Canad. Math. Bull. Vol. **47** (1), 2004 pp. 17–21

Universal Singular Inner Functions

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Abstract. We show that there exists a singular inner function S which is universal for noneuclidean translates; that is one for which the set $\{S(\frac{z+z_n}{1+\bar{z}_n z}) : n \in \mathbb{N}\}$ is locally uniformly dense in the set of all zero-free holomorphic functions in $\mathbb{D}$ bounded by one.

Almost half a century ago M. Heins [He] constructed a Blaschke product B that has the following special property: there exists a sequence (z_n) tending to the boundary, $\partial\mathbb{D}$, of $\mathbb{D}$ such that for every holomorphic function f in the unit disk $\mathbb{D}$ bounded by 1 there exists a subsequence (z_{n_k}) of (z_n) such that $B(\frac{z+z_{n_k}}{1+\bar{z}_{n_k} z})$ tends to f locally uniformly in $\mathbb{D}$. This Blaschke product is called a *universal Blaschke product*. Since that time, interest has increased in universality in many different situations (see for example, the paper [GE]). Universal functions play an important role in operator theory, and in particular, in the study of cyclic vectors for certain operators. In this paper we are interested in whether or not there is an analogous result for singular inner functions. In this case, of course, every limit of noneuclidean translates of S must either be identically zero or zero-free in $\mathbb{D}$. It is the aim of this note to prove the existence of universal singular inner functions. Thus, in this setting, a universal singular inner function is a singular inner function S such that every zero-free holomorphic function in $\mathbb{D}$ and bounded by 1 can be locally uniformly approximated by noneuclidean translates of S.

1 Approximating by Singular Inner Functions

A singular inner function is a function of the form

$$S_\mu(z) = e^{i\theta} \exp\left(-\int_{|\xi|=1} \frac{\xi + z}{\xi - z}\, d\mu(\xi) \right),$$

where μ is a singular, positive Borel measure on $\partial\mathbb{D}$. All measures considered here are assumed to be finite. The singular inner function S_μ is said to be *normalized*, if $S_\mu(0) > 0$, and *discrete*, if μ is a discrete measure.

Though the next lemma is well known, we present a short proof of it for the sake of completeness.

Lemma 1.1 *The linear span of the set of Dirac measures* $\{\delta_{k/n} : 0 \le k \le n-1\}$ *with positive coefficients is weak-$*$ dense in the set of all positive Borel measures on* $[0, 1[$.

Received by the editors March 6, 2002; revised September 10, 2002.
The authors were supported by the RIP-program Oberwolfach, 2001.
AMS subject classification: 30D50.

Proof Let μ be a positive Borel measure on $[0,1[$, $f \in C([0,1])$ and $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that for all $x, y \in [0,1]$ with $|x-y| < \delta$ we have $|f(x) - f(y)| < \frac{\varepsilon}{\mu([0,1[)}$. Choose an integer n such that $\frac{1}{n} < \delta$. Let $\sigma = \sum_{k=0}^{n-1} \mu([\frac{k}{n}, \frac{k+1}{n}[)\delta_{k/n}$. Then

$$\Big|\int f\,d\mu - \int f\,d\sigma\Big| \leq \Big|\sum_{k=0}^{n-1} \int_{[\frac{k}{n},\frac{k+1}{n}[} [f - f(k/n)]\,d\mu\Big|$$
$$\leq \sum_{k=0}^{n-1} \frac{\varepsilon}{\mu([0,1[)} \int_{[\frac{k}{n},\frac{k+1}{n}[} 1\,d\mu = \varepsilon \qquad \blacksquare$$

Let $\mathcal{H}$ denote the set of all zero-free functions in the unit ball of H^∞ that are positive at the origin.

Proposition 1.2 *The closure of the set of normalized singular inner functions in the compact-open topology is the union of the constant function* 0 *and the set* $\mathcal{H}$.

Proof Since one inclusion is obvious, it remains to show that every $f \in \mathcal{H} \cup \{0\}$ is in the closure of the set of normalized singular inner functions. If $f \equiv 0$, then we take $f_n(z) = [\exp(-\frac{1+z}{1-z})]^n$. If $f \equiv 1$, then we may choose $f_n(z) = [\exp(-\frac{1+z}{1-z})]^{1/n}$, where $f_n(0) > 0$. If $f \in \mathcal{H} \setminus \{1\}$, then $g := -\log f$ is holomorphic in $\mathbb{D}$ and not identically 0 (here $\log f(0)$ is taken to be negative). Obviously, $|f| < 1$ implies that $\operatorname{Re} g > 0$. By Herglotz's theorem ([Ho], p. 40), there exists a (unique) positive Borel measure ν on $[0, 2\pi[$ such that

$$g(z) = \int_{[0,2\pi[} \frac{e^{it}+z}{e^{it}-z}\,d\nu(t).$$

By Lemma 1.1, there exist positive numbers $\varepsilon_{k,n} > 0$ such that the measures $\sigma_n = \sum_{k=0}^{n-1} \varepsilon_{k,n}\delta_{2\pi k/n}$ converge weak-$*$ to ν. Hence

$$g_n(z) = \int_{[0,2\pi[} \frac{e^{it}+z}{e^{it}-z}\,d\sigma_n(t)$$

converges locally uniformly on $\mathbb{D}$ to g. Thus $\big(\exp(-g_n)\big)$ is a sequence of singular inner functions converging locally uniformly to f. $\blacksquare$

As a corollary of the proof of Proposition 1.2 we obtain the following useful result.

Corollary 1.3 *The set* $\mathcal{D}$ *of discrete singular inner functions of the form*

$$e^{iq}\prod_{k=0}^{n-1} \exp\Big(-\varepsilon_{k,n}\frac{e^{2\pi ik/n}+z}{e^{2\pi ik/n}-z}\Big),$$

where $q \in \mathbb{Q}$, $n \in \mathbb{N}$, $\varepsilon_{k,n} \in \mathbb{Q}$ *and* $\varepsilon_{k,n} > 0$, *is a countable, locally uniformly dense subset of the set* $\mathcal{N}$ *of all zero-free holomorphic functions in* $\mathbb{D}$ *bounded by one.*

https://doi.org/10.4153/CMB-2004-003-0 Published online by Cambridge University Press

2 Universal Singular Inner Functions

Let (a_n) be a sequence of points in $\mathbb{D}$ satisfying $\sum_n (1-|a_n|) < \infty$, and let

$$B(z) = \prod_{n=1}^{\infty} \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z}$$

denote the corresponding Blaschke product. Then a straightforward calculation shows that

$$(1-|a_n|^2)|B'(a_n)| = \prod_{j:j\neq n} \left| \frac{a_j - a_n}{1 - \bar{a}_j a_n} \right|.$$

A sequence (z_n) of distinct points in $\mathbb{D}$ is called *thin* if

$$\lim_{k\to\infty} \prod_{j:j\neq k} \left| \frac{z_j - z_k}{1 - \bar{z}_j z_k} \right| = 1.$$

It is known that (z_n) is thin if $\frac{1-|z_{n+1}|}{1-|z_n|} \to 0$ (see *e.g.* [GM]). These examples of thin sequences will be useful later in this paper. Thin sequences have been the object of much study and are sometimes referred to as "sparse" sequences in the literature (see *e.g.* [H] and [GI]).

In what follows we will let $L_a(z) = \frac{a+z}{1+\bar{a}z}$, $a \in \mathbb{D}$. The Blaschke product $B(z) = \prod_{n=1}^{\infty} \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1-\bar{a}_n z}$ is called *thin* if its zeros, (a_n), form a thin sequence. Note that for a thin Blaschke product B we have

$$|(B \circ L_{a_n})'(0)| = (1-|a_n|^2)|B'(a_n)| \to 1.$$

Proposition 2.1 *Let B be a thin Blaschke product with zero sequence (a_n). Then there exists a subsequence (a_{n_k}) of the zero sequence of B such that $B \circ L_{a_{n_k}}$ converges uniformly on compacta to a unimodular constant times the identity function.*

Proof Since $\{B \circ L_{a_n} : n \in \mathbb{N}\}$ is a bounded family of analytic functions, Montel's theorem implies that there exists a subsequence, (a_{n_k}), of (a_n) and an analytic function h in the ball of H^∞ such that $B \circ L_{a_{n_k}}$ converges to h uniformly on compacta. We note that $h(0) = 0$. Furthermore, the comments preceding the proof of this proposition imply that $|(B \circ L_{a_{n_k}})'(0)| \to 1$. Thus by Schwarz's lemma, we see that there exists a unimodular constant $e^{i\theta}$ such that $h(z) = e^{i\theta} z$, as desired. ■

From the proposition above it follows that $(f \circ e^{-i\theta} B) \circ L_{a_{n_k}}$ converges locally uniformly to f for any $f \in H^\infty$.

Our construction of universal singular inner functions will use the following elementary result. It readily follows from Lemma 1.5 in [GI] and the fact (mentioned above) that (z_n) is thin if $\frac{1-|z_{n+1}|}{1-|z_n|} \to 0$.

Proposition 2.2 *For any sequence (z_n) in $\mathbb{D}$ with $|z_n| \to 1$ there exists a thin Blaschke product B whose zeros within $\mathbb{D}$ are contained in the set $\{z_n : n \in \mathbb{N}\}$ and have the same cluster points as the sequence (z_n).*

We are now ready to prove the main result of this note.

Theorem 2.3 *Let (z_n) be a sequence in $\mathbb{D}$ having infinitely many nontangential cluster points on $\partial\mathbb{D}$. There exists a (discrete) singular inner function S such that*

$$\Big\{ S\Big(\frac{z+z_n}{1+\bar{z}_n z}\Big) : n \in \mathbb{N} \Big\}$$

is locally uniformly dense in the set $\mathcal{N}$ of all zero-free holomorphic functions in $\mathbb{D}$ bounded by one.

Proof Let (λ_n) be a sequence of distinct nontangential cluster points of (z_j) in $\partial\mathbb{D}$, say $\lim_j z'_{j,n} = \lambda_n$ nontangentially. Without loss of generality we may assume that λ_n tends to 1 and $0 < \arg \lambda_{n+1} < \arg \lambda_n$. Let B be a thin Blaschke product with zeros contained in the set $\{z'_{j,n} : n, j \in \mathbb{N}\}$ such that the cluster set of the zeros of B is precisely the set $\{\lambda_n : n \in \mathbb{N}\} \cup \{1\}$.

According to Corollary 1.3 there exists a countable set $\mathcal{D} = \{S_j : j \in \mathbb{N}\}$ of discrete, singular inner functions that is locally uniformly dense in $\mathcal{N}$. Furthermore, for each j, there exists a subsequence of the zero sequence of B, denoted $(z_{n(k,j)})_k$, that converges to λ_j and satisfies

$$(1) \qquad (S_j \circ e^{-i\theta_j} B) \circ L_{z_{n(k,j)}} \text{ tends to } S_j$$

locally uniformly on $\mathbb{D}$ as $k \to \infty$.

Since the composition of two inner functions is inner, we see that $S_j \circ e^{-i\theta_j} B$ is a singular inner function. We write

$$S_{\mu_j} = S_j \circ e^{-i\theta_j} B,$$

where μ_j is the associated positive, singular Borel measure on $\partial\mathbb{D}$. Note that S_{μ_j} is not necessarily normalized. The singularities of S_{μ_j} are determined by those of B and by the preimages of the finite number of singularities of S_j under the map $e^{-i\theta_j} B$. Since $\{\lambda_n : n \in \mathbb{N}\} \cup \{1\}$ is the set of singularities of B, we can conclude that S_{μ_j} has only countably many singularities, and so, S_{μ_j} is a discrete singular inner function. Moreover, we will show that the measure μ_j has no mass at λ_j. In fact, suppose the contrary. Then S_{μ_j} has nontangential limit zero at λ_j. On the other hand, $S_j \circ e^{-i\theta_j} B$ has the nonzero value $S_j(0)$ at every zero of B. Since, by our assumption, some of the zeros of B cluster nontangentially at λ_j, we get a contradiction.

The fact that $\mu_j(\{\lambda_j\}) = 0$ now implies that there are pairwise disjoint closed arcs I_j on $\partial\mathbb{D}$ centered at λ_j with $\mu_j(I_j) \leq 2^{-j}$ for $j \in \mathbb{N}$. We let ν_j be the restriction of μ_j to I_j. Then $\nu = \sum \nu_j$ is a positive, finite, singular measure on $\partial\mathbb{D}$. Let S_ν be the associated singular inner function. Write S_ν as

$$(2) \qquad \Big(\prod_{n \neq j} S_{\nu_n}\Big) (S_{\nu_j}/S_{\mu_j}) S_{\mu_j}.$$

Note that for every j the functions $\prod_{n\neq j} S_{\nu_n}$ and S_{μ_j}/S_{ν_j} are analytic and unimodular on I_j. Since $(L_{z_{n(k,j)}})$ converges locally uniformly in $\mathbb{D}$ to λ_j, assertions (1) and (2) now imply that there exists $\sigma_j \in [0, 2\pi[$ such that

$$\lim_k S_\nu \circ L_{z_{n(k,j)}} = e^{i\sigma_j} S_j \tag{3}$$

locally uniformly in $\mathbb{D}$. We now use an interpolation result due to Eva Decker ([De], p. 554) to obtain a (discrete) singular inner function S^* that is analytic at every λ_j and has the prescribed radial limits $e^{-i\sigma_j}$ at λ_j. The function $S = S^* S_\nu$ is now the universal singular inner function we were looking for. In fact, let d be a distance function generating local uniform convergence on the set $H(\mathbb{D})$ of all holomorphic functions in $\mathbb{D}$; for example let $\|f\|_n = \max\{|f(z)| : |z| \leq 1 - \frac{1}{n}\}$ and

$$d(f, g) = \sum_n \frac{1}{2^n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}.$$

Let $f \in \mathcal{N}$ and $\varepsilon > 0$. Choose $S_j \in \mathcal{D}$ so that $d(S_j, f) < \varepsilon/2$. Note that due to the analyticity of the function S^* at each λ_j, the sequence $(S^* \circ L_{z_{n(k,j)}})_k$ converges locally uniformly to $S^*(\lambda_j) = e^{-i\sigma_j}$. Hence, by (3) and the fact that $S = S^* S_\nu$, we see that $S \circ L_{z_{n(k,j)}}$ converges locally uniformly to S_j. Thus we find a zero, z_m, of B so that $d(S \circ L_{z_m}, S_j) < \varepsilon/2$. Then $d(S \circ L_{z_m}, f) < \varepsilon$. ■

Acknowledgements The authors thank the Mathematisches Forschungsinstitut Oberwolfach for the support and for the kind hospitality they received during their stay in the summer of 2001. The work presented here is part of their project "Approximation and interpolation problems II".

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