{b}_4^{3v} = \bar{h}_1^{16v} \bar{k}_1^{-8v} \bar{h}_2^{8v} \bar{k}_2^{-4v} \bar{h}_3^{4v} \bar{k}_3^{-2v} \bar{h}_4^{2v} \bar{k}_4^{-v} = \bar{h}_1^{15v}$. But $[\bar{b}_1, \bar{h}_1] = \bar{1}$, hence $\bar{X} = \bar{1}$ in $\bar{P}$, a contradiction.

COMMENT. Since the residual properties discussed here pass down to subgroups, the existence of polygonal products as just described would scarcely be surprising if (two factor) generalised free products (such as $A_1 *_{K_1} A_2$) didn't possess strong residual properties. In fact [1] shows that if A_1, A_2 are finitely generated and nilpotent and if K_1 is cyclic (not necessarily maximally so) then $A_1 *_{K_1} A_2$ has the very strong property of potency. (Recall that a group G is said to be *potent* [1] if to each element $g \in G$ and to each positive integer n dividing the order of g (meaning all positive integers if g has infinite order) there exists a finite homomorphic image $\bar{G}$ of G in which the image $\bar{g}$ of g has order *precisely* n.)

In relation to example 2 it is true that $\langle a, b : a^\lambda = b^\mu \rangle$ cannot be residually finite-p unless λ and μ are themselves powers of p (or λ or μ is equal to 1). Nevertheless example 2 insists that each K_i is maximal cyclic and if we make such assumptions about K_1 we find (when A_1 and A_2 are finitely generated torsion free abelian) that $A_1 *_{K_1} A_2$ is the direct product of K_1 with a free product of free abelian groups and so is certainly residually-finite-p for all primes p.

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