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ON MEAN CURVATURE FLOW OF SPACELIKE HYPERSURFACES IN ASYMPTOTICALLY FLAT SPACETIMES

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Abstract

We prove a priori estimates for the gradient and curvature of spacelike hypersurfaces moving by mean curvature in a Lorentzian manifold. These estimates are obtained under much weaker conditions than have been previously assumed. We also use mean curvature flow in the construction of maximal slices in asymptotically flat spacetimes. An essential tool is a maximum principle for sub-solutions of a parabolic operator on complete Riemannian manifolds with time-dependent metric.

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Introduction

Spacelike hypersurfaces of prescribed mean curvature have been an important tool in the study of the structure of Lorentzian manifolds; see, for example, [15] or [1] for a list of references. General existence and regularity results for such surfaces in cosmological spacetimes were first obtained by Gerhardt [9]. In [1], Bartnik proved the existence of entire spacelike maximal hypersurfaces, that is, hypersurfaces with zero mean curvature, in asymptotically flat spacetimes. These papers deal directly with the nonlinear *elliptic* differential equations describing the respective hypersurfaces. The existence proofs are non-constructive in that they use topological fixed point theorems.

In [8], an approach via a *parabolic* equation was taken. An initial hypersurface M_0, given by a spacelike embedding $x_0 : M^n \to \mathscr{V}$ of an n-dimensional manifold M^n into the spacetime $\mathscr{V}$, is deformed by the evolution equation

$$(1) \qquad \frac{d}{ds}x(p,s) = [(H - \mathscr{H})\nu](p,s) \qquad p \in M^n, \quad s \in (0, s_0)$$

subject to the initial condition $x(\cdot, 0) = x_0$, where $x_s = x(\cdot, s) : M^n \to \mathscr{V}$ denotes a one-parameter family of smooth spacelike embeddings, with images $x_s(M^n) = M_s$, mean curvature $H(\cdot, s)$ and future directed unit normal field $\nu(\cdot, s)$ and where $\mathscr{H} \in C^\infty(\mathscr{V})$ is a given function. Note that stationary solutions of (1) are spacelike hypersurfaces with mean curvature equal to $\mathscr{H}$ everywhere. Mean curvature flow problems have previously been studied in Riemannian manifolds; see [11] and [12].

In [8], existence and asymptotic convergence results for solutions of (1) in cosmological spacetimes $\mathscr{V}$ were proved. Although the parabolic approach has the advantage of providing a method of constructing hypersurfaces of prescribed mean curvature, a major technical difficulty arises due to the fact that the mean curvature of M_s is not a priori controlled by any structure conditions on $\mathscr{H}$ as in the stationary case. Obtaining such control, however, is an essential step in showing that the equation is uniformly parabolic. In [8], this problem was handled by imposing a monotonicity condition on the forcing term $\mathscr{H}$. Moreover the *timelike convergence condition*

$$\overline{\mathrm{Ric}}(X, X) \geq 0$$

for all timelike vector fields X was assumed to hold in $\mathscr{V}$.

In this paper we overcome some of these technical difficulties by estimating the gradient function and the mean curvature of the hypersurfaces M_s simultaneously, given that the height function on M_s is controlled; see Proposition 2.2. As in [8], a height bound follows by means of the strong maximum principle if we assume the existence of future and past barrier surfaces for M_0. This implies the existence of a solution of (1) for all $s \in (0, \infty)$ and asymptotic convergence of a subsequence of (M_s) to a stationary solution without monotonicity conditions on $\mathscr{H}$; see Theorem 2.1. The assumption on the Ricci curvature on $\mathscr{V}$ has been weakened to the condition

$$(2) \qquad \overline{\mathrm{Ric}}(X, X) \geq \kappa\, \overline{g}(X, X), \quad \kappa \geq 0,$$

for all timelike vector fields X, where $\overline{g}$ denotes the metric tensor on $\mathscr{V}$. Note, in particular, that in case $\mathscr{V}$ satisfies Einstein's equations, the *weak energy*

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condition (see [10, 4.3]) implies (2) with κ depending on the cosmological constant and a bound for the scalar curvature of $\mathscr{V}$.

An interesting feature of the a priori estimate implied by Proposition 2.2 is that it is *interior* in time, that is, it does not depend on the initial values of the estimated quantities. This suggests an approximation argument in order to solve (1) for initial data which are spacelike in a weaker sense. For the corresponding elliptic equation we refer to the *interior* gradient estimate in [2, Theorem 3.7] and its consequences.

In [1], the existence of spacelike hypersurfaces with prescribed mean curvature $\mathscr{H}$ was proven under very general structure conditions on $\mathscr{H}$ (for example dependence of $\mathscr{H}$ on the normal field to the hypersurface) and by merely assuming global bounds on the geometry of $\mathscr{V}$. So far we have not succeeded in adapting the parabolic problem to this more general setting. This may be related to a possible lack of global stability of the stationary solution with respect to the corresponding variational functional (area in the case $\mathscr{H} \equiv 0$; see [2, Section 6] for a general definition) in the absence of some of the restrictions we have imposed.

In Section 3, we establish convergence of a solution of (1) to an entire maximal hypersurface in an asymptotically flat spacetime, using a maximum principle for subsolutions of parabolic equations on a complete non-compact manifold with a parameter-dependent metric, proved in [8] (see also [14] for an earlier version in the case of a fixed metric), as well as a barrier argument based on the existence of maximal surfaces proved by Bartnik in [1]. For results on non-compact mean curvature flow in Euclidean space we refer to [6] and [7]. In [16], some of the interior estimation techniques of [7] have been adapted to mean curvature flow in Minkowski space.

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1. Evolution equations and maximum priciples

As in [8], we consider $(n+1)$-dimensional smooth spacetimes $\mathscr{V}$ with a Lorentzian metric $\overline{g} = (\overline{g}_{\alpha\beta})$ of signature $(-, +, +, \dots, +)$. The metric pairing will be denoted by $\langle \cdot, \cdot \rangle$, the canonical connection by $\overline{\nabla}$ and the curvature tensor by $\overline{\mathrm{Rm}} = (\overline{R}_{\alpha\beta\gamma\delta})$. Greek indices run from 0 to n. As in [1] we shall assume the existence of a global *time function* $t \in C^\infty(\mathscr{V})$ with nonzero, past-directed

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timelike vector field $\overline{\nabla} t$. The reference slices $\mathcal{S}_t = \{x \in \mathcal{V} : t(x) = t\}$ have a future-directed unit normal vector

$$T = -\psi \overline{\nabla} t$$

where the *lapse function* $\psi \in C_\infty(\mathcal{V})$ is defined by

$$\psi^{-2} = -\langle \overline{\nabla} t, \overline{\nabla} t \rangle.$$

We denote an adapted orthonormal frame for $\mathcal{S}_t$ by $e_0, e_1, \ldots, e_n$ where $e_0 = T$. For a smooth spacelike hypersurface M_n embedded into $\mathcal{V}$ by a map

$$x : M_n \to \mathcal{V}$$

we let ν denote the future-directed timelike unit normal and choose locally an adapted orthonormal frame $\tau_0, \tau_1, \ldots, \tau_n$ in $\mathcal{V}$ such that when restricted to $M = x(M^n)$ we have $\tau_0 = \nu$. We denote the induced metric and the curvature tensor on M by $g = (g_{ij})$ and $Rm = (R_{ijkl})$ respectively, where Latin indices range from 1 to n. The second fundamental form $A = (h_{ij})$ on M is defined by

$$h_{ij} = \langle \overline{\nabla}_{\tau_i} \nu, \tau_j \rangle = -\langle \overline{\nabla}_{\tau_i} \tau_j, \nu \rangle.$$

Summing over repeated indices we define

$$H = h_{ii}, \quad |A|^2 = h_{ij} h_{ij},$$

the first quantity being the mean curvature of M.

Furthermore, we consider the *height function* of M given by

$$u = t_{|M}$$

and the *gradient function*

$$v = -\langle \nu, T \rangle$$

which measures the angle between M and the reference slicing $\mathcal{S}_t$. Note, in particular, that $\langle \tau_\alpha, e_\beta \rangle \leq v$ for all $0 \leq \alpha, \beta \leq n$. Therefore, the restriction of any k-tensor $B \in T^k(\mathcal{V})$ to M can be estimated by

$$\| B_{|TM} \| \leq v^k \| B \| \tag{3}$$

where $\| \cdot \|$ denotes the tensor norm in $T^k(\mathcal{V})$. The norm in the space of m-times continuously differentiable tensorfields in a subset $K \subset \mathcal{V}$ will be denoted by $\| \cdot \|_{m,K}$. Let us recall the following evolution equations derived in [8] from equation (1).

PROPOSITION 1.1. *The metric, the normal and the volume element satisfy*

$$\text{(i)} \qquad \frac{d}{ds} g_{ij} = 2(H - \mathscr{H}) h_{ij},$$

$$\text{(ii)} \qquad \frac{d}{ds} \nu = \nabla(H - \mathscr{H}),$$

$$\text{(iii)} \qquad \frac{d}{ds} \mu_s = H(H - \mathscr{H}) \mu_s.$$

The height, gradient and mean curvature satisfy

$$\text{(iv)} \qquad \left(\frac{d}{ds} - \Delta\right) u = -\mathscr{H} \psi^{-1} v - \operatorname{div} \overline{\nabla} t,$$

$$\text{(v)} \qquad \left(\frac{d}{ds} - \Delta\right) v = -v(|A|^2 + \overline{\operatorname{Ric}}(\nu, \nu)) - T(H_T) + \langle \nabla \mathscr{H}, T \rangle - (H - \mathscr{H}) \langle \overline{\nabla}_\nu T, \nu \rangle,$$

$$\text{(vi)} \qquad \left(\frac{d}{ds} - \Delta\right) (H - \mathscr{H}) = -(H - \mathscr{H})(|A|^2 + \overline{\operatorname{Ric}}(\nu, \nu) + \langle \overline{\nabla} \mathscr{H}, \nu \rangle),$$

where $T(H_T)$ denotes the variation of H with respect to a deformation of $\mathscr{V}$ generated by T; see [1]. *The second fundamental form satisfies the estimates*

$$\text{(vii)} \qquad \left(\frac{d}{ds} - \Delta\right) |A|^2 \leq -2|\nabla A|^2 - |A|^4 + C$$

with $C = C(n, v, \| \overline{\nabla} \overline{\operatorname{Rm}} \|, \| \mathscr{H} \|, \| \overline{\nabla}^2 \mathscr{H} \|)$ and

$$\text{(viii)} \qquad \left(\frac{d}{ds} - \Delta\right) |\nabla^m A|^2 \leq -2|\nabla^{m+1} A|^2 + c(m, n) \sum_{i+j+k=m} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A| + C\left(1 + \sum_{j=0}^{m} |\nabla^j A|\right) |\nabla^m A|$$

for any $m \geq 1$, where

$$C = C\left(n, m, v, \sum_{j=0}^{m+1} \| \overline{\nabla}^j \overline{\operatorname{Rm}} \|, \sum_{j=0}^{m} \| \overline{\nabla}^j \mathscr{H} \|, \| \overline{\nabla}^{m+2} \mathscr{H} \|\right).$$

PROOF. The last inequality is derived similarly as in [12, Lemma 7.2] taking (3) into account. Note that the different signs arising in the Lorentzian setting are irrelevant for the estimate.

THEOREM 1.2. [7] *Let $(M_s)_{s\in[0,s_0]}$ be a family of geodesically complete non-compact spacelike hypersurfaces solving* (1). *Suppose the uniform volume growth condition*

$$\mathrm{vol}^s(B^s_\rho(p_0)) \le \exp(k_0(1+\rho^2))$$

holds for some point $p_0 \in M^n$ and a uniform constant $k_0 > 0$ for all $s \in [0, s_0]$ where $B^s_\rho(p_0)$ denotes a geodesic ball of radius ρ with respect to the induced metric $g(s) = (g_{ij}(s))$. Let f be a function on $M^n \times [0, s_0]$ which is smooth for $s \in (0, s_0)$ and continuous for $s \in [0, s_0]$. Assume that f satisfies

(i) $$\left(\frac{d}{ds} - \Delta\right) f \le \mathbf{a}\cdot\nabla f + bf$$

for a function b and a vector field $\mathbf{a}$ with $\sup_{M^n\times[0,s_0]}|b| < \infty$ and $\sup_{M^n\times[0,s_0]}|\mathbf{a}| < \infty$,

(ii) $f(p, 0) \le 0$ *for all* $p \in M^n$

(iii) $\int_0^{s_0}\int_{M^n} \exp(-\alpha l_s(p, p_0)^2)|\nabla f|^2(p, s)\,d\mu_s ds < \infty$

for some $\alpha > 0$ and geodesic distance l_s from $p_0 \in M^n$, as well as

(iv) $$\sup_{M^n\times[0,s_0]}\left|\frac{d}{ds}g_{ij}\right| < \infty.$$

Then we have $f \le 0$ on $M^n \times [0, s_0]$.

REMARK 1.3. The proof in [7] shows that we merely need to assume that f is a Lipschitz continuous function which satisfies inequality (i) in the weak sense.

COROLLARY 1.4. *Suppose the conditions on $(M_s)_{s\in[0,s_0]}$ in Theorem* 1.2 *hold and let f be a function on $M^n \times [0, s_0]$ satisfying condition* (iii) *and the inequality*

$$\left(\frac{d}{ds} - \Delta\right) f \le \mathbf{a}\cdot\nabla f - \delta^2 f^2 + C^2$$

for some positive numbers δ and C and vector field $\mathbf{a}$ with $\sup_{M^n\times[0,s_0]}|\mathbf{a}| < \infty$. Then f satisfies the estimates

(i) $f \le C/\delta + 1/(\delta^2 s)$ *on* $M^n \times (0, s_0]$
(ii) $f \le C/\delta + \sup_{M_0} f$ *on* $M^n \times [0, s_0]$.

REMARK 1.5. For a compact manifold M^n, this was proved in [8, Lemma 4.5]. In the non-compact case, however, we cannot use the maximum point argument employed there.

PROOF OF COROLLARY 1.4. For $g = s\,(f - C/\delta)$ we verify the inequality

$$\left(\frac{d}{ds} - \Delta\right) g \leq \mathbf{a} \cdot \nabla g - 2\delta C g + \delta^2 s^{-1} g \left(1/\delta^2 - g\right).$$

Let $g_\delta = \max(g - 1/\delta^2, 0)$. Using the relations $\nabla g_\delta = \nabla g$ a.e., $(g - 1/\delta^2) g_\delta = g_\delta^2$ as well as Young's inequality to control the term $\mathbf{a} \cdot \nabla g$ we obtain that g_δ satisfies the inequality

$$\left(\frac{d}{ds} - \Delta\right) g_\delta^2 \leq \frac{1}{2} \sup_{M^n \times [0,s_0]} |\mathbf{a}|^2 g_\delta^2$$

in the distributional sense. Since $|\nabla g_\delta|$ also satisfies condition (iii) of the theorem, inequality (i) follows in view of Remark 1.3. To prove (ii), we argue similarly that f_k^2 where $f_k = \max(f - k, 0)$ for $k = C/\delta + \sup_{M_0} f$ satisfies the same inequality as g_δ.

2. Mean curvature flow in cosmological spacetimes

In this section we assume that $\mathscr{V}$ is a cosmological spacetime, that is, it is connected, globally hyperbolic and admits a compact Cauchy surface which, in particular, implies the existence of a global time function as defined in Section 1.

Following [1] and [8], we call two compact spacelike C^2-hypersurfaces M^+ and M^- *barrier surfaces* for M_0 with respect to $\mathscr{H}$ if

$$M^\pm \subset I^\pm(M_0)$$

and

$$H_{M^+}(x) < \mathscr{H}(x) \qquad \text{for all } x \in M^+$$
$$H_{M^-}(x) > \mathscr{H}(x) \qquad \text{for all } x \in M^-$$

where $I^\pm(M_0)$ denotes the future and past of M_0, respectively.

THEOREM 2.1. *Let $\mathscr{V}$ be a cosmological spacetime satisfying condition* (2). *Assume furthermore that for a spacelike hypersurface M_0 two barrier surfaces*

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M^+ and M^- with respect to $\mathcal{H} \in C^\infty(\mathcal{V})$ exist. Then (1) *admits a smooth solution (M_s) for $s \in (0, \infty)$ with initial surface M_0. Moreover, every sequence $(s_k) \to \infty$ has a subsequence $(s_{k'}) \to \infty$ for which $(M_{s_{k'}})$ converges uniformly to a smooth spacelike hypersurface M_∞ satisfying*

$$H_{|M_\infty} = \mathcal{H}_{|M_\infty}.$$

REMARK 2.2. (i) For conditions which ensure asymptotic convergence to a unique limiting hypersurface and for a discussion of the rate of convergence we refer to [8].

(ii) The estimates on the quantities v and $|\nabla^m A|^2$, $m \geq 0$ on M_s for $s > 0$ depend only on n, $\inf_{M^+} t$, $\sup_{M^-} t$, κ, $\| \mathcal{H} \|_{\infty,\mathcal{V}}$, $\| t \|_{\infty,\mathcal{V}}$ and bounds on the geometry of $\mathcal{V}$.

(iii) Note that in contrast to [8], we do not impose any monotonicity condition on $\mathcal{H}$.

(iv) Condition (2) is only used in order to estimate

$$-(H - \mathcal{H})^2 \left(\overline{Ric}(\nu, \nu) + \langle \overline{\nabla} \mathcal{H}, \nu \rangle\right) \leq cv(H - \mathcal{H})^2$$

in the evolution equation for $(H - \mathcal{H})^2$. Note that the method in Propsition 2.3 could also handle an expression of the form $cv^{2-\epsilon}(H - \mathcal{H})^2$ for any $\epsilon > 0$. The weaker conditions in [1, Theorem 3.1], however, lead to a term of order $v^2(H - \mathcal{H})^2$.

The main step in the proof of the theorem is the following a priori estimate.

PROPOSITION 2.3. *Let (M_s) be a smooth solution of* (1) *which is contained in the region $K = \{x \in \mathcal{V}, |t(x)| \leq t_0\}$. Suppose condition* (2) *is satisfied. Then the function*

$$f = e^{\lambda u} v^2 + \mu (H - \mathcal{H})^2$$

satisfies the inequality

$$\left(\frac{d}{ds} - \Delta\right) f \leq \mathbf{a}.\nabla f - \delta^2 f^2 + C^2$$

with smooth vector field

$$\mathbf{a} = -\left(1 + \frac{1}{4n}\right)\left(v^{-2} e^{-\lambda u} \nabla f - 2\mu v^{-2} e^{-\lambda u} \nabla (H - \mathcal{H})^2\right) + \frac{\lambda}{2n} \nabla u$$

where λ, μ, δ and C depend on n, t_0, κ, $\| \psi \|_{1,K}$, $\| \mathcal{H} \|_{1,K}$ and $\| \overline{\mathrm{Rm}} \|_{0,K}$.

PROOF. From Proposition 1.1(iv) and (3) we infer as in [1] and [8] that

$$\left(\frac{d}{ds} - \Delta\right) u \leq c_0 v^2,$$

where c_0 depends on $\| \psi \|_{1,K}$ *and* $\| \mathscr{H} \|_{0,K}$. This gives

$$\left(\frac{d}{ds} - \Delta\right) e^{\lambda u} \leq c_0 \lambda e^{\lambda u} v^2 - \lambda^2 e^{\lambda u} |\nabla u|^2. \tag{4}$$

Proposition 1.1(v) implies as in [1] and [8] that

$$\left(\frac{d}{ds} - \Delta\right) v \leq -v|A|^2 + c(v^3 + v^2|A|),$$

where c depends on n, $\| \psi \|_{1,K}$ and $\| \mathscr{H} \|_{1,K}$ and $\| \overline{\mathrm{Rm}} \|_{0,K}$.

Estimating

$$|A|^2 v^2 \geq \left(1 + \frac{1}{2n}\right) |\nabla v|^2 - H^2 v^2 - c(n) v^4$$

as in [1] and using Young's inequality for the term $v^2|A|$ yields

$$\left(\frac{d}{ds} - \Delta\right) v^2 \leq -4\left(1 + \frac{1}{4n}\right) |\nabla v|^2 + c v^4 + 2H^2 v^2.$$

From $H^2 \leq 2(H - \mathscr{H})^2 + 2\mathscr{H}^2$ and the fact that $v \geq 1$ we thus obtain

$$\left(\frac{d}{ds} - \Delta\right) v^2 \leq -4\left(1 + \frac{1}{4n}\right) |\nabla v|^2 + c_1 v^4 + 4(H - \mathscr{H})^2 v^2 \tag{5}$$

where c_1 depends on n, $\| \psi \|_{1,K}$ and $\| \mathscr{H} \|_{1,K}$ and $\| \overline{\mathrm{Ric}} \|_{0,K}$.

Proposition 1.1(vi), assumption (2), the inequalities $|A|^2 \geq H^2/n$ and $H^2 \geq \frac{1}{2}(H - \mathscr{H})^2 - \mathscr{H}^2$ as well as (3) applied to the term $\langle \overline{\nabla} \mathscr{H}, \nu \rangle$ imply

$$\left(\frac{d}{ds} - \Delta\right) (H - \mathscr{H})^2 \leq -\frac{1}{n}(H - \mathscr{H})^4 + cv(H - \mathscr{H})^2 + \frac{2}{n}\mathscr{H}^2 - 2|\nabla(H - \mathscr{H})|^2$$

with c depending on κ and $\| \overline{\nabla} \mathscr{H} \|_{0,K}$ where we again used $v \geq 1$.

Using the inequality $v(H - \mathscr{H})^2 \leq \frac{1}{2n}(H - \mathscr{H})^4 + c(n)v^2$ and $v \geq 1$ yields

$$\left(\frac{d}{ds} - \Delta\right) (H - \mathscr{H})^2 \leq -\frac{1}{2n}(H - \mathscr{H})^4 - 2|\nabla(H - \mathscr{H})|^2 + c_2 v^2 \tag{6}$$

where c_2 depends on n, $\| \mathscr{H} \|_{1,K}$ and κ.

Define

$$f = e^{\lambda u} v^2 + \mu (H - \mathscr{H})^2$$

where λ and μ will be chosen later. We then compute from (4), (5) and (6)

$$\begin{aligned}\left(\frac{d}{ds} - \Delta\right) f \le\ & c_0 \lambda e^{\lambda u} v^4 - \lambda^2 e^{\lambda u} |\nabla u|^2 v^2 - 4(1 + \frac{1}{4n}) e^{\lambda u} |\nabla v|^2 \\ & + 4 e^{\lambda u} (H - \mathscr{H})^2 v^2 + c_1 e^{\lambda u} v^4 - 2 \nabla e^{\lambda u} . \nabla v^2 \\ & - \frac{\mu}{2n} (H - \mathscr{H})^4 + c_2 \mu v^2 - 2\mu |\nabla (H - \mathscr{H})|^2 .\end{aligned}$$

Observe the identities

$$-2 \nabla e^{\lambda u} . \nabla v^2 = -2 \lambda \nabla u . \nabla f + 2 \lambda^2 e^{\lambda u} |\nabla u|^2 v^2 + 2 \lambda \mu \nabla u . \nabla (H - \mathscr{H})^2$$

and

$$\begin{aligned}-4 e^{\lambda u} |\nabla v|^2 =\ & \mathbf{b} . \nabla f - \lambda^2 e^{\lambda u} |\nabla u|^2 v^2 - \mu^2 v^{-2} e^{-\lambda u} |\nabla (H - \mathscr{H})^2|^2 \\ & - 2 \lambda \mu \nabla u . \nabla (H - \mathscr{H})^2 ,\end{aligned}$$

where

$$\mathbf{b} = v^{-2} e^{-\lambda u} \nabla f - 2 \mu v^{-2} e^{-\lambda u} \nabla (H - \mathscr{H})^2 - 2 \lambda \nabla u .$$

This implies

$$\begin{aligned}\left(\frac{d}{ds} - \Delta\right) f \le\ & \mathbf{a} . \nabla f + (c_0 \lambda + c_1) e^{\lambda u} v^4 + c_2 \mu v^2 + 4 e^{\lambda u} (H - \mathscr{H})^2 v^2 \\ & - \frac{\mu}{2n} (H - \mathscr{H})^4 - 2 \mu |\nabla (H - \mathscr{H})|^2 - \frac{1}{4n} \lambda^2 e^{\lambda u} |\nabla u|^2 v^2 \\ & - (1 + \frac{1}{4n}) \mu^2 v^{-2} e^{-\lambda u} |\nabla (H - \mathscr{H})^2|^2 - \frac{1}{2n} \lambda \mu \nabla u . \nabla (H - \mathscr{H})^2\end{aligned}$$

where

$$\mathbf{a} = -(1 + \frac{1}{4n}) \left(v^{-2} e^{-\lambda u} \nabla f - 2 \mu v^{-2} e^{-\lambda u} \nabla (H - \mathscr{H})^2 \right) + \frac{\lambda}{2n} \nabla u .$$

In view of

$$\begin{aligned}\left| -\frac{1}{2n} \lambda \mu \nabla u . \nabla (H - \mathscr{H})^2 \right| \le\ & \left(1 + \frac{1}{4n}\right) \mu^2 v^{-2} e^{-\lambda u} |\nabla (H - \mathscr{H})^2|^2 \\ & + \frac{1}{4n(1 + 4n)} \lambda^2 e^{\lambda u} v^2 |\nabla u|^2\end{aligned}$$

and by discarding the term $-2\mu|\nabla(H-\mathscr{H})|^2$ we obtain

$$\left(\frac{d}{ds}-\Delta\right)f \le \mathbf{a}.\nabla f - \frac{1}{1+4n}\lambda^2 e^{\lambda u}|\nabla u|^2 v^2 - \frac{\mu}{2n}(H-\mathscr{H})^4 + (c_0\lambda + c_1)e^{\lambda u}v^4 + c_2\mu v^2 + 4e^{\lambda u}(H-\mathscr{H})^2 v^2.$$

We observe that

$$4e^{\lambda u}(H-\mathscr{H})^2 v^2 \le \frac{1}{4n\lambda}e^{\lambda u}(H-\mathscr{H})^4 + 16n\lambda e^{\lambda u}v^4$$

and

$$c_2\mu v^2 \le \lambda v^4 e^{\lambda u} + \frac{1}{4}c_2^2\mu^2\lambda^{-1}e^{-\lambda u},$$

we use the identity $|\nabla u|^2 = \psi^{-2}(v^2-1)$ (see [1]), the fact that $v \ge 1$ and we assume $\lambda \ge 1$ to arrive at

$$\left(\frac{d}{ds}-\Delta\right)f \le \mathbf{a}.\nabla f - \frac{1}{1+4n}\lambda(\lambda\psi^{-2}-c_3)e^{\lambda u}v^4 - \frac{1}{2n}\left(\mu - \frac{1}{2\lambda}e^{\lambda u}\right)(H-\mathscr{H})^4 + c_4\mu^2\lambda^{-1}e^{-\lambda u} + \frac{1}{1+4n}\lambda^2\psi^{-2}e^{\lambda u}v^2,$$

where c_3 and c_4 depend on c_0, c_1, c_2 and n. Upon estimating

$$\frac{1}{1+4n}\lambda^2\psi^{-2}e^{\lambda u}v^2 \le \lambda e^{\lambda u}v^4 + ce^{\lambda u}$$

with $c = c(n, \lambda, \|\psi\|_{0,K})$ we choose $\lambda \ge 1$ such that $\lambda(\lambda\psi^{-2}-c_3) \ge 2(1+4n)$. We furthermore set $\mu = \lambda^{-1}\sup_K e^{\lambda u}$. Thus

$$\left(\frac{d}{ds}-\Delta\right)f \le \mathbf{a}.\nabla f - e^{\lambda u}v^4 - \frac{\mu}{4n}(H-\mathscr{H})^4 + C^2,$$

where C depends on λ, μ and all the previous constants.

From the inequality $f^2 \le 2\max\{e^{\lambda u}, 4n\mu\}(e^{\lambda u}v^4 + (\mu/4n)(H-\mathscr{H})^4)$, we finally conclude that

$$\left(\frac{d}{ds}-\Delta\right)f \le \mathbf{a}.\nabla f - \delta^2 f^2 + C^2$$

where δ and C depend on all the above quantities.

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PROOF OF THEOREM 2.1. We essentially follow the proof in [8, Theorem 4.1]. The only difference arises due to the stronger assumptions on $\overline{\mathrm{Ric}}$ and $\mathscr{H}$ in [8] which were imposed in order to first obtain bounds on $|H|$ and then estimate v in terms of $|H|$ and $|u|$. Here we obtain a priori bounds on v and $|H|$ simultaneously by combining Proposition 2.2 with Corollary 1.4 for compact M^n, see Remark 1.5. The height estimate

$$\sup_{M_s}|t| \leq t_0 \equiv \max\{\inf_{M^+} t, \sup_{M^-} t\}$$

follows from the strong maximum principle as in the proof [8, Theorem 4.2]. Note, in particular, that by Corollary 1.4(i) the estimates on v and $|H|$ do not depend on the initial values of these quantities.

Estimates on $|A|^2$ and $|\nabla^m A|^2$ are now derived from Proposition 1.1(vii) and (viii) and Corollary 1.4 in exactly the same manner as in [8, Proposition 4.7].

To obtain a bound on $|\nabla^m A|^2, m \geq 1$ which is independent of the initial values, we use Proposition 1.1(vii) and (viii) to verify inductively as in [7, Proof of Theorem 4.1] that the function

$$f = \varphi^{m+1}|\nabla^m A|^2(\Lambda + \varphi^m|\nabla^{m-1}A|^2),$$

with $\varphi(s) = s/(s+1)$ which vanishes for $s = 0$, satisfies an inequality of the form

$$\left(\frac{d}{ds} - \Delta\right) f \leq -\varphi^{-1}(\delta^2 f^2 - C^2).$$

Note that similarly as in the maximum point argument given in [8, Lemma 4.5], the right hand side of the inequality becomes negative where the maximum of f reaches a value greater than C/δ for the first positive time.

3. Mean curvature flow in asymptotically flat spacetimes

In this section we adopt the definitions and assumptions of [1, Section 5]: let $\mathscr{V}$ be a 4-dimensional spacetime with non-negative radius function $r \in C^\infty(\mathscr{V})$ and time function $t \in C^\infty(\mathscr{V})$ satisfying the conditions of Section 1.

Following [1], we call $\mathscr{V}$ *asymptotically flat* if there is a constant $R_0 \geq 1$ such that the *exterior region* $\mathscr{V}_E = \{x \in \mathscr{V} : r(x) \geq R_0\}$ has coordinates (y^i, t) such that

$$r = \left(\sum_1^3 (y^i)^2\right)^{1/2},$$

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$$ds^2 = -(\psi^2 - \sigma^2)dt^2 + 2\sigma_i dy^i dt + \overline{g}_{ij} dy^i dy^j = \overline{g}_{\alpha\beta} dy^\alpha dy^\beta,$$

where σ is the *shift vector* and there are constants C_3 and C_4 such that

$$r \sum_{\alpha,\beta} |\overline{g}_{\alpha\beta} - \eta_{\alpha\beta}| + r^2 \sum_{\alpha,\beta,\gamma} \left| \frac{\partial \overline{g}_{\alpha\beta}}{\partial y^\gamma} \right| \leq C_3 \text{ and } r^3 |H_{\mathscr{S}}| \leq C_4$$

with $C_3 R_0^{-1} \leq 10^{-2}$, where $\eta_{\alpha\beta}$ is the Minkowski metric and $H_{\mathscr{S}}$ denotes the mean curvature of the slices $\mathscr{S}_t$.

The *interior region* of $\mathscr{V}$ is defined by $\mathscr{V}_I = \{x \in \mathscr{V}, r(x) \leq R_0\}$. $\mathscr{V}$ satisfies the *uniform interior condition* if there is a constant C_5 such that for all $z \in \mathscr{V}$ with $r(z) = R_0$,

$$\sup_{x \in \mathscr{V}_I - I^+(z)} (t(x) - t(z)) \leq C_5 \quad \text{if } t(z) \geq 0;$$
$$\sup_{x \in \mathscr{V}_I - I^-(z)} (t(x) - t(z)) \leq C_5 \quad \text{if } t(z) \leq 0.$$

In the following we will additionally assume that there are coordinates (y, t) covering a region of $\mathscr{V}$ large enough to admit entire maximal hypersurfaces asymptotic to the slices $\mathscr{S}_t$ for $t \in (-2C_8 - 1, 2C_8 + 1)$ where C_8 is the height bound obtained in [1, Theorem 5.3]. For a statement of the conditions required to ensure this we refer to [1, Section 5].

THEOREM 3.1. *Suppose $\mathscr{V}$ satisfies the above assumptions and condition* (2). *Then the initial value problem* (1) *with $M_0 = S_0$ and $\mathscr{H} \equiv 0$ has a smooth solution for all $s \in [0, \infty)$. In case the timelike convergence condition*

$$\overline{\mathrm{Ric}}(X, X) \geq 0$$

for all timelike vector fields X is satisfied, every sequence $(s_k) \to \infty$ has a subsequence $(s_{k'}) \to \infty$ for which $(M_{s_{k'}})$ converges uniformly on compact subsets to a smooth entire maximal hypersurface in the region $\{x \in \mathscr{V} : |t(x)| \leq 2C_8 + 1\}$.

REMARK 3.2. (i) Note that Theorem 3.1 relies on the existence of maximal hypersurfaces which are used as barriers for the mean curvature evolution. For weaker asymptotic flatness conditions which still ensure the existence of maximal slices we refer to [3].

(ii) The long-time asymptotic behaviour of the hypersurfaces M_s at spatial infinity has not yet been studied.

LEMMA 3.3. *Let $(M_s)_{s\in[0,s_0]}$, $s_0 < \infty$ be a solution of* (1) *in an asymptotically flat spacetime $\mathscr{V}$ with $M_0 = S_0$ and suppose*

$$\sup_{M^3\times[0,s_0]} v \le \alpha_0 < \infty \text{ and } \sup_{M^3\times[0,s_0]} |A| \le \alpha_1 < \infty.$$

Then $(M_s)_{s\in[0,s_0]}$ satisfies the conditions of Theorem 1.2. *In particular,*

$$\int_0^{s_0}\int_{M^3} \exp(-l_s(p, p_0))d\mu_s ds < \infty$$

for fixed $p_0 \in M^3$ which implies that any function f with $\sup_{M^3\times[0,s_0]}|\nabla f| < \infty$ satisfies condition (iii) *of Theorem* 1.2.

PROOF. We will only consider the case where $\mathscr{H} \equiv 0$. From $|\nabla r| \le |\overline{\nabla} r| v$ we see that

$$\sup_{M^n\times[0,s_0]} |\nabla r| \le \alpha_2 < \infty$$

where $\alpha_2 = \alpha_2(\alpha_0, \|\ \overline{\nabla} r\ \|_{0,\mathscr{V}})$. Integrating along a geodesic with respect to $g(s)$ from p_0 to p therefore yields

$$r(x_s(p)) \le r(x_s(p_0)) + \alpha_2 l_s(p, p_0). \tag{7}$$

We furthermore note that by (1) we have for any $p \in M^3$ and $s \in [0, s_0]$,

$$|r(x_s(p)) - r(x_0(p))| \le \alpha_3 s_0, \tag{8}$$

where α_3 depends on α_0, α_1 and $\|\ \overline{\nabla} r\ \|_{0,\mathscr{V}}$. Combining (8) for p_0 with (7), we see that geodesic balls in M_s are contained in the compact sets $M_s \cap \{x \in \mathscr{V} : r(x) \le R\}$. This implies geodesic completeness of the M_s for $s \in [0, s_0]$.

From (7), (8) and Proposition 1.1(iii) we infer

$$\int_{M^3} \exp(-l_s(p, p_0))d\mu_s \le \alpha_4 \int_{\mathscr{S}_0} \exp(-\alpha_2^{-1} r)$$

where α_4 depends on α_1, α_2, α_3, s_0 and $r(x_0(p_0))$. Asymptotic flatness therefore implies

$$\int_0^{s_0}\int_{M^3} \exp(-l_s(p, p_0))d\mu_s ds < \infty$$

for fixed $p_0 \in M^3$.

In view of Proposition 1.1(i), we also have

$$\sup_{M^3\times[0,s_0]} \left|\frac{d}{ds} g_{ij}\right| \leq \alpha_5 < \infty,$$

with α_5 depending on α_1, which establishes condition (iv) of Theorem 1.2. The Gauss equations imply

$$\mathrm{Ric}_{ij} = \overline{\mathrm{Ric}}_{ij} - Hh_{ij} + h_{ik}h_{kj} + \overline{R}_{i0j0}$$

where Ric denotes the Ricci tensor on M_s. Using (3) and the finiteness assumption on v and $|A|$ therefore yields

$$\mathrm{Ric}_{M_s} \geq \alpha_6 > -\infty$$

uniformly in s where α_6 depends on α_0, α_1 and $\| \overline{\mathrm{Rm}} \|_{0,\mathscr{V}}$. The volume growth condition now follows from a standard argument (see for example [4]).

PROOF OF THEOREM 3.1. We assume for simplicity that $M_0 = \mathscr{S}_0$ satisfies

$$\sup_{M_0} |\nabla^m A|^2 < \infty$$

for all $m \geq 0$. Note also that $v = 1$ on $\mathscr{S}_0$ although a uniform bound on v at $s = 0$ would suffice for the argument. We solve the initial value problem on a small time interval $[0, s_0]$ by working in the class of hypersurfaces with uniformly bounded quantities v and $|\nabla^m A|^2$ for all $m \geq 0$ (note that uniform $C^{2,\alpha}$-bounds would be sufficient to ensure short-time existence). For any solution (M_s) of (1) in this class, it is shown by direct computation from (1) that the supremum of all geometric quantities including the height u on M_s depends continuously on s. Using standard arguments, one may therefore reduce the short-time existence problem for (1) to a problem for a uniformly parabolic equation with uniformly bounded coefficients, see for example [13].

Having thus obtained a smooth solution (M_s) on some interval $[0, s_0]$ which satisfies

(9) $$\sup_{M^3\times[0,s_0]} (|u| + v) < \infty$$

and

(10) $$\sup_{M^3\times[0,s_0]} |\nabla^m A|^2 < \infty \quad \text{for all } m \geq 0,$$

we are, in view of the previous lemma, in a position to employ the maximum principle to obtain a priori estimates on these quantities for all $s \geq 0$. We begin with a height bound: from (1) we compute for $s \in (0, s_0)$,

$$\left(\frac{d}{ds} - \Delta\right) r = -\mathrm{div}\overline{\nabla} r \leq \beta_1,$$

where β_1 depends on $\sup_{M^3\times[0,s_0]} v$ and $\| \overline{\nabla}^2 r \|_{0,\mathscr{V}}$. Combining this with Proposition 1.1(vi) gives in view of the inequality $|A|^2 \geq H^2/n$,

$$\left(\frac{d}{ds} - \Delta\right) H^2(1+r^2) \leq -(\frac{2}{n}H^4 + 2|\nabla H|^2 - \beta_2 H^2 v^2)(1+r^2)$$
$$+2\beta_1 H^2 r - 2H^2|\nabla r|^2 - 2\nabla r^2.\nabla H^2,$$

where β_2 depends on $\| \overline{\mathrm{Ric}} \|_{0,\mathscr{V}}$. We then estimate

$$|2\nabla r^2.\nabla H^2| \leq 2|\nabla H|^2(1+r^2) + 8|\nabla r|^2 H^2$$

and use the inequality $|\nabla r| \leq |\overline{\nabla} r| v$ to obtain for $f = H^2(1+r^2)$

$$\left(\frac{d}{ds} - \Delta\right) f \leq \beta_3 f,$$

where β_3 depends on $\sup_{M^3\times[0,s_0]} v$ and $\| \overline{\mathrm{Ric}} \|_{0,\mathscr{V}}$. By (9) and (10) we also have $\sup_{M^3\times[0,s_0]} |\nabla f| < \infty$. Furthermore, we observe the inequality $\sup_{\mathscr{S}_0}(1+r^2)H^2 \leq \beta_4$ which follows from asymptotic flatness. By Lemma 3.3, we may therefore apply Theorem 1.2 to the function $e^{-\beta_3 s} f - \beta_4$ to arrive at

(11) $$\sup_{M^3\times[0,s_0]} (1+r^2)H^2 \leq \beta_4 e^{\beta_3 s_0}$$

On the other hand, we compute from (1),

$$\left|\frac{d}{ds} u^2(1+r)\right| \leq 2\psi^{-1} v|u||H|(1+r) + u^2|\overline{\nabla} r||H|v$$

such that in view of (9), (10), (11) and the fact that $u \equiv 0$ on $\mathscr{S}_0$ we obtain

$$\sup_{M^3\times[0,s_0]} u^2(1+r) < \infty$$

This implies that for fixed $s \in [0, s_0]$

(12) $$u(x_s(p), s) \to 0 \text{ as } r(x_s(p)) \to \infty.$$

The assumptions of the theorem allow us to apply [1, Theorem 5.4] to obtain two entire maximal hypersurfaces M^+ and M^- which are asymptotic to the reference slices $\mathcal{S}_{C_8+1}$ and $\mathcal{S}_{-(C_8+1)}$ respectively where C_8 is the height bound obtained in [1, Theorem 5.3]. This height estimate when applied to M^+ and M^- also yields that

$$\inf_{M^+} t \geq 1 \text{ and } \sup_{M^-} t \leq -1.$$

Therefore we infer from (12) that, unless

$$\inf_{M^+} t > t_{|M_s} > \sup_{M^-} t$$

for all $s \in [0, s_0]$, there must be a first parameter $s_1 \in (0, s_0]$ for which M_{s_1} touches either M^+ or M^-. By the strong maximum principle for parabolic equations, this is impossible in view of the fact that M^+ and M^- are stationary solutions of (1). This establishes the estimate

$$\sup_{M^3\times[0,s_0]} |u| \leq t_0 \equiv \max\{\inf_{M^+} t, \sup_{M^-} t\}.$$

Condition (2) now enables us to invoke Proposition 2.3. Note, in particular, that for $f = e^{\lambda u}v^2 + \mu H^2$, the inequality $\sup_{M^3\times[0,s_0]}(|\mathbf{a} + |\nabla f|) < \infty$ holds in view of (9) and (10). Taking Lemma 3.3 into account we may therefore apply Corollary 1.4 to infer

$$\sup_{M_s}(v + |H|) \leq c(1 + \frac{1}{\sqrt{s}}) \text{ for } s \in (0, s_0]$$

or

$$\sup_{M_s}(v + |H|) \leq c(1 + \sup_{\mathcal{S}_0}(v + |H|)) \text{ for } s \in [0, s_0]$$

where c depends on $t_0, \kappa, \| \psi \|_{1,K}$ and $\| \overline{\mathrm{Rm}} \|_{0,K}$ in $K = \{x \in \mathcal{V} : |t(x)| \leq t_0\}$.

To obtain bounds on $|\nabla^m A|^2$ for $m \geq 0$ we proceed as in the proof of Theorem 2.2, this time, however, using (9), (10), Lemma 3.3 and Corollary 1.4. In particular, for $m \geq 1$, we define as in the proof of Theorem 2.1 the function $f = \varphi^{m+1}|\nabla^m A|^2(\Lambda + \varphi^m|\nabla^{m-1}A|^2)$ which vanishes at $s = 0$. The inequality

$$\left(\frac{d}{ds} - \Delta\right) f \leq -\varphi^{-1}\delta^2(f^2 - \frac{C^2}{\delta^2})$$

then implies that $f_\delta = \max(f - C/\delta, 0)$ satisfies

$$\left(\frac{d}{ds} - \Delta\right) f_\delta^2 \leq 0$$

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in the distributional sense so that we can apply Theorem 1.2.

Since our estimates are independent of s_0 we conclude the existence of the solution (M_s) for all $s \in [0, \infty)$. Note again that the a priori estimates on all geometric quantities of M_s depend only on the height bound t_0, the time function t and bounds on the geometry of $\mathcal{V}$ in $K = \{x \in \mathcal{V} : |t(x)| \le t_0\}$.

If we additionally assume the timelike convergence condition, we obtain from Proposition 1.1(vi) and the inequality $|A|^2 \ge H^2/n$ that

$$\left(\frac{d}{ds} - \Delta\right) H^2 \le -\frac{2}{n} H^4,$$

which yields

$$\sup_{M_s} H^2 \le \frac{n}{2s}$$

for $s \in (0, \infty)$, in view of Corollary 1.4(i).

This decay estimate and the uniform bounds for all geometric quantities on M_s imply that for every sequence $(s_k) \to \infty$ we can select a subsequence $(s_{k'}) \to \infty$ such that $(M_{s_{k'}})$ converges uniformly on compact subsets to a smooth entire maximal hypersurface in the region $\{x \in \mathcal{V} : |t(x)| \le t_0\}$.

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