QUOTIENT RINGS, CHAIN CONDITIONS AND INJECTIVE RING ENDOMORPHISMS

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1. Introduction. In this paper, the situation we shall be concerned with is that of a ring R, with a ring monomorphism $\alpha: R \to R$, which will not be assumed to be surjective.

Much work has been done on the skew polynomial ring $R[x, \alpha]$ and the skew Laurent polynomial ring $R[x, x^{-1}, \alpha]$, where α is an automorphism—see [**3**] for example. However, the fact that α is not surjective renders the study of these objects much more difficult.

It is with this in mind that D. A. Jordan [**4**] constructs a minimal overring $A(R, \alpha)$ to which α extends as an automorphism $\bar{\alpha}$ say. Using the fact that $A(R, \alpha)[x, x^{-1}, \bar{\alpha}]$ coincides with $R[x, x^{-1}, \alpha]$ (see [**4**]), it is clear that existing results for the case where α is an automorphism can be used in the non-surjective case, provided we can handle the relationship between R and $A(R, \alpha)$. This paper studies that relationship, with particular regard to chain conditions and quotient rings for $A(R, \alpha)$.

The paper is divided into three main sections, the first of which deals with conditions on a left Noetherian ring which are equivalent to $A(R, \alpha)$ being a left order in a left Artinian ring. The second section answers in the negative a question raised by Jordan in [**4**], where he asks whether R having left Krull dimension 1 is sufficient to ensure that $A(R, \alpha)$ has left Krull dimension. The final section presents an example which shows that it is possible for R to have acc on annihilator left ideals, but for this condition to fail in $A(R, \alpha)$.

2. Preliminaries. The purpose of this section is to present the relevant definitions and results concerning $A(R, \alpha)$. These come mainly from [**4**], and deal with the relationship between the left ideal structure of $A(R, \alpha)$ and the left ideal structure of R.

All rings are assumed to have unity, and it will be assumed that all monomorphisms $\alpha: R \to R$ satisfy $\alpha(1) = 1$. To say that $I \subseteq R$ is an *ideal* will mean that it is both a left ideal and a right ideal—a similar interpretation will be placed on the words *Artinian*, *Noetherian*, and so on. The nilpotent radical $N(R)$ of a ring R will be taken to be the sum of all the nilpotent left ideals of R, and if S is a subset of R then $C_R(S)$ will denote all the elements of R which are regular modulo S.

We begin by defining the ring $A(R, \alpha)$. A more detailed construction may be found in [**4**].

Definition 2.1 [**4**]. Let R be a ring, $\alpha: R \to R$ a ring monomorphism, and let $R[x, x^{-1}, \alpha]$ be the skew Laurent polynomial ring, having as elements finite sums of elements of the form $x^{-j} r x^{i}$, where $i, j \geqq 0$ and $r \in R$.

Then $A(R, \alpha)$ is the subring $\{x^{-i} r x^{i} \mid r \in R, i \geqq 0\}$ of $R[x, x^{-1}, \alpha]$.

REMARK. It can be shown (see [4]) that $A(R, \alpha)$ is, up to isomorphism, the minimal overring of R to which α extends as an automorphism. The action of α on $A(R, \alpha)$ is defined by $\alpha(x^{-i}rx^i)=x^{-i}\alpha(r)x^i$, and no confusion should arise from the fact that α denotes both the original monomorphism on R and the automorphism on $A(R, \alpha)$.

DEFINITION 2.2 [4]. Let $\alpha:R\to R$ be a ring monomorphism. Then a left ideal I of R is said to be *closed* if $\bigcup_{n\geqq 0} \alpha^{-n}(R\alpha^n(I))\subseteq I$.

DEFINITION 2.3 [4]. A sequence $(I_i)_{i\geqq 0}$ of subsets of R such that, for all $i\geqq 0$, $\alpha^{-1}(I_{i+1})=I_i$ is called an *α-sequence.*

REMARK. It is easily shown that, given an α-sequence $(I_i)_{i\geqq 0}$ of left ideals of R, I_i is closed for each $i\geqq 0$.

DEFINITION 2.4 [4]. Let $(I_i)_{i\geqq 0}$ and $(J_i)_{i\geqq 0}$ be α-sequences of closed left ideals of R. Then define a relation $\leqq$ on the set of α-sequences of closed left ideals of R by putting $(I_i)_{i\geqq 0}\leqq (J_i)_{i\geqq 0}$ if and only if $I_i\subseteq J_i$ for all $i\geqq 0$.

It is clear that $\leqq$ defines a partial ordering on the set of all α-sequences of closed left ideals of R. The significance of these three definitions is made precise by the following theorem.

THEOREM 2.5 [4]. *There exists an order-preserving bijection Γ from the lattice of left ideals of $A(R, \alpha)$ to the partially ordered set of α-sequences of closed left ideals of R given by*

$$\Gamma(I)=(I_i)_{i\geqq 0}, \quad \textit{where} \quad I_i=\{r\in R \mid x^{-i}rx^i\in I\}.$$

The inverse map Δ is given by

$$\Delta(I_i)_{i\geqq 0}=\bigcup_{i\geqq 0} x^{-i}I_ix^i$$

and is also order-preserving.

Proof. Theorem 4.7 of [4].

Among the consequences of Theorem 2.5 are the following results.

THEOREM 2.6. *If R is left Artinian then $A(R, \alpha)$ is also left Artinian.*

Proof. See [4, Corollary 5.3].

THEOREM 2.7. *If R is a semiprime left Goldie ring then $A(R, \alpha)$ is a semiprime left Goldie ring.*

Proof. Corollary 7.4 of [4].

DEFINITION 2.8. Let R be a ring and $\alpha:R\to R$ a monomorphism. Then a left, right, or two sided ideal I is said to be *α-invariant* if $\alpha(I)\subseteq I$. It is said to be *α-stable* if I is α-invariant and $\alpha^{-1}(I)\subseteq I$.

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The above definitions often prove useful when dealing with certain ideals of $A(R, \alpha)$. In particular, it is easy to see that the nilpotent radical of a ring is stable under any automorphism of that ring—so $N(A(R, \alpha))$ is stable under the automorphism $\alpha: A(R, \alpha) \to A(R, \alpha)$.

PROPOSITION 2.9. *Let I be an α-stable left ideal of $A(R, \alpha)$, with corresponding α-sequence $\Gamma(I) = (I_i)_{i \geqq 0}$. Then $I_i = I_j$ for all $i, j \geqq 0$.*

Proof. Let $i \geqslant 0$ and let $r \in I_i$. Then $x^{-i}rx^i \in I$, and since I is an α-stable left ideal of $A(R, \alpha)$, $\alpha^i(x^{-i}rx^i) \in I$, i.e. $r \in I_0$.

Now, if $r \in I_0$ then $r \in I$ and, since I is α-stable, $\alpha^{-i}(r) \in I$, i.e. $x^{-i}rx^i \in I$, or $r \in I_i$.

3. Artinian quotient rings. The aim of this section is to obtain necessary and sufficient conditions on a left Noetherian ring R so that $A(R, \alpha)$ is a left order in a left Artinian ring. In addition to R being left Noetherian, it will be assumed that the nilpotent radical $N(R)$ of R is α-invariant—this assumption is not very restrictive, as there are no known examples of a left Noetherian ring whose nilpotent radical is not α-invariant. Moreover, it has been shown by Dean **[1]** that if R is left Noetherian with acc on right annihilators then $N(R)$ is invariant under any monomorphism $\alpha: R \to R$.

It is important to note that $A(R, \alpha)$ is not assumed to be left Noetherian, and the proof of the main result relies on a non-Noetherian version of Small's theorem. Before presenting this variation of Small's theorem, we recall the definition of reduced rank for a left R-module.

DEFINITION 3.1. Let R be a ring, M a left R-module, and $Z(M)$ the singular submodule of M.

(i) If R is a semiprime left Goldie ring then the *reduced rank* $\rho(M)$ of M is defined to be the Goldie dimension of $M/Z(M)$.

(ii) If R is such that the nilpotent radical N of R is nilpotent and R/N is a left Goldie ring then the *reduced rank* $\rho(M)$ of M is given by

$$\rho_R(M) = \sum_{i=0}^{k-1} \rho_{R/N}\left(\frac{N^iM}{N^{i+1}M}\right),$$

where $N^k = 0$, $N^0 = R$, and the reduced ranks on the right are calculated as in (i).

THEOREM 3.2. *Let R be a ring with nilpotent radical N. Then R has a left Artinian left quotient ring if and only if*:

(i) *N is nilpotent*;
(ii) *R/N is a left Goldie ring*;
(iii) *$\rho_R({}_RR)$ is finite*;
(iv) *$C_R(N) = C_R(0)$.*

Proof. This is Theorem 3 of **[6]**.

We shall prove that conditions (i), (ii) and (iii) hold automatically for $A(R, \alpha)$ in the case where R is left Noetherian with α-invariant nilpotent radical.

LEMMA 3.3. *Let R be a left Noetherian ring with nilpotent radical $N(R)$ such that $\alpha(N(R)) \subseteq N(R)$. Then*

(i) *the nilpotent radical N of $A(R, \alpha)$ is given by $N = \bigcup_{i \geqq 0} x^{-i}N(R)x^i$,*

(ii) *N is nilpotent.*

Proof. (i) Denote $\bigcup_{i \geqq 0} x^{-i}N(R)x^i$ by I. It is straightforward to show that I is an ideal of $A(R, \alpha)$ and, since R is left Noetherian, $N(R)$ is nilpotent—say $N(R)^k = 0$.

Let $x^{-i_j}a_jx^{i_j} \in I$, where $a_j \in N(R)$ and $i_j \geqq 0$ for $j = 1, \ldots, k$, $i = \max\{i_1, \ldots, i_k\}$. Then

$$\prod_{j=1}^{k} x^{-i_j}a_jx^{i_j} = \prod_{j=1}^{k} x^{-i}\alpha^{i-i_j}(a_j)x^i$$

$$= x^{-i}\left(\prod_{j=1}^{k} \alpha^{i-i_j}(a_j)\right)x^i$$

$$= 0$$

since $N(R)$ is α-invariant. Thus $I^k = 0$ and $I \subseteq N$.

Now, since N is an α-stable ideal of $A(R, \alpha)$, Proposition 2.9 gives $N_i = N_0 = N \cap R$, where $\Gamma(N) = (N_i)_{i \geqq 0}$. But $N \cap R$ is a nilpotent ideal of R; so $N \cap R \subseteq N(R)$ and, by Theorem 2.5,

$$N = \bigcup_{i \geqq 0} x^{-i}(N \cap R)x^i \subseteq I.$$

(ii) This is immediate from (i).

LEMMA 3.4. *Let R be a left Noetherian ring such that $N(R)$ is α-invariant. Then $A(R, \alpha)/N$ is a semiprime left Goldie ring.*

Proof. Let k be such that $N(R)^k = 0$ and let $r \in \alpha^{-1}(N(R))$. Then $\alpha(r^k) = \alpha(r)^k \in N(R)^k = 0$, and $\alpha^{-1}(N(R))$ is a nilpotent ideal of R. Thus $\alpha^{-1}(N(R)) \subseteq N(R))$; so that $N(R)$ is α-stable.

Thus it is possible to define a ring monomorphism $\bar{\alpha}: R/N(R) \to R/N(R)$ by

$$\bar{\alpha}(r + N(R)) = \alpha(r) + N(R).$$

Now, it can be shown that $\psi: A(R/N(R), \bar{\alpha}) \to A(R, \alpha)/N$ defined by

$$\psi(x^{-i}(r + N(R))x^i) = x^{-i}rx^i + N$$

is a well-defined ring homomorphism. In fact, ψ can be seen to be an isomorphism; so that $A(R, \alpha)/N$ is isomorphic to $A(R/N(R), \bar{\alpha})$. But $R/N(R)$ is a semiprime left Goldie ring; so, by Theorem 2.7, $A(R/N(R), \bar{\alpha})$, and hence $A(R, \alpha)/N$, are also semiprime left Goldie.

LEMMA 3.5. *Let R be a left Noetherian ring and let J be an α-stable left ideal of $A(R, \alpha)$. Then $A(R, \alpha)/J$ has finite left Goldie dimension.*

Proof. Assume that $A(R, \alpha)/J$ does not have finite left Goldie dimension. Then there exists a sequence $(K_i)_{i\geqq 0}$ of left ideals of $A(R, \alpha)$ such that $J \subsetneqq K_i$ and the sum $\sum_{i=0}^{\infty} K_i/J$ is direct. If $(K_{ij})_{j\geqq 0}$ denotes the α-sequence $\Gamma(K_i)$ and $(J_j)_{j\geqq 0}$ denotes the α-sequence $\Gamma(J)$ then, by Theorem 2.5, $J_j \subseteq K_{ij}$ for all $i, j \geqq 0$. It is now claimed that, for each $j \geqq 0$, the sum $\sum_{i=0}^{\infty} K_{ij}/J_j$ is direct.

Indeed, for $j \geqq 0$, let $r_i \in K_{ij}$ (for $i = 0, \ldots, p$) be such that $\sum_{i=0}^{p} r_i + J_j = 0$. Then $\sum_{i=0}^{p} r_i \in J_j$; so that $x^{-j} \sum_{i=0}^{p} r_i x^j \in J$, or $\sum_{i=0}^{p} x^{-j} r_i x^j \in J$. Since $x^{-j} r_i x^j \in K_i$, directness of the sum $\sum_{i=0}^{\infty} K_i/J$ means that $x^{-j} r_i x^j \in J$, i.e. $r_i \in J_j$, for each $i = 1, \ldots, p$ and this proves directness of the sum $\sum_{i=0}^{\infty} K_{ij}/J_j$.

Now, since $J \subsetneqq K_i$ for each $i \geqq 0$, there exists $l \geqq 0$ with $J_l \subsetneqq K_{il}$. Furthermore, if $r \in K_{il} - J_l$ then $x^{-l} r x^l \in K_i - J$. Therefore, for any $k \geqq 0$, $x^{-(l+k)} \alpha^k(r) x^{l+k} \in K_i - J$; so that $\alpha^k(r) \in K_{i,l+k} - J_{l+k}$. Thus, for each $i \geqq 0$, there exists $l_0 \geqq 0$ such that, for all $l \geqq l_0$, $K_{il}/J_l \neq 0$. Therefore, there exists $j_0 \geqq 0$ such that $K_{0,j_0}/J_{j_0} \neq 0$.

By the above argument, there exists $j_1 \geqq j_0$ with $K_{1,j_1}/J_{j_1} \neq 0$, $K_{0,j_1}/J_{j_1} \neq 0$ and the sum $K_{0,j_1}/J_{j_1} + K_{1,j_1}/J_{j_1}$ direct.

The procedure can be repeated indefinitely to yield, for any $n \geqq 0$, a direct sum

$$\frac{K_{0,j_n}}{J_{j_n}} \oplus \frac{K_{1,j_n}}{J_{j_n}} \oplus \ldots \oplus \frac{K_{n,j_n}}{J_{j_n}}$$

of non-zero submodules of R/J_{j_n}. But J is an α-stable left ideal of $A(R, \alpha)$; so by Proposition 2.9, $J_j = J_0 = J \cap R$ for all $j \geqq 0$. This means that, given any $n \geqq 0$, there exists a direct sum

$$\frac{K_{0,j_n}}{J \cap R} \oplus \ldots \oplus \frac{K_{n,j_n}}{J \cap R}$$

of non-zero submodules of $R/(J \cap R)$. This is impossible since R is left Noetherian, and $R/(J \cap R)$, therefore, has finite Goldie dimension.

Lemma 3.6. *Let R be a left Noetherian ring with $N(R)$ α-invariant. Then $\rho(A(R, \alpha)) < \infty$, where ρ denotes the reduced rank of a left $A(R, \alpha)$-module.*

Proof. By Proposition 3.3, the nilpotent radical N of $A(R, \alpha)$ is nilpotent, of index k say. By Lemma 3.4, $A(R, \alpha)/N$ is a semiprime left Goldie ring. Then, from Definition 3.1, the reduced rank of $A(R, \alpha)$ is given by

$$\rho(A(R, \alpha)) = \sum_{i=1}^{k-1} \rho_{A(R,\alpha)/N}\left(\frac{N^i}{N^{i+1}}\right).$$

It is therefore sufficient to show that $\rho_{A(R,\alpha)/N}\left(\frac{N^i}{N^{i+1}}\right)$ is finite, for each $i=0,\ldots,k-1$.

Consider the singular submodule $Z(N^i/N^{i+1})$ of the $A(R,\alpha)/N$-module N^i/N^{i+1}, and denote the set $C_{A(R,\alpha)}(N)$ by $C(N)$. By definition,

$$Z\left(\frac{N^i}{N^{i+1}}\right)=\{r+N^{i+1}\mid cr\in N^{i+1}\text{ for some } c\in C(N)\}$$
$$=\frac{A_i}{N^{i+1}},$$

where $A_i=\{r\in N^i\mid cr\in N^{i+1}\text{ for some } c\in C(N)\}$. Using the fact that N is α-stable, it can be shown that $\alpha(C(N))=C(N)$ and then that A_i is an α-stable left ideal of $A(R,\alpha)$. By Lemma 3.5, N^i/A_i has finite Goldie dimension for each $i=0,\ldots,k-1$. Thus,

$$\rho_{A(R,\alpha)/N}\left(\frac{N^i}{N^{i+1}}\right)=\dim\frac{N^i/N^{i+1}}{Z(N^i/N^{i+1})}$$
$$=\dim\frac{N^i/N^{i+1}}{A_i/N^{i+1}}$$
$$=\dim N^i/A_i<\infty$$

for each $i=0,\ldots,k-1$.

NOTATION. Denote by $'C_\alpha(0)$ the set $\{r\in R\mid \alpha^n(r)\in {}'C_R(0)\text{ for all } n\geqq 0\}$, by $C'_\alpha(0)$ the set $\{r\in R\mid \alpha^n(r)\in C'_R(0)\text{ for all } n\geqq 0\}$ and by $C_\alpha(0)$ the set $'C_\alpha(0)\cap C'_\alpha(0)$. Note that $C'_R(0)$ and $'C_R(0)$ refer to the right regular and left regular elements of R respectively.

Also, the sets $C_{A(R,\alpha)}(0)$ and $C_{A(R,\alpha)}(N)$ will be denoted by $C(0)$ and $C(N)$ respectively.

We are now in a position to prove the main result of this section.

THEOREM 3.7. *Let R be a left Noetherian ring such that $N(R)$ is α-invariant. Then $A(R,\alpha)$ has a left Artinian left quotient ring if and only if $C_\alpha(0)=C_R(N(R))$.*

Proof. First note that, by Lemma 3.3, the nilpotent radical N of $A(R,\alpha)$ is nilpotent; by Lemma 3.4, $A(R,\alpha)/N$ is a left Goldie ring and, by Lemma 3.6, $A(R,\alpha)$ has finite reduced rank, as a left $A(R,\alpha)$-module. By Theorem 3.2, it is only necessary to show that $C_\alpha(0)=C_R(N(R))$ if and only if $C(0)=C(N)$.

As in the proof of Lemma 3.4, $N(R)$ is α-stable; so it is possible to define a monomorphism $\bar\alpha:R/N(R)\to R/N(R)$ by $\bar\alpha(r+N(R))=\alpha(r)+N(R)$. But $R/N(R)$ is a semiprime left Noetherian ring; so, by Goldie's theorem and Proposition 2.4 of **[3]**, $\bar\alpha(C_{R/N(R)}(0))\subseteq C_{R/N(R)}(0)$ or $\alpha(C_R(N(R))\subseteq C_R(N(R))$.

Now let $x^{-i}rx^i\in C(N)$ and let $s\in R$ be such that $rs\in N(R)$. Then $x^{-i}rsx^i=$

$(x^{-i}rx^i)(x^{-i}sx^i) \in \bigcup_{j \geqq 0} x^{-j}N(R)x^j = N$ by Lemma 3.3. Therefore $x^{-i}sx^i \in N$ and $s \in N$ since N is α-stable. Thus $s \in N \cap R \subseteq N(R)$, and $r \in C'_R(N(R))$. A similar argument on the left yields that $r \in {}'C_R(N(R))$, and so $C(N) \subseteq \bigcup_{i \geqq 0} x^{-i}C_R(N(R))x^i$.

On the other hand, let $r \in C_R(N(R))$, let $i \geqq 0$ and let $s \in R$, $j \geqq 0$ such that $(x^{-i}rx^i)(x^{-j}sx^j) \in N$. Then $x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j} \in N$ and, since N is α-stable, $\alpha^j(r)\alpha^i(s) \in N \cap R \subseteq N(R)$. Since $\alpha(C_R(N(R))) \subseteq C_R(N(R))$, $\alpha^j(r) \in C_R(N(R))$; so that $\alpha^i(s) \in N(R)$, and $x^{-(i+j)}\alpha^i(s)x^{i+j} \in N$ (by Lemma 3.3). Thus $x^{-j}sx^j \in N$ and $x^{-i}rx^i \in C'(N)$. Similarly, it can be shown that $x^{-i}rx^i \in {}'C(N)$; whence $\bigcup_{i \geqq 0} x^{-i}C_R(N(R))x^i = C(N)$.

By Proposition 3.1 of [**4**], $\bigcup_{i \geqq 0} x^{-i}C_\alpha(0)x^i = C(0)$, and it is now routine to prove that $C(0) = C(N)$ if and only if $C_\alpha(0) = C_R(N(R))$. By Theorem 3.2, $A(R, \alpha)$ has a left Artinian left quotient ring if and only if $C_\alpha(0) = C_R(N(R))$.

4. Krull dimension. We now turn away from the question of left Artinian left quotient rings to consider the effect on chain conditions as we pass from R to $A(R, \alpha)$. The first of these chain conditions is Krull dimension, as defined in [**2**].

In his paper [**4**], Jordan shows that it is possible to have a ring R of Krull dimension 2, and a monomorphism $\alpha: R \to R$ such that $A(R, \alpha)$ does not have Krull dimension. However, he also shows (see Theorem 2.6) that if R has left Krull dimension zero then so does $A(R, \alpha)$. Thus, the question arises as to what happens when $K\dim_R R = 1$. The following example settles this question by providing a commutative ring R with Krull dimension 1 and a monomorphism $\alpha: R \to R$ such that $A(R, \alpha)$ does not have Krull dimension.

Example 4.1. Let R be the polynomial ring $K[y]$ over the field K, and let $\alpha: R \to R$ be the K-monomorphism such that $\alpha(y) = y^2$. Note that R is a commutative, Noetherian domain of Krull dimension 1.

With $\langle y^n \rangle$ denoting the ideal of R generated by y^n, it is easily seen that, for any $n \in \mathbb{N}$,

$$\alpha^{-1}(\langle y^n \rangle) = \begin{cases} \langle y^{n/2} \rangle & \text{for } n \text{ even}, \\ \langle y^{n+1/2} \rangle & \text{for } n \text{ odd}. \end{cases} \qquad (1)$$

Now let $k \geqq 0$ and let Z_k denote the set

$$Z_k = \{(n_0, n_1, \ldots, n_k) \in \mathbb{N}^{k+1} \mid n_0 = 1 \quad \text{and}, \\ 1 \leqq i \leqq k, \quad \text{either} \quad n_i = 2n_{i-1} \quad \text{or} \quad n_i = 2n_{i-1} - 1\}.$$

Define, for each $k \in \mathbb{N}$, a map $f_k: \mathbb{N} \to \mathbb{N}$ by $f_1(n) = 2n - 1$ and $f_k(n) = 2f_{k-1}(n) - 1$ for $k \geqq 2$. Finally, for $k \geqq 0$ and $N \in Z_k$, put

$$(B_{N,k})_i = \begin{cases} \langle y^{n_i} \rangle & \text{for} \quad 0 \leqq i \leqq k, \\ \langle y^{f_{i-k}(n_k)} \rangle & \text{for} \quad i \geqq k+1, \end{cases}$$

where $N = (n_0, n_1, \ldots, n_k)$.

From (1), $((B_{N,k})_i)_{i\geqq 0}$ is an α-sequence of ideals of R and, by the remark following Definition 2.3, each $(B_{N,k})_i$ is closed. By Theorem 2.5, it therefore defines an ideal of $A(R, \alpha)$, which will be denoted by $B_{N,k}$. The collection $\{B_{N,k} \mid k \geqq 0, N \in Z_k\}$ of ideals of $A(R, \alpha)$ will be denoted by X. It is now claimed that, given $B_{N,k}$, $B_{N_1,k_1} \in X$, with $B_{N,k} \subsetneqq B_{N_1,k_1}$, there exists an infinite descending chain of ideals in X between $B_{N,k}$ and B_{N_1,k_1}. Indeed, since $B_{N,k} \neq B_{N_1,k_1}$, by Theorem 2.5, there exists $m \geqq k + k_1$ such that $(B_{N,k})_m \subsetneqq (B_{N_1,k_1})_m$. Assume that $(B_{N,k})_m = \langle y^{m_0}\rangle$ and $(B_{N_1,k_1})_m = \langle y^{m_1}\rangle$. Since $m \geqq k + k_1$, $(B_{N,k})_{m+1} = \langle y^{2m_0-1}\rangle$ and $(B_{N_1,k_1})_{m+1} = \langle y^{2m_1-1}\rangle$. Define

$$(B_{N_2,k_2})_i = \begin{cases} (B_{N_1,k_1})_i & \text{for } 0 \leqq i \leqq m, \\ \langle y^{2m_1}\rangle & \text{for } i = m+1, \\ \langle y^{f_{i-m-1}(2m_1)}\rangle & \text{for } i \geqq m+2; \end{cases}$$

so that $k_2 = m + 1$, and $N_2 \in Z_{k_2}$ has jth entry n_j such that $(B_{N_2,k_2})_j = \langle y^{n_j}\rangle$. Then, since $m_0 \geqq m_1 + 1$, $2m_1 - 1 < 2m_1 < 2m_0 - 1$, and $(B_{N,k})_{m+1} \subsetneqq (B_{N_2,k_2})_{m+1} \subsetneqq (B_{N_1,k_1})_{m+1}$. Also, since each f_k is an increasing function, $(B_{N,k})_i \subsetneqq (B_{N_2,k_2})_i \subsetneqq (B_{N_1,k_1})_i$ for all $i \geqq m+2$. Hence $B_{N,k} \subsetneqq B_{N_2,k_2} \subsetneqq B_{N_1,k_1}$. The process can be repeated for $B_{N,k} \subsetneqq B_{N_2,k_2}$, and repeated application yields the required infinite descending chain.

Now assume that $A(R, \alpha)$ has Krull dimension. Then, by Lemma 1.1 of [**2**], the $A(R, \alpha)$-module I/J has Krull dimension, for any ideals $I \supsetneqq J$ of $A(R, \alpha)$. Let I, $J \in X$ be such that $I \supsetneqq J$ and $\operatorname{K dim} I/J = \min\{\operatorname{K dim} A/B \mid A \supsetneqq B,\ A, B \in X\}$. As shown above, there exists an infinite descending chain $(I_j)_{j\geqq 0}$ of ideals in X with $J \subsetneqq I_j \subsetneqq I$ for all $j \geqq 0$. By definition of Krull dimension, there must exist $k \geqq 0$ such that, for all $j \geqq k$, $\operatorname{K dim}(I_j/I_{j+1}) < \operatorname{K dim} I/J$. This is a contradiction; so $A(R, \alpha)$ cannot have Krull dimension.

5. Ascending chain condition on annihilator ideals. It has been shown [**7**, Corollary 2.23] that if R has finite left Goldie dimension then so must $A(R, \alpha)$. It is natural, therefore, to ask whether the other Goldie condition, the ascending chain condition for annihilator left ideals, is passed from R to $A(R, \alpha)$. The following example shows that this need not be the case—the ring R concerned was first used by J. W. Kerr [**5**] as an example of a ring with acc on annihilators but no bound on the lengths of chains of annihilators.

Example 5.1. Let K be a field and let

$$\hat{Y} = \{\hat{y}_{ij} \mid i, j \in \mathbb{N}, j \leqq i\}$$

be a collection of commuting indeterminates. Let $\hat{\alpha} : K[\hat{Y}] \to K[\hat{Y}]$ be the K-monomorphism such that $\hat{\alpha}(\hat{y}_{ij}) = \hat{y}_{i+1,j+1}$, and consider the ideal I of $K[\hat{Y}]$ generated by

$$\{\hat{Y}^3, \hat{y}_{ij}\hat{y}_{ik} \mid i, j, k \in \mathbb{N}, k \neq j\}.$$

It is clear that $\hat{\alpha}(I) \subseteq I$, and it can also be shown that I is $\hat{\alpha}$-stable. Therefore, $\hat{\alpha}$ defines, in a natural way, a monomorphism

$$\alpha:\frac{K[\hat{Y}]}{I}\to\frac{K[\hat{Y}]}{I}.$$

The commutative ring $K[\hat{Y}]/I$ will be denoted by R, Y will denote the image of $\hat{Y}$ in R, and y_{ij} will denote the image of $\hat{y}_{ij}$ in R. It is shown by Kerr [5] that R has acc on annihilator ideals.

Now, consider the ring $A(R, \alpha)$, and consider an element of the form $x^{-m}y_{m+1,1}x^m$. Then $(x^{-m}y_{m+1,1}x^m)^2 \neq 0$; but, for $n \geqq 0$ with $n \neq m$,

$$\begin{aligned}(x^{-n}y_{n+1,1}x^n)(x^{-m}y_{m+1,1}x^m) &= x^{-(m+n)}\alpha^m(y_{n+1,1})\alpha^n(y_{m+1,1})x^{m+n}\\ &= x^{-(m+n)}y_{m+n+1,m+1}\,y_{m+n+1,n+1}x^{m+n}\\ &= 0\end{aligned}$$

because of the definition of the ideal I.

Now let $B_n = \{x^{-m}y_{m+1,1}x^m \mid m \geqq n\}$ for each $n \geqq 0$. Since $B_n \supseteq B_{n+1}$ for all $n \geqq 0$, certainly $l(B_n) \subseteq l(B_{n+1})$. But, from above, $x^{-n}y_{n+1,1}x^n \in l(B_{n+1})$ but $x^{-n}y_{n+1,1}x^n \notin l(B_n)$. Thus $(l(B_n))_{n\geqq 0}$ is an infinite ascending sequence of annihilators of $A(R, \alpha)$.

Remark. Although this example shows that acc on left annihilators need not be passed from R to $A(R, \alpha)$, the ring R has infinite Goldie dimension, and it is not known what happens if R has finite Goldie dimension. In other words, it is not known whether R being left Goldie forces $A(R, \alpha)$ also to be left Goldie.

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