EXTENSIONS OF THE FUGLEDE-PUTNAM-TYPE
THEOREMS TO SUBNORMAL OPERATORS

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At first we investigate the similarity between the Kleinecke-
Shirokov theorem for subnormal operators and the Fuglede-Putnam
theorem and also we show an asymptotic version of this
similarity. These results generalize results of Ackermans, van
Eijndhoven and Martens. Also we show two theorems on degree of
approximation on subnormal derivation ranges. These results
generalize results of Stampfli on degree of approximation on
normal derivation ranges. The purpose of this paper is to show
that the Fuglede-Putnam-type theorem on normal operators can
certainly be generalized to subnormal operators.

1. Introduction

An operator means a bounded linear operator on a complex Hilbert
space. Let $B(H)$ denote the set of all bounded linear operators on a
complex Hilbert space H . An operator T is called subnormal if T has
a normal extension.

Following [1], we introduce the following two definitions.

DEFINITION 1. $[C_j, B]_* = C_1B - BC_2$ for C_1, C_2, $B \in B(H)$.

DEFINITION 2. A topology τ on $B(H)$ is called an algebra

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topology, if

  (i) τ is not finer than the uniform operator topology,

  (ii) $B(H)$ with topology τ is a locally convex topological
     vector space,

  (iii) the mapping $X \to BXC$ from $B(H)$ into $B(H)$ is
     τ-continuous for each pair $B, C \in B(H)$.

Recently Ackermans, van Eijndhoven and Martens in [1] investigated the
similarity between the Kleinecke-Shirokov theorem for normal operators and
the Fuglede theorem and also they have shown an asymptotic version of this
similarity as follows.

  THEOREM A *[1]. Let N_1, $N_2 \in B(H)$ be normal. If*
$\left[N_j, \ \left[N_j, \ B \right]_* \right]_* = 0$ *then* $\left[N_j, \ B \right]_* = 0$.

  THEOREM B *[1]. Let N_1, $N_2 \in B(H)$ be normal and let τ be an*
algebra topology on $B(H)$. Let W be a τ-open neighbourhood of 0 and
let $K > 0$. Then there exists a τ-open neighbourhood V of 0 such
that $\left[N_j, \ B \right]_* \in W$ for all $B \in B(H)$ with $\|B\| \leq k$ and
$\left[N_j, \ \left[N_j, \ B \right]_* \right]_* \in V$.

  In [11], Stampfli has estimated the degree of approximation on normal
derivation ranges as follows.

  THEOREM C *[11]. Let A be a normal operator in $B(H)$ with*
$\sigma(A) \subset \Gamma$ *where Γ is a rectifiable curve and length $\Gamma = l$. Let*
$AX - XA = W$. *Then there exists a $Y \in B(H)$ and a natural number n such*
that

$$\|Y\| \leq n\|X\|$$

and

$$\|(A^*Y - YA^*) - W\| \leq 3n^{-1} l \|X\| .$$

  THEOREM D *[10]. Let A be a normal operator with $\sigma(A) \subset \Gamma$ where*
Γ *is a rectifiable curve. Let $\|X\| \leq 1$. If $\|AX - XA\| < \delta$, then*
$\|A^*X - XA^*\| < C\delta^{\frac{1}{2}}$ *where C is a universal constant.*

  Our Theorem 1 shows the similarity between the Kleinecke-Shirokov

theorem [9], [10], for subnormal operators and the Fuglede theorem [8] and
Theorem 1 is an extension of Theorem A. Theorem 2 shows an asymptotic
version of this similarity and Theorem 2 is also an extension of Theorem B.
Theorem 3 and Theorem 4 are extensions of Theorem C and Theorem D
respectively. Combining the idea in [3], [4], [5] and [6] for subnormal
operators with the results and techniques in Theorem A, Theorem B, Theorem
C and Theorem D, we show our results. Our main purpose is to show that
Fuglede-Putnam type theorems for normal operators such as Theorem A,
Theorem B, Theorem C and Theorem D can certainly be generalized to sub-
normal operators.

2. Statement of theorems

First we show Theorem 1 and Theorem 2. Theorem 1 is an extension of
Theorem A and Theorem 2 is an extension of Theorem B.

THEOREM 1. *Let* A_1, $A_2^* \in B(H)$ *be subnormal. If*
$\left[A_j, \left[A_j, B\right]_*\right]_* = 0$, *then* $\left[A_j, B\right]_* = 0$.

THEOREM 2. *Let* A_1, $A_2^* \in B(H)$ *be subnormal and let* τ *be an
algebra topology on* $B(H)$. *Let* W *be a* τ-*open neighbourhood of* 0 *and
let* $k > 0$. *Then there exists a* τ-*open neighbourhood* V *of* 0 *such
that* $\left[A_j, B\right]_* \in W$ *for all* $B \in B(H)$ *with* $\|B\| \le K$ *and*
$\left[A_j, \left[A_j, B\right]_*\right]_* \in V$.

Next we show Theorem 3 and Theorem 4. Theorem 3 is an extension of
Theorem C and Theorem 4 is an extension of Theorem D.

THEOREM 3. *Let* A *and* B^* *be subnormal operators in* $B(H)$ *with
$\sigma\left(N_A\right) \subset \Gamma$ and $\sigma\left(N_{B*}^*\right) \subset \Gamma$ where N_A and N_{B*} are normal extensions on
$H \oplus H$ of A and B^* respectively, and Γ is a rectifiable curve and
length $\Gamma = l$. Let $AX - XB = W$. Then there exists a $Y \in B(H)$ and a
natural number n such that*

$$\|Y\| \lesssim n\|X\|$$

and

$$\|(A^*Y - YB^*) - W\| \le 3n^{-1}l\|X\| \ .$$

THEOREM 4. *Let A and B^* be subnormal operators with $\sigma(N_A) \subset \Gamma$
and $\sigma(N_{B^*}^*) \subset \Gamma$ where N_A and N_{B^*} are normal extensions on $H \oplus H$ of
A and B^* respectively and Γ is a rectifiable curve. Let $\|X\| \leq 1$.
If $\|AX-XB\| < \delta$ then $\|A^*X-XB^*\| < C\delta^{\frac{1}{2}}$ where C is a universal constant.*

REMARK I. In Theorem 3 and Theorem 4, assume that A and B^* are
subnormal operators with $\sigma(A) \subset \Gamma$ and $\sigma(B) \subset \Gamma$ and also assume that
they have, respectively, minimal normal extensions N_A and N_{B^*} on
$H \oplus H$, instead of the hypotheses $\sigma(N_A) \subset \Gamma$ and $\sigma(N_{B^*}^*) \subset \Gamma$. Then we
have the same conclusions as Theorem 3 and Theorem 4 respectively, because
$\sigma(N_A) \subset \sigma(A) \subset \Gamma$ and $\sigma(N_{B^*}^*) \subset \sigma(B^{**}) = \sigma(B) \subset \Gamma$ [δ, Problem 157, p.102],
so that A and B^* satisfy the hypotheses in Theorem 3 and Theorem 4
respectively.

3. Proofs of theorems

Proof of Theorem 1. A normal extension N_A of A on the Hilbert
space H is given by

$$N_{A_1} = \begin{pmatrix} A_1 & A_{12}^{(1)} \\ 0 & A_{22}^{(2)} \end{pmatrix}$$

acting on the Hilbert space $H \oplus H$ whose restriction to $H \oplus \{0\}$ is A ,
[7], and a normal one $N_{A_2^*}$ of A_2^* on H is also given by

$$N_{A_2^*} = \begin{pmatrix} A_2^* & A_{12}^{(2)} \\ 0 & A_{22}^{(2)} \end{pmatrix}$$

acting on $H \oplus H$. We put $M_1 = N_{A_1}$, $M_2 = (N_{A_2^*})^*$, and $\tilde{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ on
$H \oplus H$. Then M_1, M_2 are both normal on $H \oplus H$. We have
$[M_j, [M_j, \tilde{B}]_*]_* = 0$ by the assumption as follows:

$$[M_j, \; [M_j, \; \tilde{B}]_*]_* = M_1 (M_1 \tilde{B} - \tilde{B} M_2) - (M_1 \tilde{B} - \tilde{B} M_2) M_2$$

$$= \begin{pmatrix} A_1 (A_1 B - BA_2) - (A_1 B - BA_2) A_2 & 0 \\ \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} [A_j, [A_j, B]_*]_* & 0 \\ \\ 0 & 0 \end{pmatrix}$$

$$= 0$$

so that

$$[M_j, \; \tilde{B}]_* = \begin{pmatrix} [A_j, B]_* & 0 \\ \\ 0 & 0 \end{pmatrix} = 0$$

by Theorem A; that is, $[A_j, B]_* = 0$, so the proof is complete.

   Proof of Theorem 2. As stated in the proof of Theorem 1, we have normal extensions N_{A_1} and $N_{A_2^*}$ on $H \oplus H$ and we employ the same notation as in the proof of Theorem 1. Also we put $M_1 = N_{A_1}$ and $M_2 = (N_{A_2^*})^*$. We define the subset W as follows:

$$\tilde{W} = \left\{ \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} : Y_k \text{ in } W \; (k = 1, 2, 3, 4) \right\}$$

in the set of operators on $H \oplus H$. Then $\tilde{W}$ easily turns out to be a τ-open neighbourhood of 0 on $H \oplus H$ because W is so on H . By Theorem B, there exists a neighbourhood $\tilde{V}$ of 0 on $H \oplus H$ such that $[M_j, \tilde{B}]_* \in \tilde{W}$ for all $\tilde{B} \in B(H \oplus H)$ with $\|\tilde{B}\| \leq K$ and $[M_j, \; [M_j, \; \tilde{B}]_*]_* \in \tilde{V}$.

   Define $V = \left\{ Y : \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} \text{ is in } \tilde{V} \right\}$. Then this set V turns out to be a τ-neighbourhood of 0 on H because $\tilde{V}$ is so on $H \oplus H$. Assume B is an operator on H with $[A_j, \; [A_j, \; B]_*]_* = Y$ in V and $\|B\| \leq K$. Put $\tilde{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ on $H \oplus H$. Then $\|\tilde{B}\| \leq K$ and in the same way as in the proof of Theorem 1, we have

$$[M_j, \ [M_j, \ \tilde{B}]_*]_* = \begin{pmatrix} [A_j,[A_j,B]_*]_* & 0 \\ \\ 0 & 0 \end{pmatrix}$$

is in $\tilde{V}$. Hence we have

$$[M_j, \ \tilde{B}]_* = \begin{pmatrix} [A_j,B]_* & 0 \\ \\ 0 & 0 \end{pmatrix}$$

is in $\tilde{W}$, which implies that $[A_j, \ B]_*$ is in W , so the proof is complete.

In order to prove Theorem 3 we first prove Lemma 1 by employing Berberian's trick [2].

LEMMA 1. *Let A and B be normal operators in $B(H)$ with $\sigma(A) \subset \Gamma$ and $\sigma(B) \subset \Gamma$ where Γ is a rectifiable curve and length $\Gamma = l$. Let $AX - XB = W$. Then there exists a $Y \in B(H)$ and a natural number n such that*

$$\|Y\| \leq n\|X\|$$

and

$$\|(A^*Y-YB^*)-W\| \leq 3n^{-1}l\|X\| .$$

Proof. Consider $\tilde{A}, \tilde{X}$ and $\tilde{W}$ on $H \oplus H$ as follows:

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} , \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{W} = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix} .$$

Clearly $\tilde{A}$ is a normal operator on $H \oplus H$ and $\sigma(\tilde{A}) = \sigma(A) \cup \sigma(B) \subset \Gamma$ by the hypotheses and

$$\tilde{A}\tilde{X} - \tilde{X}\tilde{A} = \begin{pmatrix} 0 & AX-XB \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix} = \tilde{W} .$$

By Theorem C, there exists a $\tilde{Y} \in B(H \oplus H)$ and a natural number n such that $\|\tilde{Y}\| \leq n\|\tilde{X}\|$ and

(1) $$\|(\tilde{A}^*\tilde{Y}-\tilde{Y}\tilde{A}^*)-\tilde{W}\| \leq 3n^{-1}l\|\tilde{X}\|$$

and by scrutinising the proof of Theorem C, we have $\tilde{Y} = [\theta_{ij}\tilde{X}_{ij}]$, where [ ] denotes an operator valued matrix and $|\theta_{ij}| = 1$ for

$i, j = 1, 2, \ldots, n$ (see the proof of Theorem C), that is, there exists a $Y \in B(H)$ such that

$$\tilde{Y} = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \quad , \quad Y = \left[\theta_{ij} X_{ij} \right] \quad \text{and} \quad \|Y\| \leq n\|X\| \ .$$

(1) implies

$$\|(\tilde{A}^* \tilde{Y} - \tilde{Y} \tilde{A}^*) - \tilde{W}\| = \left\| \begin{pmatrix} 0 & (A^* Y - Y B^*) - W \\ 0 & 0 \end{pmatrix} \right\| \leq 3n^{-1} l \|\tilde{X}\| = 3n^{-1} l \|X\| \ ,$$

so the proof is complete.

Proof of Theorem 3. A normal extension N_A of A and a normal one N_{B^*} of B^* are given by

$$N_A = \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad , \quad N_{B^*} = \begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix} \ ,$$

respectively acting on the Hilbert space $H \oplus H$.

Put $\hat{X} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ and $\hat{W} = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$ on $H \oplus H$, respectively. Then

$$N_A \hat{X} - \hat{X} N_{B^*}^* = \begin{pmatrix} AX - XB & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} = \hat{W} \ ,$$

so that by Lemma 1, there exists a $\hat{Y} \in B(H \oplus H)$ and a natural number such that $\|\hat{Y}\| \leq n\|\hat{X}\|$ and

$$(2) \qquad \qquad \| (N_A^* \hat{Y} - \hat{Y} N_{B^*}) - \hat{W} \| \leq 3n^{-1} l \|\hat{X}\| \ .$$

By the same way as in the proof of Lemma 1, we have $\hat{Y} = \left[\theta_{ij} \hat{X}_{ij} \right]$, where [ ] denotes an operator valued matrix and $|\theta_{ij}| = 1$ for $i, j = 1, 2, \ldots, n$, that is, there exists a $Y \in B(H)$ such that

$$\hat{Y} = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad Y = \left[\theta_{ij} X_{ij} \right] \quad \text{and} \quad \|Y\| \leq n\|X\| \ .$$

(2) yields

$$\| \left(N_A^* \widehat{Y} - \widehat{Y} N_{B^*} \right) - \widehat{W} \| = \left\| \begin{pmatrix} (A^*Y - YB^*) - W & -YB_{12} \\ A_{12}^* Y & 0 \end{pmatrix} \right\| \leq 3n^{-1}l \|\widehat{X}\| = 3n^{-1}l \|X\|$$

hence

$$\| (A^*Y - YB^*) - W \| \leq 3n^{-1}l \|X\| \ ,$$

so the proof is complete.

LEMMA 2. *Let A and B be normal operators with $\sigma(A) \subset \Gamma$ and $\sigma(B) \subset \Gamma$ where Γ is a rectifiable curve. Let $\|X\| \leq 1$. If $\|AX - XB\| < \delta$ then $\|A^*X - XB^*\| \leq C\delta^{\frac{1}{2}}$ where C is a universal constant.*

The proof of Lemma 2 is almost the same as that of Lemma 1 using Berberian's trick, so we omit it.

Proof of Theorem 4. We retain the notation of Theorem 3. We have $\|\widehat{X}\| \leq 1$ and

$$\| N_A \widehat{X} - \widehat{X} N_{B^*}^* \| = \left\| \begin{pmatrix} AX - XB & 0 \\ 0 & 0 \end{pmatrix} \right\| < \delta \ ,$$

by the hypotheses, then by Lemma 2,

$$\| N_A^* X - X N_{B^*} \| = \left\| \begin{pmatrix} A^*X - XB^* & -XB_{12} \\ A_{12}^* X & 0 \end{pmatrix} \right\| < C\delta^{\frac{1}{2}}$$

so that

$$\| A^*X - XB^* \| < C\delta^{\frac{1}{2}}$$

and the proof is complete.

REMARK 2. We remark that Theorem 3 and Theorem 4 do not involve symmetric hypotheses on A and B but rather on A and B^*. In view of this point, it is natural and reasonable in Lemma 1 and Lemma 2 to interpret the hypothesis of normality of A and B as that of normality of A and B^*.

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