

Completely semi-stable trees

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Completely semi-stable trees are characterised by the complete absence of three types of subtrees.

1. Introduction

All graphs G are on a finite set of vertices V , and have no loops, multiple edges or directed edges. For graph concepts used here see [1], and for extra permutation group concepts see [7].

A graph G is *semi-stable*, [3], if there exists a vertex $v \in V$ such that $\Gamma(G)_v = \Gamma(G_v)$, where $\Gamma(G)$ is the automorphism group of G . In such a case G is said to be *semi-stable at v* . From [3] we see that all regular graphs are semi-stable, and clearly they must be semi-stable at every vertex. Similarly in [4] it is shown that $P_m \times P_n$ is semi-stable at every vertex provided

- (i) neither m nor n is 1,
- (ii) if $m = 2$ then so is n , and *vice-versa*, and
- (iii) m and n are not both equal to 3,

where P_r is the path on r vertices.

Graphs which are semi-stable at every vertex we call *completely semi-stable*. We now consider which trees are completely semi-stable.

2. Completely semi-stable trees

In the remainder of this paper T is a tree on a vertex set V .

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The following characterisation of semi-stability was proved in [6].

THEOREM 1. *G is semi-stable at v if and only if v is adjacent to all vertices in a fixed block of $\Gamma(G_v)$.*

Now it is clear that if we wish to build up a semi-stable tree T from a subforest T_v , then the new vertex v can be adjacent to exactly one vertex from each of the trees of the subforest. (Obviously if we join v to more than one vertex of a given tree we produce a cycle.) So the fixed block of $\Gamma(T_v)$ to which v is joined must consist of a union of single vertices from each tree of the subforest, and these vertices then must be fixed vertices in the automorphism group of the tree to which they belong. Hence if one of the trees of the subforest T_v contains no fixed vertices, then T is not semi-stable at v .

LEMMA 1. *If T_v contains a component with no fixed vertices, then this component is a bicentral tree of the type shown in Figure 1, and T is not semi-stable at v .*

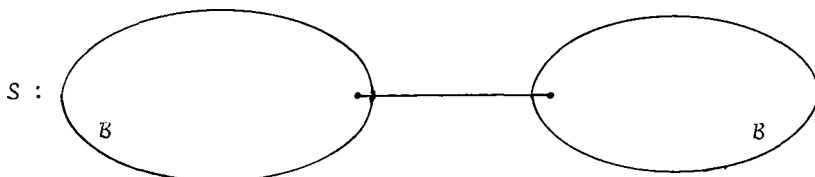


Figure 1

Proof. If a tree S has no fixed vertices then it must be bicentral, and the branches attached to each centre vertex must be the same. The argument used prior to the statement of the Lemma then gives the required result. And in a similar way

LEMMA 2. *If T is of the form shown in Figure 2, then T is not semi-stable at v .*

Proof. In Figure 2, A , B , and C , are all subtrees of T . The vertices w_1 and x_1 , w_2 and x_2 correspond in the two copies of B . Clearly w_1 and x_1 are in the same fixed block of $\Gamma(T_v)$.

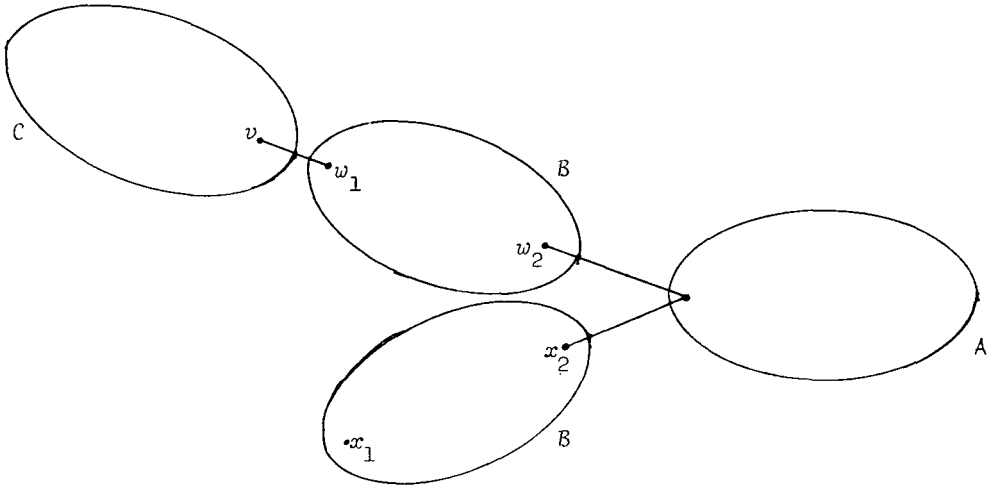


Figure 2

LEMMA 3. *If T is as in Figure 3, where v is not necessarily fixed in $\Gamma(T)$, but B_1 together with v and rooted at v is not isomorphic to B_2 together with v and rooted at v , whereas $(B_1)_v = (B_2)_v$, then T is not semi-stable at v .*

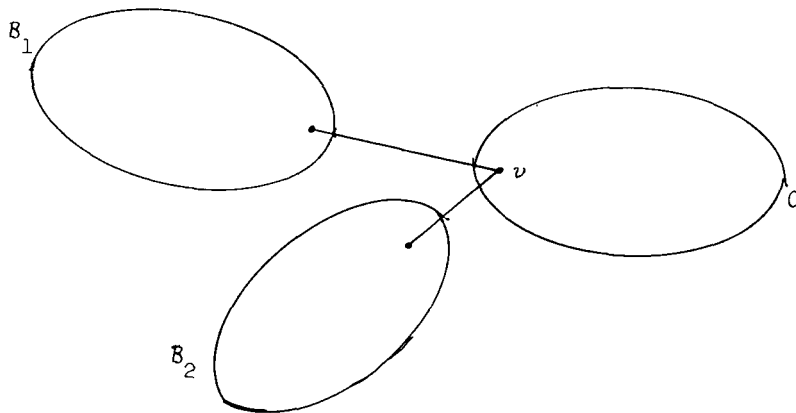
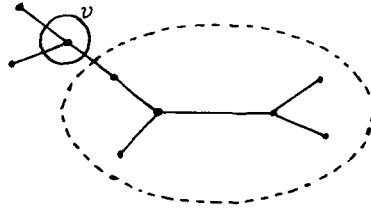


Figure 3

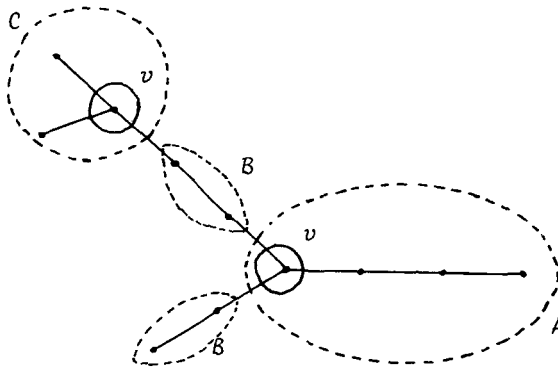
Proof. Clearly removing v introduces extra automorphisms.

Specific examples of trees with each type of inadmissible subtree are

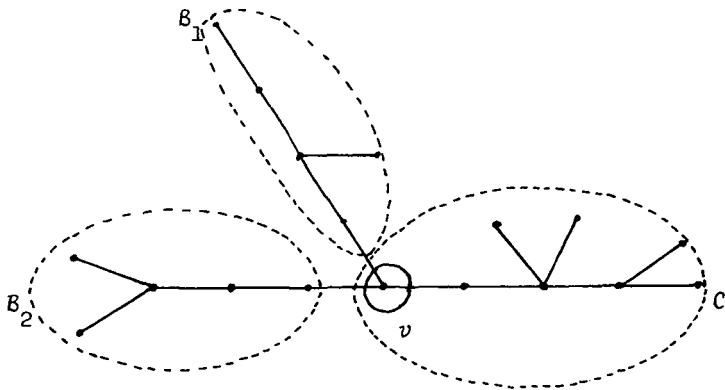
given, in order, in Figure 4.



(a) - Lemma 1



(b) - Lemma 2



(c) - Lemma 3

Figure 4

The main theorem can now be stated.

THEOREM 2. *T is completely semi-stable if and only if*

- (i) *there is no vertex v in V such that T_v contains a component of the type shown in Figure 1,*
- (ii) *T is not of the form of Lemma 2, and*
- (iii) *T is not of the form of Lemma 3.*

Proof. (⇒) follows from Lemmas 1, 2 and 3.

(⇐). Assume there exists a vertex v at which T is not semi-stable. Then there exists an automorphism g in Γ(T_v), which does not occur in Γ(T)_v. (Since in general Γ(T_v) ≤ Γ(T_v) - see [3].) Let w be adjacent to v in T and assume w^g ≠ w. Such a vertex must exist by Theorem 1.

(1) If w and w^g are in the same component of T_v and this component, which is a tree, has no fixed vertices under the action of g, then the subtree contradicts (i). So assume that x is the nearest fixed vertex to w in the same component of T_v as w and w^g. From elementary properties of automorphisms, the branch of this component which contains w, when rooted at x, is isomorphic to the branch of the component which contains w^g, when rooted at x. Hence (ii) is contradicted.

(2) Suppose then w and w^g are in different components B₁, B₂ of T_v. Since g ∈ Γ(T_v) then B₁ = B₂.

(a) If w is not fixed in Γ(B₁) then there exists h ∈ Γ(B₁) ≤ Γ(T_v) such that w^h ≠ w, and w^h ∈ B₁. Applying the argument of (1) we obtain a contradiction. (h ∉ Γ(T) since otherwise w, v and w^h form a cycle in T.)

(b) If w is fixed in Γ(B₁) then w^g is fixed in Γ(B₂). Now either B₁, (B₁ together with v and rooted at v) and B₂, (B₂

together with v and rooted at v), are isomorphic or they are not. If they are not then (iii) is contradicted. If they are then the non semi-stability at v is caused by some vertex adjacent to v , other than w .

Considering all vertices adjacent to v in T in order via (1) and (2), we either produce a contradiction of (i), (ii) or (iii), or we find that T is semi-stable at v . But this is a contradiction. Hence the theorem holds.

3. Stable graphs

In [5] the concept of stable graphs was introduced. G is *stable* if there exists a sequence of all the vertices of V , $\{v_1, v_2, \dots, v_n\}$, such that $\Gamma(G_{v_1 v_2 \dots v_k}) = \Gamma(G)_{v_1 v_2 \dots v_k}$ for $k = 1, 2, 3, \dots, n$. It was shown in [2], that a tree T is stable if and only if $\Gamma(T)$ contains a transposition.

THEOREM 3. *If T is completely semi-stable then T is stable.*

Proof. T is stable if and only if it contains a bunch ([2]). A bunch is a set of r (> 1) vertices of degree one adjacent to a common vertex.

Suppose T has no bunch. Then it must have a vertex x of degree 2 adjacent to an end vertex y (Figure 5). (This arises since $|V|$ is finite. If the degree of x is more than 2, move along the tree away from w to the next penultimate vertex. If this has degree greater than 2, continue away from this vertex over edges not so far traversed. Eventually we must reach a penultimate vertex of degree 2.)

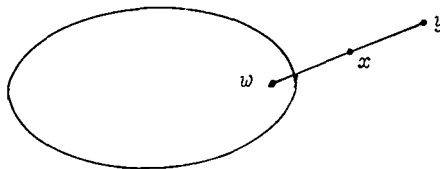


Figure 5

Since x exists as in Figure 5 consider T_w . This contains an inadmissible bicentral subtree on the vertex set $\{x, y\}$, and so T is

not semi-stable at w - a contradiction.

However T stable does not imply T completely semi-stable. For instance the tree of Figure 6 is stable, but is not semi-stable at v by Theorem 2.

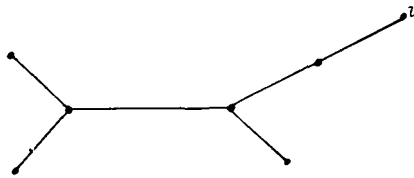


Figure 6

In conclusion we note that for graphs other than trees it is not possible to say that G completely semi-stable implies G stable. In Section 1 we saw that regular graphs are completely semi-stable, but in general they are not stable. Take, for instance, the cycle on n vertices. Similarly the graphs $P_m \times P_n$ which are again completely semi-stable (for

- (i) neither m nor n equal to 1 ,
- (ii) if $m = 2$ then $n = 2$, and *vice-versa*,
- (iii) m and n not both equal to 3)

are not in general stable. In fact only $P_2 \times P_2$ is stable.

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