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Sharp smoothing properties of averages over curves

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Abstract

We prove sharp smoothing properties of the averaging operator defined by convolution with a measure on a smooth nondegenerate curve γ in $\mathbb{R}^d$, $d \geq 3$. Despite the simple geometric structure of such curves, the sharp smoothing estimates have remained largely unknown except for those in low dimensions. Devising a novel inductive strategy, we obtain the optimal L^p Sobolev regularity estimates, which settle the conjecture raised by Beltran–Guo–Hickman–Seeger [1]. Besides, we show the sharp local smoothing estimates on a range of p for every $d \geq 3$. As a result, we establish, for the first time, nontrivial L^p boundedness of the maximal average over dilations of γ for $d \geq 4$.

Contents

1. Introduction

The regularity property of integral transforms defined by averages over submanifolds is a fundamental subject in harmonic analysis, which has been extensively studied since the 1970s. There is an immense

body of literature devoted to the subject (see, for example, [33, 21, 32, 8] and references therein). However, numerous problems remain wide open. The regularity property is typically addressed in the frameworks of L^p improving, L^p Sobolev regularity, and local smoothing estimates, to which L^p boundedness of the maximal average is also closely related. In this paper, we study the smoothing estimates for the averaging operator given by convolution with a measure supported on a curve.

Let $I = [-1, 1]$ and γ be a smooth curve from I to $\mathbb{R}^d$. We define a measure $\mathfrak{m}_t$ supported on $t\gamma$ by

$$\langle \mathfrak{m}_t, f \rangle = \int f(t\gamma(s))\psi(s)ds,$$

where $\psi \in C_c^\infty((-1, 1))$. We are concerned with $d \geq 3$ since all the problems we address in the current paper are well understood when $d = 2$. We consider the averaging operator

$$\mathcal{A}_t f(x) = f * \mathfrak{m}_t(x)$$

and study the above-mentioned regularity problems on $\mathcal{A}_t$ under the assumption that γ is nondegenerate, that is to say,

$$\det(\gamma'(s), \ldots, \gamma^{(d)}(s)) \neq 0, \quad s \in I. \tag{1.1}$$

The L^p improving property of $\mathcal{A}_t$ for a fixed $t \neq 0$ now has a complete characterization; see [7, 34] (also, see [36] for generalizations to variable coefficient settings). However, L^p Sobolev and local smoothing estimates for $\mathcal{A}_t$ turned out to be more involved and are far less well understood. Recently, there has been progress in low dimensions $d = 3, 4$ ([24, 14, 1, 2]), but it does not seem feasible to extend the approaches in the recent works to higher dimensions. We discuss this matter in detail near the end of the introduction. By devising an inductive strategy, we prove the optimal L^p Sobolev regularity and sharp local smoothing estimates in any dimension $d \geq 3$. As a result, we also obtain L^p boundedness of the associated maximal function which was unknown for $d \geq 4$.

L^p Sobolev regularity

Let $2 \leq p \leq \infty$. We set $\mathcal{A}f = \mathcal{A}_1 f$ and consider the L^p Sobolev regularity estimate

$$\|\mathcal{A}f\|_{L^p_\alpha(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}. \tag{1.2}$$

When $d = 2$, the estimate holds if and only if $\alpha \leq 1/p$ (e.g., see [6]). In higher dimensions, however, the problem of obtaining (1.2) with the sharp smoothing order α becomes highly nontrivial except for the $L^2 \to L^2_{1/d}$ estimate which is an easy consequence of the decay property of the Fourier transform of $\mathfrak{m}_t$:

$$|\widehat{\mathfrak{m}_t}(\xi)| \leq C(1 + |t\xi|)^{-1/d}. \tag{1.3}$$

It was conjectured by Beltran, Guo, Hickman and Seeger [2, Conjecture 1] that the estimate (1.2) holds for $\alpha \leq 1/p$ if $2d - 2 < p < \infty$. When $d = 3$, the conjecture was verified by the conditional result of Pramanik and Seeger [24] and the decoupling inequality due to Bourgain and Demeter [4] (see [20, 35] for earlier results). The case $d = 4$ was recently obtained by Beltran et al. [2]. Our first result proves the conjecture for every $d \geq 5$.

Theorem 1.1. *Let $d \geq 3$. Suppose γ is a smooth nondegenerate curve. Then, the estimate (1.2) holds for $\alpha \leq 1/p$ if $p > 2(d - 1)$.*

Interpolation with the $L^2 \to L^2_{1/d}$ estimate gives (1.2) for $\alpha < (p+2)/(2dp)$ when $2 < p \leq 2(d-1)$. It is also known that (1.2) fails if $\alpha > \alpha(p) := \min(1/p, (p + 2)/(2dp))$ (see [2, Proposition 1.2]). Thus, only the estimate (1.2) with $\alpha = \alpha(p)$ remains open for $2 < p \leq 2(d - 1)$. Those endpoint

estimates seem to be a subtle problem. The argument in this paper provides simpler alternative proofs of the previous results for $d = 3, 4$. Theorem 1.1 remains valid as long as $\gamma \in C^{2d}(I)$ (see Theorem 4.1). However, we do not try to optimize the regularity assumption.

The result in Theorem 1.1 can be easily generalized to curves of different types. We say a smooth curve γ from I to $\mathbb{R}^d$ is of finite type if there is an ℓ such that $\operatorname{span}\{\gamma^{(1)}(s), \ldots, \gamma^{(\ell)}(s)\} = \mathbb{R}^d$ for each $s \in I$. The type at s is defined to be the smallest of such ℓ and the maximal type is the supremum over $s \in I$ of the type at s. (See, e.g., [24, 12].) Using Theorem 1.1 and a rescaling argument ([24, 12]), one can obtain the following, which proves the Conjecture 2 in [2].

Corollary 1.2. *Let $d \geq 3$, $\ell > d$ and $2 \leq p < \infty$. Suppose γ is a curve of maximal type ℓ. Then the estimate (1.2) holds for $\alpha \leq \min\big(\alpha(p), 1/\ell\big)$ if $p \neq \ell$ when $\ell \geq 2d - 2$, and if $p \in [2, 2\ell/(2d - \ell)) \cup (2d - 2, \infty)$ when $d < \ell < 2d - 2$.*

By interpolation, (1.2) holds for $\alpha < \min\big(\alpha(p), 1/\ell\big)$ if $p = \ell$ when $\ell \geq 2d - 2$, and if $2\ell/(2d - \ell) \leq p \leq 2d - 2$ when $d < \ell < 2d - 2$. These estimates are sharp. Since a finite type curve contains a nondegenerate subcurve and the $L^2 \to L^2_{1/\ell}$ estimate is optimal, (1.2) fails if $\alpha > \min\big(\alpha(p), 1/\ell\big)$. In particular, when $\ell \geq 2d - 2$, Corollary 1.2 resolves the problem of the Sobolev regularity estimate (1.2). In fact, failure of the $L^\ell \to L^\ell_{1/\ell}$ bound was shown in [2] using Christ's example [6]. By [28, Theorem 1.1], Corollary 1.2 also gives $H^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ bound on the lacunary maximal function $f \to \sup_{k \in \mathbb{Z}} |f * \mathfrak{m}_{2^k}|$ whenever γ is of finite type.

Sharp local smoothing

We now consider the estimate

$$\|\chi(t)\mathcal{A}_t f\|_{L^p_\alpha(\mathbb{R}^{d+1})} \leq C\|f\|_{L^p(\mathbb{R}^d)}, \tag{1.4}$$

where χ is a smooth function supported in $(1/2, 4)$. Compared with the L^p Sobolev estimate (1.2), the additional integration in t is expected to yield extra smoothing. Such a phenomenon is called *local smoothing*, which has been studied for the dispersive equations to a great extent (e.g., see [29, 9]). However, the local smoothing for the averaging operators exhibits considerably different nature.

In particular, there is no local smoothing when $p = 2$. Besides, a bump function example shows (1.4) holds only if $\alpha \leq 1/d$. As we shall see, the estimate (1.4) fails unless $\alpha \leq 2/p$ (Proposition 3.9 below). So, it seems to be plausible to conjecture that (1.4) holds for $\alpha < \min(2/p, 1/d)$ if $2 < p < \infty$. For $d = 2$, the conjecture follows by the recent result on Sogge's local smoothing conjecture for the wave operator ([30, 38, 16, 4]), which is due to Guth, Wang and Zhang [11]. When $d = 3$, some local smoothing estimates were utilized by Pramanik and Seeger [24] and Beltran et al. [1] to prove L^p maximal bound. Nevertheless, for $d \geq 3$, no local smoothing estimate up to the sharp order $2/p$ has been known previously.

Theorem 1.3. *Let $d \geq 3$. Suppose γ is a smooth nondegenerate curve. Then, if $p \geq 4d - 2$, the estimate (1.4) holds true for $\alpha < 2/p$.*

Theorem 1.3 remains valid as far as $\gamma \in C^{3d+1}(I)$ (see Theorem 2.2 below).

Maximal estimate

The local smoothing estimate (1.4) has been of particular interest in connection to L^p boundedness of the maximal operator

$$Mf(x) = \sup_{0 < t} |\mathcal{A}_t f(x)|$$

([19, 27, 24, 1]) and problems in geometric measure theory (see, e.g., [38] and Corollary 1.6 below). If the estimate (1.4) holds for some $\alpha > 1/p$, L^p boundedness of M follows by a standard argument relying on the Sobolev embedding ([24]).

The study of the maximal functions generated by dilations of submanifolds goes back to Stein's spherical maximal theorem [31] (see, also, [32, Ch.10] and [13]). The circular maximal theorem was later proved by Bourgain [3] (also, see [30, 19, 26, 27, 15]). Afterwards, a natural question was whether the maximal operator M under consideration in the current paper is bounded on L^p for some $p \neq \infty$ when $d \geq 3$. In view of Stein's interpolation argument based on L^2 estimate [31], proving L^p boundedness of M becomes more challenging as d increases since the decay of the Fourier transform of $\mathfrak{m}_t$ gets weaker (see (1.3)). Though the question was raised as early as in the late 1980s, it remained open for any $d \geq 3$ until recently. In $\mathbb{R}^3$, the first positive result was obtained by Pramanik and Seeger [24] and the range of p was further extended to $p > 4$ thanks to the decoupling inequality for the cone [4]. Very recently, the authors [14] proved L^p boundedness of M on the optimal range, that is, M is bounded on L^p if and only if $p > 3$. The same result was independently obtained by Beltran et al. [1].

However, no nontrival L^p bound on M has been known in higher dimensions. The following establishes existence of such maximal bounds for every $d \geq 4$.

Theorem 1.4. *Let $d \geq 4$. Suppose γ is a smooth nondegenerate curve. Then, for $p > 2(d-1)$ we have*

$$\|Mf\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}. \tag{1.5}$$

The result is a consequence of Theorem 1.3. Since the estimate (1.4) holds for $p = 2$ and $\alpha = 1/d$, interpolation gives (1.4) for some $\alpha > 1/p$ when $2d - 2 < p < \infty$. So, the maximal estimate (1.5) follows, as mentioned before, by a standard argument. A natural conjecture is that M is bounded on L^p if and only if $p > d$. M cannot be bounded on L^p if $p \leq d$, as can be seen by a simple adaptation of the argument in [14, Proposition 4.4]. Theorem 1.4 also extends to the finite type curves by the rescaling argument. The following result is sharp when $\ell \geq 2(d-1)$.

Corollary 1.5. *Let $d \geq 4$ and $\ell > d$. Suppose γ is a curve of maximal type ℓ. Then the estimate (1.5) holds if $p > \max(\ell, 2(d-1))$.*

Packing of curves in $\mathbb{R}^d$

The sharp local smoothing estimate (1.4) in Theorem 1.3 has interesting measure theoretic consequences concerning unions of curves generated by translation and dilation of a nondegenerate curve. The following generalizes Wolff's result [38, Corollary 3], where unions of circles in $\mathbb{R}^2$ were considered (see also [17, 18, 37] for earlier results).

Corollary 1.6. *Let γ be a smooth nondegenerate curve in $\mathbb{R}^d$, $d \geq 3$, and let $E \subset \mathbb{R}^{d+1}$ be a set of Hausdorff dimension greater than $d - 1$. Suppose F is a set in $\mathbb{R}^d$ such that $(x + t\gamma(I)) \cap F$ has positive one-dimensional outer measure for all $(x, t) \in E$. Then F has positive outer measure.*

Corollary 1.6 follows by Theorem 1.3 and the argument in [38]. The result does not hold in general without the nondegeneracy assumption on γ as one can easily see considering a curve contained in a lower dimensional vector space. The same result continues to be valid for the finite type curves. Consequently, Corollary 1.6 implies the following.

Corollary 1.6′. *Let γ be a smooth finite type curve in $\mathbb{R}^d$, $d \geq 3$, and let E and F be compact subsets in $\mathbb{R}^d$. Suppose E has Hausdorff dimension greater than $d - 1$ and for each $x \in E$ there is $t(x) > 0$ such that $x + t(x)\gamma(I) \subset F$. Then, F has positive measure.*

Our approach

To prove L^p ($p \neq 2$) smoothing properties of $\mathcal{A}_t$, we need more than the decay of $\widehat{\mathfrak{m}}_t$, that is, (1.3). When $d = 2$, we have rather a precise asymptotic expansion of $\widehat{\mathfrak{m}}_t$, which makes it possible to relate $\mathcal{A}_t$

to other forms of operators. In fact, one can use the estimate for the wave operator (e.g., [27, 35, 15]) to obtain local smoothing estimate. However, in higher dimensions $d \geq 3$, to compute $\widehat{m}_t$ explicitly is not a simple matter. Even worse, this becomes much more complicated as d increases since one has to take into account the derivatives $\gamma^{(k)}(s) \cdot \xi$, $k = 2, \ldots, d$. The common approach in the previous works ([24, 1, 2]) to get around this difficulty was to use detailed decompositions (of various scales) on the Fourier side away from the conic sets where $\widehat{m}_t$ decays slowly. The consequent decompositions were then combined with the decoupling or square function estimate ([20, 23, 24, 25, 1, 2]). However, this type of approach based on fine scale decomposition becomes exceedingly difficult to manage as the dimension d gets larger and, consequently, does not seem to be tractable in higher dimensions.

To overcome the difficulty, we develop a new strategy which allows us to dispense with such sophisticated decomposition. Before closing the introduction, we briefly discuss the key ingredients of our approach.

- The main novelty of this paper lies in an induction argument which we build on the local nondegeneracy assumption:

$$\sum_{\ell=1}^{L} |\langle \gamma^{(\ell)}(s), \xi \rangle| \geq B^{-1}|\xi| \qquad\qquad \mathfrak{N}(L,B)$$

for a constant $B \geq 1$. To prove our results, we consider the operator $\mathcal{A}_t[\gamma, a]$ (see (2.2) below for its definition). Clearly, $\mathfrak{N}(d, B')$ holds for a constant $B' > 0$ if γ satisfies the condition (1.1). However, instead of considering the case $L = d$ alone, we prove the estimate for all $L = 2, \ldots, d$ under the assumption that $\mathfrak{N}(L, B)$ holds on the support of a. See Theorem 2.2 and 4.1. A trivial (yet, important) observation is that $\mathfrak{N}(L - 1, B)$ implies $\mathfrak{N}(L, B)$, so we may think of $\mathcal{A}_t[\gamma, a]$ as being more degenerate as L gets larger. Thanks to this hierarchical structure, we may use an inductive strategy along the number L. See Proposition 2.3 and 4.2 below.
- We extend the rescaling [12, 14] and iteration [24] arguments. Roughly speaking, we combine the first with the induction assumption in Proposition 2.3 (or 4.2) to handle the less degenerate parts, and use the latter to deal with the remaining part. In order to generalize those arguments, we introduce a class of symbols which are naturally adjusted to a small subcurve (Definition 2.4). We also use the decoupling inequalities for the nondegenerate curves obtained by Beltran et al. [2] (Corollary 2.15). Their inequalities were deduced from those due to Bourgain, Demeter and Guth [5]. Instead of applying the inequalities directly, we use modified forms which are adjusted to the sharp smoothing orders of the specific estimates (see (2.40) and (2.41) below). This makes it possible to obtain the sharp estimates on extended ranges.

Organization of the paper. We first prove Theorem 1.3 whose proof is more involved than that of Theorem 1.1. In Section 2, the proof of Theorem 1.3 is reduced to that of Proposition 2.9, which we prove while assuming Proposition 2.10. The proof of Proposition 2.10 is given in Section 3. We prove Theorem 1.1 in Section 4.

2. Smoothing estimates with localized frequency

In this section, we consider an extension of Theorem 1.3 via microlocalization (see Theorem 2.2 below) so that we can prove it in an inductive manner. We then reduce the matter to proving Proposition 2.9, which we show by applying Proposition 2.10. We also obtain some preparatory results.

Let $1 \leq L \leq d$ be a positive integer and $B \geq 1$ be a large number. For quantitative control of estimates we consider the following two conditions:

$$\max_{0 \leq j \leq 3d+1} |\gamma^{(j)}(s)| \leq B, \qquad\qquad s \in I, \qquad\qquad (2.1)$$

$$\text{Vol}\left(\gamma^{(1)}(s), \ldots, \gamma^{(L)}(s)\right) \geq 1/B, \qquad\qquad s \in I, \qquad\qquad \mathfrak{B}(L,B)$$

where $\mathrm{Vol}(v_1, \ldots, v_L)$ denotes the L-dimensional volume of the parallelepiped generated by $v_1, \ldots, v_L$. By finite decomposition and a standard reduction using rescaling and a change of variables, the constant B can be taken to be close to 1 (see Section 2.2).

Notation. For nonnegative quantities A and D, we denote $A \lesssim D$ if there exists an independent positive constant C such that $A \leq CD$, but the constant C may differ at each occurrence depending on the context, and $A \lesssim_B D$ means the inequality holds with an implicit constant depending on B. Throughout the paper, the constant C mostly depends on B. However, we do not make it explicit every time since it is clear in the context. By $A = O(D)$ we denote $|A| \lesssim D$.

Definition 2.1. For $k \geq 0$, let $\mathbb{A}_k = \{\xi \in \mathbb{R}^d : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. We say $a \in C^{d+L+2}(\mathbb{R}^{d+2})$ is a symbol of type (k, L, B) relative to γ if $\mathrm{supp}\, a \subset I \times [2^{-1}, 4] \times \mathbb{A}_k$, $\mathfrak{R}(L, B)$ holds for γ whenever $(s, t, \xi) \in \mathrm{supp}\, a$ for some t, and

$$|\partial_s^j \partial_t^l \partial_\xi^\alpha a(s, t, \xi)| \leq B|\xi|^{-|\alpha|}$$

for $(j, l, \alpha) \in \mathcal{I}_L := \{(j, l, \alpha) : 0 \leq j \leq 1, 0 \leq l \leq 2L, |\alpha| \leq d + L + 2\}$.

We define an integral operator by

$$\mathcal{A}_t[\gamma, a]f(x) = (2\pi)^{-d} \iint_{\mathbb{R}} e^{i(x - t\gamma(s)) \cdot \xi} a(s, t, \xi) ds\, \widehat{f}(\xi)\, d\xi. \tag{2.2}$$

Note $\mathcal{A}_t f = \mathcal{A}_t[\gamma, \psi]f$. Theorem 1.3 is a consequence of the following.

Theorem 2.2. *Let $\gamma \in C^{3d+1}(I)$ satisfy (2.1) and $\mathfrak{B}(L, B)$ for some $B \geq 1$. Suppose a is a symbol of type (k, L, B) relative to γ. Then, if $p \geq 4L - 2$, for $\epsilon > 0$ there is a constant $C_\epsilon = C_\epsilon(B)$ such that*

$$\|\mathcal{A}_t[\gamma, a]f\|_{L^p(\mathbb{R}^{d+1})} \leq C_\epsilon 2^{(-\frac{2}{p} + \epsilon)k} \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.3}$$

Theorem 2.2 is easy to prove when $L = 1$. Indeed, (2.3) follows from the estimate

$$|\mathcal{A}_t[\gamma, a]f(x)| \lesssim_B \int_I K * |f|(x - t\gamma(s))\, ds,$$

where $K(x) = 2^{(d-1)k}(1 + |2^k x|)^{-d-3}$. Note $|\gamma'(s) \cdot \xi| \sim 2^k$ if $(s, t, \xi) \in \mathrm{supp}\, a$ for some t. By integration by parts in s, $\mathcal{A}_t[\gamma, a] = t^{-1}\mathcal{A}_t[\gamma, \tilde{a}]$, where $\tilde{a} = i(\gamma'(s) \cdot \xi \partial_s a - \gamma''(s) \cdot \xi a)/(\gamma'(s) \cdot \xi)^2$. Since $|\partial_\xi^\alpha \tilde{a}| \lesssim |\xi|^{-|\alpha|-1}$ for $|\alpha| \leq d + 3$, routine integration by parts in ξ gives the desired estimate (e.g., see *Proof of Lemma 2.7* below). When $L = 2$, Theorem 2.2 follows by the result in [24, Theorem 4.1] and the decoupling inequality in [4].

Once we have Theorem 2.2, the proof of Theorem 1.3 is straightforward. By the Littlewood–Paley decomposition it is sufficient to show the estimate (2.3) for $p \geq 4d-2$ with $a_k(s, t, \xi) = \psi(s)\chi(t)\beta(2^{-k}|\xi|)$, where $\beta \in C_c^\infty((1/2, 2))$. This can be made rigorous using $\iint e^{-it(\tau + \gamma(s) \cdot \xi)}\psi(s)\chi(t)dsdt = O((1 + |\tau|)^{-N})$ for any N if $|\tau| \geq (1 + \max_{s \in \mathrm{supp}\, \psi} |\gamma(s)|)|\xi|$. Since γ satisfies the condition (1.1), a_k is of type (k, d, B) relative to γ for a large B. Therefore, Theorem 1.3 follows from Theorem 2.2.

Theorem 2.2 is immediate from the next proposition, which places Theorem 2.2 in an inductive framework.

Proposition 2.3. *Let $2 \leq N \leq d$. Suppose Theorem 2.2 holds for $L = N - 1$. Then, Theorem 2.2 holds true with $L = N$.*

To prove Proposition 2.3, from this section to Section 3 we fix $N \in [2, d]$, γ satisfying $\mathfrak{B}(N, B)$, and a symbol a of type (k, N, B) relative to γ.

One of the main ideas is that by a suitable decomposition of the symbol we can separate from $\mathcal{A}_t[\gamma, a]$ the less degenerate part which corresponds to $L = N - 1$. To this part we apply the assumption

combined with a rescaling argument. To do this, we introduce a class of symbols which are adjusted to short subcurves of γ.

2.1. Symbols associated to subcurves

We begin with some notations. Let $N \geq 2$, and let δ and B' denote numbers such that

$$2^{-k/N} \leq \delta \leq 2^{-7dN} B^{-6N}, \qquad B \leq B' \leq B^C$$

for a large constant $C \geq 3d + 1$. We note that $\mathfrak{B}(N - 1, B')$ holds for some B'. In fact, $\mathfrak{B}(N - 1, B^2)$ follows by (2.1) and $\mathfrak{B}(N, B)$.

For $s \in I$, we define a linear map $\widetilde{\mathcal{L}}_s^\delta : \mathbb{R}^d \mapsto \mathbb{R}^d$ as follows:

$$
\begin{aligned}
(\widetilde{\mathcal{L}}_s^\delta)^\top \gamma^{(j)}(s) &= \delta^{N-j} \gamma^{(j)}(s), & j &= 1, \ldots, N-1, \\
(\widetilde{\mathcal{L}}_s^\delta)^\top v &= v, & v &\in \left(V_s^{\gamma, N-1}\right)^\perp,
\end{aligned}
\tag{2.4}
$$

where $V_s^{\gamma, \ell} = \operatorname{span}\left\{\gamma^{(j)}(s) : j = 1, \ldots, \ell\right\}$. $\widetilde{\mathcal{L}}_s^\delta$ is well defined since $\mathfrak{B}(N - 1, B^2)$ holds for γ. The linear map $\widetilde{\mathcal{L}}_s^\delta$ naturally appears when we rescale a subcurve of length about δ (see the proofs of Lemma 2.7 and 2.8). We denote

$$\mathcal{L}_s^\delta(\tau, \xi) = \left(\delta^N \tau - \gamma(s) \cdot \widetilde{\mathcal{L}}_s^\delta \xi, \ \widetilde{\mathcal{L}}_s^\delta \xi\right), \qquad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d. \tag{2.5}$$

We set $G(s) = (1, \gamma(s))$ and define

$$\Lambda_k(s, \delta, B') = \bigcap_{0 \leq j \leq N-1} \left\{(\tau, \xi) \in \mathbb{R} \times \mathbb{A}_k : |\langle G^{(j)}(s), (\tau, \xi)\rangle| \leq B' 2^{k+5} \delta^{N-j}\right\}.$$

Definition 2.4. Let $(s_0, \delta) \in (-1, 1) \times (0, 1)$ such that $I(s_0, \delta) := [s_0 - \delta, s_0 + \delta] \subset I$. Then, by $\mathfrak{A}_k(s_0, \delta)$ we denote the set of $\mathfrak{a} \in C^{d+N+2}(\mathbb{R}^{d+3})$ such that

$$\operatorname{supp} \mathfrak{a} \subset I(s_0, \delta) \times [2^{-1}, 2^2] \times \Lambda_k(s_0, \delta, B), \tag{2.6}$$

$$\left|\partial_s^j \partial_t^l \partial_{\tau, \xi}^\alpha \mathfrak{a}\left(s, t, \mathcal{L}_{s_0}^\delta(\tau, \xi)\right)\right| \leq B \delta^{-j} |(\tau, \xi)|^{-|\alpha|}, \qquad (j, l, \alpha) \in \mathcal{I}_N. \tag{2.7}$$

We define $\operatorname{supp}_\xi \mathfrak{a} = \bigcup_{s, t, \tau} \operatorname{supp} \mathfrak{a}(s, t, \tau, \cdot)$ and $\operatorname{supp}_{s, \xi} \mathfrak{a} = \bigcup_{t, \tau} \operatorname{supp} \mathfrak{a}(\cdot, t, \tau, \cdot)$, and $\operatorname{supp}_s \mathfrak{a}$ and $\operatorname{supp}_{\tau, \xi} \mathfrak{a}$ are defined likewise. Note that *a statement $S(s, \xi)$, depending on (s, ξ), holds on $\operatorname{supp}_{s, \xi} \mathfrak{a}$ if and only if $S(s, \xi)$ holds whenever $(s, t, \tau, \xi) \in \operatorname{supp} \mathfrak{a}$ for some t, τ.*

Denote $V_s^{G, \ell} = \operatorname{span}\{(1, 0), G'(s), \ldots, G^{(\ell)}(s)\}$. We take a close look at the map $\mathcal{L}_s^\delta$. By the equations (2.4) and (2.5) we have

$$
\begin{cases}
(\mathcal{L}_s^\delta)^\top G(s) = \delta^N (1, 0), & \\
(\mathcal{L}_s^\delta)^\top G^{(j)}(s) = \delta^{N-j} G^{(j)}(s), & j = 1, \ldots, N-1, \\
(\mathcal{L}_s^\delta)^\top v = v, & v \in (V_s^{G, N-1})^\perp.
\end{cases}
\tag{2.8}
$$

The first identity is clear since $(\mathcal{L}_s^\delta)^\top (\tau, \xi) = (\delta^N \tau, (\widetilde{\mathcal{L}}_s^\delta)^\top \xi - \tau (\widetilde{\mathcal{L}}_s^\delta)^\top \gamma(s))$. The second and the third follow from (2.4) since $G^{(j)} \in \{0\} \times \mathbb{R}^d$, $1 \leq j \leq N - 1$, $(V_s^{G, N-1})^\perp \subset \{0\} \times \mathbb{R}^d$, and $(\mathcal{L}_s^\delta)^\top (0, \xi) = (0, (\widetilde{\mathcal{L}}_s^\delta)^\top \xi)$. Furthermore, there is a constant $C = C(B)$, independent of s and δ, such that

$$|\mathcal{L}_s^\delta(\tau, \xi)| \leq C |(\tau, \xi)|. \tag{2.9}$$

Note that (2.9) is equivalent to $|(\mathcal{L}_s^\delta)^\top (\tau, \xi)| \le C|(\tau, \xi)|$. The inequality is clear from (2.4) because $\mathfrak{B}(N-1, B^2)$ holds and all the eigenvalues of $(\widetilde{\mathcal{L}}_s^\delta)^\top$ are contained in the interval $(0, 1]$.

Lemma 2.5. *Let $\mathcal{L}_s^\delta(\tau, \xi) \in \Lambda_k(s, \delta, B')$ and $\mathfrak{B}(N-1, B')$ hold for γ. Then, there exists a constant $C = C(B')$ such that*

$$C^{-1}|(\tau, \xi)| \le 2^k \le C|\xi|. \tag{2.10}$$

Proof. Since $\mathcal{L}_s^\delta(\tau, \xi) \in \Lambda_k(s, \delta, B')$, by (2.5) we have $2^{k-1} \le |\widetilde{\mathcal{L}}_s^\delta \xi| \le 2^{k+1}$. So, the second inequality in (2.10) is clear from (2.9) if we take $\tau = 0$.

To show the first inequality, from (2.8) we have $|\langle (1, 0), (\tau, \xi) \rangle| \le B' 2^{k+5}$ and $|\langle G^{(j)}(s), (\tau, \xi) \rangle| \le B' 2^{k+5}$, $1 \le j \le N-1$, because $\mathcal{L}_s^\delta(\tau, \xi) \in \Lambda_k(s, \delta, B')$. Also, if $v \in (V_s^{G, N-1})^\perp$ and $|v| = 1$, by (2.8) we see $|\langle v, (\tau, \xi) \rangle| = |\langle v, \mathcal{L}_s^\delta(\tau, \xi) \rangle| \le 2^{k+1}$. Therefore, we get $|(\tau, \xi)| \le C2^k$ for some $C = C(B')$ since $\mathfrak{B}(N-1, B')$ holds and $V_s^{G, N-1} \oplus (V_s^{G, N-1})^\perp = \mathbb{R}^{d+1}$. □

The following shows the matrices $\mathcal{L}_s^\delta$, $\mathcal{L}_{s_0}^\delta$ are close to each other if so are s, s_0.

Lemma 2.6. *Let $s, s_0 \in (-1, 1)$ and γ satisfy $\mathfrak{B}(N-1, B')$. If $|s - s_0| \le \delta$, then there exists a constant $C = C(B') \ge 1$ such that*

$$C^{-1}|(\tau, \xi)| \le |(\mathcal{L}_{s_0}^\delta)^{-1} \mathcal{L}_s^\delta (\tau, \xi)| \le C|(\tau, \xi)|. \tag{2.11}$$

Proof. It suffices to prove that (2.11) holds if $|s - s_0| \le c\delta$ for a constant $c > 0$, independent of s and s_0. Applying this finitely many times, we can remove the additional assumption. Moreover, it is enough to show

$$\|(\mathcal{L}_s^\delta)^\top (\mathcal{L}_{s_0}^\delta)^{-\top} - I\| \lesssim_{B'} c \tag{2.12}$$

when $|s - s_0| \le c\delta$. Here, $\|\cdot\|$ denotes a matrix norm. Taking $c > 0$ sufficiently small, we get (2.11).

By (2.8), $(\mathcal{L}_s^\delta)^\top (\mathcal{L}_{s_0}^\delta)^{-\top} G^{(j)}(s_0) = (\mathcal{L}_s^\delta)^\top \delta^{-(N-j)} G^{(j)}(s_0)$ for $j = 1, \ldots, N-1$. Let $s_0 = s + c'\delta$, $|c'| \le c$. Expanding $G^{(j)}$ in Taylor series at s, by the condition (2.1) we have

$$(\mathcal{L}_s^\delta)^\top (\mathcal{L}_{s_0}^\delta)^{-\top} G^{(j)}(s_0) = (\mathcal{L}_s^\delta)^\top \left(\sum_{\ell=j}^{N-1} \delta^{-(N-j)} G^{(\ell)}(s) \frac{(c'\delta)^{\ell-j}}{(\ell-j)!} + O(c^{N-j} B') \right)$$

for $j = 1, \ldots, N-1$. By (2.8) and the mean value theorem, we get

$$(\mathcal{L}_s^\delta)^\top (\mathcal{L}_{s_0}^\delta)^{-\top} G^{(j)}(s_0) = G^{(j)}(s_0) + O(cB'), \quad j = 1, \ldots, N-1.$$

From (2.8), we also have $(\mathcal{L}_s^\delta)^\top (\mathcal{L}_{s_0}^\delta)^{-\top} (1, 0) = \delta^{-N} (\mathcal{L}_s^\delta)^\top G(s_0)$. A similar argument also shows $(\mathcal{L}_s^\delta)^\top (\mathcal{L}_{s_0}^\delta)^{-\top} (1, 0) = (1, 0) + O(cB')$.

Let $\{v_N, \ldots, v_d\}$ denote an orthonormal basis of $(V_{s_0}^{G, N-1})^\perp$. By $\mathfrak{B}(N-1, B')$ and (2.1), it follows that $|\gamma^{(j)}(s_0)| \ge (B')^{-1-N}$, $j = 1, \ldots, N-1$. Since $|\gamma^{(j)}(s) - \gamma^{(j)}(s_0)| \le cB'\delta$, there is an orthonormal basis $\{v_N(s), \ldots, v_d(s)\}$ of $(V_s^{G, N-1})^\perp$ such that $|v_j(s) - v_j| \lesssim_{B'} c\delta$, $j = N, \ldots, d$. So, we have $|(\mathcal{L}_s^\delta)^\top v_j - v_j| \lesssim_{B'} c\delta$ by (2.9). Since $(\mathcal{L}_{s_0}^\delta)^{-\top} v_j = v_j$, it follows that $|(\mathcal{L}_s^\delta)^\top (\mathcal{L}_{s_0}^\delta)^{-\top} v_j - v_j| \lesssim_{B'} c\delta$, $j = N, \ldots, d$.

We denote by M the matrix $[(1, 0), G'(s_0), \ldots, G^{(N-1)}(s_0), v_N, \ldots, v_d]$. Then, combining all together, we have $\|(\mathcal{L}_s^\delta)^\top (\mathcal{L}_{s_0}^\delta)^{-\top} M - M\| \lesssim_{B'} c$. Note that $\mathfrak{B}(N-1, B')$ gives $|M^{-1} v| \lesssim_{B'} |v|$ for $v \in \mathbb{R}^{d+1}$. Therefore, we obtain (2.12). □

For a continuous function $\mathfrak{a}$ supported in $I \times [1/2, 4] \times \mathbb{R} \times \mathbb{A}_k$, we set

$$m[\mathfrak{a}](\tau, \xi) = \iint e^{-it'(\tau + \gamma(s) \cdot \xi)} \mathfrak{a}(s, t', \tau, \xi) \, ds \, dt', \tag{2.13}$$

$$\mathcal{T}[\mathfrak{a}] f(x, t) = (2\pi)^{-d-1} \iint e^{i(x \cdot \xi + t\tau)} m[\mathfrak{a}](\tau, \xi) \widehat{f}(\xi) \, d\xi \, d\tau. \tag{2.14}$$

Lemma 2.7. *Suppose* $\mathfrak{a} \in C^{d+3}(\mathbb{R}^{d+3})$ *satisfies* (2.6) *and* (2.7) *for* $j = l = 0$ *and* $|\alpha| \leq d + 3$. *Then, there is a constant* $C = C(B)$ *such that*

$$\|\mathcal{T}[\mathfrak{a}] f\|_{L^\infty(\mathbb{R}^{d+1})} \leq C\delta \|f\|_{L^\infty(\mathbb{R}^d)}, \tag{2.15}$$

$$\|(1 - \tilde{\chi}) \mathcal{T}[\mathfrak{a}] f\|_{L^p(\mathbb{R}^{d+1})} \leq C 2^{-k} \delta^{1-N} \|f\|_{L^p(\mathbb{R}^d)}, \quad p > 1, \tag{2.16}$$

where $\tilde{\chi} \in C_c^\infty((2^{-2}, 2^3))$ *such that* $\tilde{\chi} = 1$ *on* $[3^{-1}, 6]$.

Proof. We first note

$$\mathcal{T}[\mathfrak{a}] f(x, t) = \int K[\mathfrak{a}](s, t, \cdot) * f(x) \, ds, \tag{2.17}$$

where

$$K[\mathfrak{a}](s, t, x) = (2\pi)^{-d-1} \iiint e^{i(t-t', x-t'\gamma(s)) \cdot (\tau, \xi)} \mathfrak{a}(s, t', \tau, \xi) \, d\xi \, d\tau \, dt'. \tag{2.18}$$

Since $\text{supp}_s \, \mathfrak{a} \subset I(s_0, \delta)$, to prove the estimate (2.15) we need only to show

$$\|K[\mathfrak{a}](s, \cdot)\|_{L_t^\infty L_x^1} \leq C, \qquad s \in I(s_0, \delta) \tag{2.19}$$

for some $C = C(B) > 0$. To this end, changing variables $(\tau, \xi) \to 2^k \mathcal{L}_s^\delta(\tau, \xi)$ in the right-hand side of (2.18) and noting $|\det \mathcal{L}_s^\delta| = \delta^N |\det \widetilde{\mathcal{L}}_s^\delta| = \delta^{N(N+1)/2}$, we get

$$K[\mathfrak{a}](s, t, x) = C_* \iiint e^{i2^k(t-t', x-t\gamma(s)) \cdot (\delta^N \tau, \widetilde{\mathcal{L}}_s^\delta \xi)} \mathfrak{a}(s, t', 2^k \mathcal{L}_s^\delta(\tau, \xi)) \, d\xi \, d\tau \, dt',$$

where $C_* = (2\pi)^{-d-1} \delta^{N(N+1)/2} 2^{k(d+1)}$. Since $\mathfrak{a}$ satisfies (2.6), by (2.11) and Lemma 2.5 we have $\text{supp} \, \mathfrak{a}(s, t, 2^k \mathcal{L}_s^\delta \cdot) \subset \{(\tau, \xi) : |(\tau, \xi)| \lesssim_B 1\}$. Besides, by (2.7) and (2.11) it follows that $|\partial_{\tau, \xi}^\alpha (\mathfrak{a}(s, t, 2^k \mathcal{L}_s^\delta(\tau, \xi)))| \lesssim_B 1$ for $|\alpha| \leq d + 3$. Thus, repeated integration by parts in τ, ξ yields

$$|K[\mathfrak{a}](s, t, x)| \lesssim C_* \int_{1/2}^4 \left(1 + 2^k |(\delta^N(t - t'), (\widetilde{\mathcal{L}}_s^\delta)^\top (x - t\gamma(s)))| \right)^{-d-3} dt',$$

by which we obtain (2.19) as desired.

It is easy to show the estimate (2.16). The above estimate for $K[\mathfrak{a}]$ gives

$$\|(1 - \tilde{\chi}) K[\mathfrak{a}](s, t, \cdot)\|_{L_x^1} \lesssim \delta^{-N} 2^{-k} |t - 1|^{-1} |1 - \tilde{\chi}(t)|.$$

Since $\text{supp}_s \, \mathfrak{a} \subset I(s_0, \delta)$, (2.16) for $p > 1$ follows by (2.17) and Minkowski's and Young's convolution inequalities. $\square$

2.2. Rescaling

Let $\mathfrak{a} \in \mathfrak{A}_k(s_0, \delta)$. Suppose that

$$\sum_{j=1}^{N-1} \delta^j |\langle \gamma^{(j)}(s), \xi \rangle| \geq 2^k \delta^N / B' \tag{2.20}$$

holds on $\operatorname{supp}_{s,\xi} \mathfrak{a}$ for some $B' > 0$. Then, via decomposition and rescaling, we can bound the L^p norm of $\mathcal{T}[\mathfrak{a}]f$ by those of the operators given by symbols of type $(j, N-1, \tilde{B})$ relative to a curve for some $\tilde{B}$ and j (see Lemma 2.8 below).

To do so, we define a rescaled curve $\gamma_{s_0}^\delta : I \to \mathbb{R}^d$ by

$$\gamma_{s_0}^\delta(s) = \delta^{-N} (\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal \big(\gamma(\delta s + s_0) - \gamma(s_0) \big). \tag{2.21}$$

As $\delta \to 0$, the curves $\gamma_{s_0}^\delta$ get close to a nondegenerate curve in an N-dimensional vector space, so the curves behave in a uniform manner. In particular, (2.1) and $\mathfrak{B}(N, B)$ hold for some B for $\gamma = \gamma_{s_0}^\delta$ if $\delta < \delta'$ for a constant $\delta' = \delta'(B)$ small enough.

Note $(\gamma_{s_0}^\delta)^{(j)}(s) = \delta^{j-N} (\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal \gamma^{(j)}(\delta s + s_0)$, $1 \leq j \leq N-1$, and $|(\gamma_{s_0}^\delta)^{(j)}(s)| \lesssim B\delta$, $N+1 \leq j \leq 3d+1$. Thus, Taylor series expansion and (2.4) give

$$(\gamma_{s_0}^\delta)^{(j)}(s) = \sum_{k=0}^{N-j-1} \frac{\gamma^{(j+k)}(s_0)}{k!} s^k + \frac{(\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal \gamma^{(N)}(s_0)}{(N-j)!} s^{N-j} + O(B\delta)$$

for $j = 1, \ldots, N-1$. By (2.21), we have $(\gamma_{s_0}^\delta)^{(N)}(s) = (\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal \gamma^{(N)}(s_0) + O(B\delta)$. We write $\gamma^{(N)}(s_0) = v + v'$, where $v \in V_{s_0}^{\gamma, N-1}$ and $v' \in (V_{s_0}^{\gamma, N-1})^\perp$. So, $(\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal \gamma^{(N)}(s_0) = (\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal v + v'$. Since $|(\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal v| \lesssim_B \delta$ and $|v'| \leq B$, $|(\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal \gamma^{(N)}(s_0)| \leq B + C\delta$ for some $C = C(B)$. Thus, $\gamma = \gamma_{s_0}^\delta$ satisfies (2.1) with B replaced by $3B$ if $\delta < \delta'$.

An elementary argument (elimination) shows

$$\operatorname{Vol}\big((\gamma_{s_0}^\delta)^{(1)}(s), \ldots, (\gamma_{s_0}^\delta)^{(N)}(s) \big) = \operatorname{Vol}\big(\gamma^{(1)}(s_0), \ldots, \gamma^{(N)}(s_0) \big) + O(\delta)$$

since $(\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal \gamma^{(N)}(s_0) = (\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal v + v'$ and $\gamma^{(N)}(s_0) = v + v'$. Taking δ' small enough, from $\mathfrak{B}(N, B)$ for γ we see that $\mathfrak{B}(N, 3B)$ holds for $\gamma = \gamma_{s_0}^\delta$ if $0 < \delta < \delta'$.

The next lemma (cf. [14, Lemma 2.9]) plays a crucial role in what follows.

Lemma 2.8. *Let* $2 \leq N \leq d$, $\mathfrak{a} \in \mathfrak{A}_k(s_0, \delta)$, *and* $j_* = \log(2^k \delta^N)$. *Suppose* (2.20) *holds on* $\operatorname{supp}_{s,\xi} \mathfrak{a}$. *Then, there exist constants* $C, l_*, \tilde{B} \geq 1$ *and* $\delta' > 0$ *depending on* B, *and symbols* $a_1, \ldots, a_{l_*}$ *of type* $(j, N-1, \tilde{B})$ *relative to* $\gamma_{s_0}^\delta$ *such that*

$$\big\| \tilde{\chi} \, \mathcal{T}[\mathfrak{a}]f \big\|_{L^p(\mathbb{R}^{d+1})} \leq C\delta \sum_{1 \leq l \leq l_*} \big\| \mathcal{A}_t[\gamma_{s_0}^\delta, a_l] \tilde{f}_l \big\|_{L^p(\mathbb{R}^{d+1})},$$

$\|\tilde{f}_l\|_p = \|f\|_p$, *and* $j \in [j_* - C, j_* + C]$ *as long as* $0 < \delta < \delta'$.

Proof. We set $\mathfrak{a}_{\delta, s_0}(s, t, \tau, \xi) = \mathfrak{a}(\delta s + s_0, t, \tau, \xi)$. Combining the identities (2.13) and (2.14), we write $\mathcal{T}[\mathfrak{a}]f$ as an integral (e.g., see (2.17) and (2.18)). Subsequently, the change of variables $s \to \delta s + s_0$ and $(\tau, \xi) \to (\tau - \gamma(s_0) \cdot \xi, \xi)$ gives

$$\mathcal{T}[\mathfrak{a}]f(x, t) = (2\pi)^{-d-1} \delta \iint e^{i\langle x - t\gamma(s_0), \xi \rangle} \mathcal{J}(s, t, \xi) \widehat{f}(\xi) \, ds d\xi,$$

where

$$\mathcal{J}(s,t,\xi) = \iint e^{it\tau} e^{-it'(\tau + (\gamma(\delta s + s_0) - \gamma(s_0)) \cdot \xi)} \, \mathfrak{a}_{\delta,s_0}(s,t',\tau - \gamma(s_0) \cdot \xi, \xi) \, dt' d\tau.$$

Let $\tilde{f}$ be given by $\mathcal{F}(\tilde{f}) = |\det \delta^{-N} \widetilde{\mathcal{L}}_{s_0}^{\delta}|^{1-1/p} \widehat{f}(\delta^{-N} \widetilde{\mathcal{L}}_{s_0}^{\delta} \cdot)$ where $\mathcal{F}(\tilde{f})$ denotes the Fourier transform of $\tilde{f}$. Then, $\|\tilde{f}\|_p = \|f\|_p$. Changing variables $\xi \to \delta^{-N} \widetilde{\mathcal{L}}_{s_0}^{\delta} \xi$ gives

$$\mathcal{T}[\mathfrak{a}]f(x,t) = C_d \iint e^{i\langle x - t\gamma(s_0), \delta^{-N} \widetilde{\mathcal{L}}_{s_0}^{\delta} \xi\rangle} \mathcal{J}(s,t,\delta^{-N} \widetilde{\mathcal{L}}_{s_0}^{\delta} \xi) \mathcal{F}(\tilde{f})(\xi) \, ds d\xi,$$

where $C_d = (2\pi)^{-d-1} \delta |\det \delta^{-N} \widetilde{\mathcal{L}}_{s_0}^{\delta}|^{1/p}$. This leads us to set

$$\tilde{a}(s,t,\xi) = \frac{1}{2\pi} \iint e^{-it'(\tau + \gamma_{s_0}^{\delta}(s) \cdot \xi)} \tilde{\chi}(t) \mathfrak{a}_{\delta,s_0}(s, t+t', \delta^{-N} \mathcal{L}_{s_0}^{\delta}(\tau,\xi)) \, dt' d\tau. \qquad (2.22)$$

It is easy to check $\tilde{a} \in C^{d+N+2}(\mathbb{R}^{d+2})$, since so is $\mathfrak{a}$ and $\gamma \in C^{3d+1}$. By (2.21) and (2.5), we note $\tilde{\chi}(t)\mathcal{J}(s,t,\delta^{-N}\widetilde{\mathcal{L}}_{s_0}^{\delta}\xi) = 2\pi e^{-it\gamma_{s_0}^{\delta}(s) \cdot \xi} \tilde{a}(s,t,\xi)$. Therefore,

$$\tilde{\chi}(t)\mathcal{T}[\mathfrak{a}]f(x,t) = \delta |\det \delta^{-N} \widetilde{\mathcal{L}}_{s_0}^{\delta}|^{\frac{1}{p}} \mathcal{A}_t[\gamma_{s_0}^{\delta}, \tilde{a}]\tilde{f}(\delta^{-N}(\widetilde{\mathcal{L}}_{s_0}^{\delta})^{\intercal}(x - t\gamma(s_0))),$$

and a change of variables gives

$$\|\tilde{\chi} \, \mathcal{T}[\mathfrak{a}]f\|_{L^p(\mathbb{R}^{d+1})} = \delta \|\mathcal{A}_t[\gamma_{s_0}^{\delta}, \tilde{a}]\tilde{f}\|_{L^p(\mathbb{R}^{d+1})}. \qquad (2.23)$$

We shall obtain symbols of type $(j, N-1, \tilde{B})$ from $\tilde{a}$ via decomposition and rescaling. To this end, we first note

$$\text{supp}_\xi \, \tilde{a} \subset \{\xi \in \mathbb{R}^d : C^{-1}\delta^N 2^k \le |\xi| \le C\delta^N 2^k\} \qquad (2.24)$$

for a constant $C = C(B) \ge 1$. This follows by Lemma 2.5 since there exists τ such that $\delta^{-N}\mathcal{L}_{s_0}^{\delta}(\tau,\xi) \in \Lambda_k(s_0,\delta,B)$ if $\xi \in \text{supp}_\xi \, \tilde{a}$. We claim

$$|\partial_s^j \partial_t^l \partial_\xi^\alpha \tilde{a}(s,t,\xi)| \lesssim_B |\xi|^{-|\alpha|}, \qquad (j,l,\alpha) \in \mathcal{I}_{N-1}. \qquad (2.25)$$

To show (2.25), let us set

$$\mathfrak{b}(s,t,t',\tau,\xi) = \tilde{\chi}(t)\mathfrak{a}_{\delta,s_0}(s, t+t', \delta^{-N}\mathcal{L}_{s_0}^{\delta}(\tau,\xi)).$$

Note that $0 \le j \le 1$. Taking derivatives on both sides of the equation (2.22), we have

$$\partial_s^j \partial_t^l \partial_\xi^\alpha \tilde{a}(s,t,\xi) = \mathcal{I}[\mathfrak{b}_1] := \frac{1}{2\pi} \iint e^{-it'(\tau + \gamma_{s_0}^{\delta}(s) \cdot \xi)} \mathfrak{b}_1(s,t,t',\tau,\xi) \, dt' d\tau,$$

where

$$\mathfrak{b}_1 = \sum_{\substack{u_1+u_2=j, \\ \alpha_1+\alpha_2+\alpha_3=\alpha}} C_{\alpha,u} (t'\gamma_{s_0}^{\delta} \, ' \cdot \xi)^{u_1-|\alpha_1|} (t'\gamma_{s_0}^{\delta} \, ')^{\alpha_1} (t'\gamma_{s_0}^{\delta})^{\alpha_2} \partial_s^{u_2} \partial_t^l \partial_\xi^{\alpha_3} \mathfrak{b}$$

with $0 \le u_1 \le 1$, $0 \le |\alpha_1| \le u_1$, and constants $C_{\alpha,u}$ satisfying $|C_{\alpha,u}| = 1$. Integration by parts $u_1 + |\alpha_2|$ times in τ gives $\partial_s^j \partial_t^l \partial_\xi^\alpha \tilde{a} = \mathcal{I}[\mathfrak{b}_2]$, where

$$\mathfrak{b}_2 = \sum_{\substack{u_1+u_2=j, \\ \alpha_1+\alpha_2+\alpha_3=\alpha}} C'_{\alpha,u} (\gamma_{s_0}^{\delta}{}' \cdot \xi)^{u_1-|\alpha_1|} (\gamma_{s_0}^{\delta}{}')^{\alpha_1} (\gamma_{s_0}^{\delta})^{\alpha_2} \partial_\tau^{u_1+|\alpha_2|} \partial_s^{u_2} \partial_t^l \partial_\xi^{\alpha_3} \mathfrak{b}$$

with constants $C'_{\alpha,u}$ satisfying $|C'_{\alpha,u}| = 1$. We decompose $\mathcal{I}[\mathfrak{b}_2] = \mathcal{I}[\chi_E \mathfrak{b}_2] + \mathcal{I}[\chi_{E^c} \mathfrak{b}_2]$, where $E = \{(\tau, \xi) : |\tau + \gamma_{s_0}^{\delta}(s) \cdot \xi| \le 1\}$. Then, integrating by parts in t' for $\mathcal{I}[\chi_{E^c} \mathfrak{b}_2]$, we obtain

$$|\mathcal{I}[\mathfrak{b}_2]| \lesssim \iint \chi_E |\mathfrak{b}_2| + \frac{\chi_{E^c} |\partial_{t'}^2 \mathfrak{b}_2|}{|\tau + \gamma_{s_0}^{\delta}(s) \cdot \xi|^2} \, dt' d\tau.$$

Since $\mathfrak{a} \in \mathfrak{A}_k(s_0, \delta)$, $|\partial_s^{j'} \partial_t^{l'} \partial_{\tau,\xi}^{\alpha'} \mathfrak{b}| \lesssim_B |\xi|^{-|\alpha'|}$ for $(j', l', \alpha') \in \mathcal{I}_N$. It is also clear that $|\gamma_{s_0}^{\delta}{}'(s)| \lesssim 1$ if $\delta < \delta'$. Thus, $|\mathfrak{b}_2| = O(|\xi|^{-|\alpha|})$, and $|\partial_{t'}^2 \mathfrak{b}_2| = O(|\xi|^{-|\alpha|})$ for $l \le 2(N-1)$. Since $\partial_s^j \partial_t^l \partial_\xi^\alpha \tilde{a} = \mathcal{I}[\mathfrak{b}_2]$, we obtain the inequality (2.25).

Now, we decompose $\tilde{a}$. Let $\tilde{\chi}_1, \tilde{\chi}_2$ and $\tilde{\chi}_3 \in \mathrm{C}_c^\infty(\mathbb{R})$ such that $\tilde{\chi}_1 + \tilde{\chi}_2 + \tilde{\chi}_3 = 1$ on $\mathrm{supp}\,\tilde{\chi}$ and $\mathrm{supp}\,\tilde{\chi}_\ell \subset [2^{\ell-3}, 2^\ell]$ for $\ell = 1, 2, 3$. Also, let $\beta \in \mathrm{C}_c^\infty((2^{-1}, 2))$ such that $\sum \beta(2^{-k}\cdot) = 1$ on $\mathbb{R}_+$. We set

$$a_{\ell,j}(s, t, \xi) = \tilde{\chi}_\ell(t) \beta(2^{-j}|\xi|) \tilde{a}(s, t, \xi),$$

so $\sum_{\ell,j} a_{\ell,j} = \tilde{a}$. By (2.24), $a_{\ell,j} = 0$ if $|j - j_*| > C$ for some $C > 0$.

Denoting $(a)_\rho(s, t, \xi) = a(s, \rho t, \rho^{-1} \xi)$, via rescaling we observe $\mathcal{A}_{\rho t}[\gamma_{s_0}^{\delta}, a] g(x) = \mathcal{A}_t[\gamma_{s_0}^{\delta}, (a)_\rho] g(\rho \cdot)(x/\rho)$. Thus, changes of variables yield

$$\|\mathcal{A}_t[\gamma_{s_0}^{\delta}, a_{\ell,j}] \tilde{f}\|_{L^p(\mathbb{R}^{d+1})} = 2^{(\ell-2)/p} \|\mathcal{A}_t[\gamma_{s_0}^{\delta}, (a_{\ell,j})_{2^{\ell-2}}] \tilde{f}_\ell\|_{L^p(\mathbb{R}^{d+1})},$$

where $\tilde{f}_\ell = 2^{(\ell-2)d/p} \tilde{f}(2^{\ell-2} \cdot)$. Since $\mathcal{A}_t[\gamma_{s_0}^{\delta}, \tilde{a}] = \sum_{\ell,j} \mathcal{A}_t[\gamma_{s_0}^{\delta}, a_{\ell,j}]$, by (2.23) we get

$$\left\| \tilde{\chi} \mathcal{T}[\mathfrak{a}] f \right\|_{L^p(\mathbb{R}^{d+1})} \lesssim \delta \sum_{\ell,j} \left\| \mathcal{A}_t[\gamma_{s_0}^{\delta}, (a_{\ell,j})_{2^{\ell-2}}] \tilde{f}_\ell \right\|_{L^p(\mathbb{R}^{d+1})}.$$

To complete the proof, we only have to relabel $(a_{\ell,j})_{2^{\ell-2}}$, $\ell = 1, 2, 3$, $j_* - C \le j \le j_* + C$. Indeed, since $\tilde{a} \in \mathrm{C}^{d+N+2}$, $(a_{\ell,j})_{2^{\ell-2}} \in \mathrm{C}^{d+N+2}$, which is supported in $I \times [2^{-1}, 4] \times \mathbb{A}_{j+\ell-2}$. Obviously, (2.25) holds for $\tilde{a} = (a_{\ell,j})_{2^{\ell-2}}$ because $\ell = 1, 2, 3$. Changing variables $s \to \delta s + s_0$ and $\xi \to \delta^{-N} \widetilde{\mathcal{L}}_{s_0}^{\delta} \xi$ in (2.20), by the identity (2.21) we see that (2.20) on $\mathrm{supp}_{s,\xi} \mathfrak{a}$ is equivalent to $\sum_{j=1}^{N-1} |\langle (\gamma_{s_0}^{\delta})^{(j)}(s), \xi \rangle| \ge 2^k \delta^N / B'$ for $(s, \xi) \in \mathrm{supp}_{s,\xi} \mathfrak{a}_{\delta, s_0}(\cdot, \delta^{-N} \mathcal{L}_{s_0}^{\delta} \cdot)$. Note that $\mathrm{supp}_{s,\xi} \mathfrak{a}_{\delta, s_0}(\cdot, \delta^{-N} \mathcal{L}_{s_0}^{\delta} \cdot) \supset \mathrm{supp}_{s,\xi} \tilde{a}$. So, the same holds on $\mathrm{supp}_{s,\xi} \tilde{a}$ and hence on $\mathrm{supp}_{s,\xi} (a_{\ell,j})_{2^{\ell-2}}$ if B' replaced by $2B'$. Therefore, $C^{-1}(a_{\ell,j})_{2^{\ell-2}}$ is of type $(j + \ell - 2, N - 1, \tilde{B})$ relative to $\gamma_{s_0}^{\delta}$ for a large constant $C = C(B)$. □

2.3. Preliminary decomposition and reduction

For the proof of Proposition 2.3, we make some reductions by decomposing the symbol a. We fix a sufficiently small positive constant

$$\delta_* < \min\{2^{-10} B^{-3} \delta', \ (2^{7d} B^6)^{-N}\}, \tag{2.26}$$

which is to be specified in what follows. Here, δ' is the number given in Lemma 2.8.

Recall γ satisfies the condition (2.1), $\mathfrak{B}(N, B)$ and a is of type (k, N, B) relative to γ. We set

$$\eta_N(s, \xi) = \prod_{1 \le j \le N-1} \beta_0 \Big(B 2^{-k-1} \delta_*^{j-N} \langle \gamma^{(j)}(s), \xi \rangle \Big), \tag{2.27}$$

where $\beta_0 \in C_c^\infty((-1, 1))$ such that $\beta_0 = 1$ on $[-1/2, 1/2]$. It is easy to see $|\partial_s^j \partial_t^l \partial_\xi^\alpha (a\eta_N)| \le C|\xi|^{-|\alpha|}$ for $(j, l, \alpha) \in \mathcal{I}_N$, and the same holds for $a(1 - \eta_N)$.

Note $\sum_{j=1}^{N-1} |\gamma^{(j)}(s) \cdot \xi| \ge (2B)^{-1} \delta_*^N |\xi|$ on $\text{supp}_{s, \xi}(a(1 - \eta_N))$. So, we see $a(1 - \eta_N)$ is a symbol of type $(k, N - 1, B')$ for $B' = CB^2 \delta_*^{-C}$ with a large C. Applying the assumption (Theorem 2.2 with $L = N - 1$ and $B = B'$), we obtain

$$\|\mathcal{A}_t[\gamma, a(1 - \eta_N)]f\|_{L^p(\mathbb{R}^{d+1})} \le C 2^{(-\frac{2}{p} + \epsilon)k} \|f\|_{L^p(\mathbb{R}^d)}, \quad p \ge 4N - 6.$$

Thus, it suffices to consider $\mathcal{A}_t[\gamma, a\eta_N]$. Since $\mathfrak{R}(N, B)$ holds on $\text{supp}_{s, \xi} a$,

$$|\gamma^{(N)}(s) \cdot \xi| \ge (2B)^{-1} |\xi| \tag{2.28}$$

holds whenever $(s, t, \xi) \in \text{supp} \, a\eta_N$ for some t.

Basic assumption. Before we continue to prove the estimate for $\mathcal{A}_t[\gamma, a\eta_N]$, we make several assumptions which are clearly permissible by elementary decompositions.

Decomposing a, we may assume that $\text{supp}_\xi a$ is contained in a narrow conic neighborhood and $\text{supp}_s a \subset I(s_0, \delta_*)$ for some s_0. Let us set

$$\Gamma_k = \Big\{ \xi \in \mathbb{A}_k : \text{dist}\big(|\xi|^{-1}\xi, |\xi'|^{-1}\xi'\big) < \delta_* \text{ for some } \xi' \in \text{supp}_\xi(a\eta_N) \Big\}.$$

We may also assume $\gamma^{(N-1)}(s') \cdot \xi' = 0$ for some $(s', \xi') \in I(s_0, \delta_*) \times \Gamma_k$. Otherwise, $|\gamma^{(N-1)}(s) \cdot \xi| \gtrsim |\xi|$ on $\text{supp}_{s, \xi} a\eta_N$ and hence $a\eta_N = 0$ if we take B large enough. By (2.28) and the implicit function theorem, there exists σ such that

$$\gamma^{(N-1)}(\sigma(\xi)) \cdot \xi = 0 \tag{2.29}$$

on a narrow conic neighborhood of ξ' where $\sigma \in C^{2d+2}$, since $\gamma \in C^{3d+1}(I)$. So, decomposing a further and taking δ_* small enough, we may assume that $\sigma \in C^{2d+2}(\Gamma_k)$ and $\sigma(\xi) \in I(s_0, \delta_*)$ for $\xi \in \Gamma_k$. Moreover, since σ is homogeneous of degree zero, we have

$$|\partial_\xi^\alpha \sigma(\xi)| \le C|\xi|^{-|\alpha|}, \qquad \xi \in \Gamma_k \tag{2.30}$$

for a constant $C = C(B)$ if $|\alpha| \le 2d + 2$. Any symbol which appears in what follows is to be given by decomposing the symbol a with appropriate cutoff functions. So, *the s, ξ-supports of the symbols are assumed to be contained in* $I(s_0, \delta_*) \times \Gamma_k$.

We break a to have further localization on the Fourier side. Let

$$\mathfrak{a}_1(s, t, \tau, \xi) = a\eta_N \, \beta_0\big(2^{-2k} \delta_*^{-2N} |\tau + \langle \gamma(s), \xi \rangle|^2\big)$$

and $\mathfrak{a}_0 = a\eta_N - \mathfrak{a}_1$. Then, by Fourier inversion

$$\mathcal{A}_t[\gamma, a\eta_N]f = \mathcal{T}[\mathfrak{a}_1]f + \mathcal{T}[\mathfrak{a}_0]f.$$

It is easy to show $\|\mathcal{T}[\mathfrak{a}_0]f\|_p \lesssim_B 2^{-2k}\|f\|_p$ for $1 \le p \le \infty$. Indeed, we consider $\tilde{\mathfrak{a}}_0 = -(\tau + \gamma(s) \cdot \xi)^{-2} \partial_t^2 \mathfrak{a}_0$. By (2.13) and integration by parts in t', $m[\mathfrak{a}_0] = m[\tilde{\mathfrak{a}}_0]$ and hence $\mathcal{T}[\mathfrak{a}_0] = \mathcal{T}[\tilde{\mathfrak{a}}_0]$. Thanks

to (2.17), it is sufficient to show

$$|K[\tilde{\mathfrak{a}}_0](s,t,x)| \leq C\, 2^{k(d-1)} \int \left(1 + 2^k|t-t'| + 2^k|x-t'\gamma(s)|\right)^{-d-3} dt'$$

for a constant $C = C(B, \delta_*)$. Note $|\tau + \langle \gamma(s), \xi \rangle| \gtrsim 2^k$ on supp $\tilde{\mathfrak{a}}_0$, and recall (2.18). Rescaling and integration by parts in τ, ξ, as in the proof of Lemma 2.7, show the estimate.

The difficult part is to obtain the estimate for $\mathcal{T}[\mathfrak{a}_1]$. Since δ_* is a fixed constant, it is obvious that $C^{-1}\mathfrak{a}_1 \in \mathfrak{A}_k(s_0, \delta_*)$ for some $C = C(B, \delta_*)$. So, the desired estimate for $\mathcal{T}[\mathfrak{a}_1]$ follows once we have the next proposition.

Proposition 2.9. *Let* $\mathfrak{a} \in \mathfrak{A}_k(s_0, \delta_*)$ *with* $\text{supp}_\xi\, \mathfrak{a} \subset \Gamma_k$. *Suppose Theorem 2.2 holds for* $L = N - 1$. *Then, if* $p \geq 4N - 2$, *for* $\epsilon > 0$, *we have*

$$\left\| \mathcal{T}[\mathfrak{a}]f \right\|_{L^p(\mathbb{R}^{d+1})} \leq C_\epsilon 2^{-\frac{2}{p}k + \epsilon k} \|f\|_{L^p(\mathbb{R}^d)}.$$

Therefore, the proof of Proposition 2.3 is completed if we prove Proposition 2.9. For the purpose, we use Proposition 2.10 below, which allows us to decompose $\mathcal{T}[\mathfrak{a}]$ into the operators given by symbols with smaller s-supports while the consequent minor parts have acceptable bounds. A similar argument was used in [24] when $L = 2$.

Let δ_0 and δ_1 be positive numbers such that

$$2^{7d}B^6\delta_0^{(N+1)/N} \leq \delta_1 \leq \delta_0 \leq \delta_*, \qquad 2^{-k/N} \leq \delta_1. \tag{2.31}$$

Then, it is clear that

$$B^{6N}\delta_0^{j+1} \leq 2^{-7dN}\delta_1^j, \qquad j = 1, \ldots, N. \tag{2.32}$$

For $n \geq 0$, we denote $\mathfrak{J}_n^\mu = \{\nu \in \mathbb{Z} : |2^n\delta_1\nu - \delta_0\mu| \leq \delta_0\}$.

Proposition 2.10. *For* μ *such that* $\delta_0\mu \in I(s_0, \delta_*) \cap \delta_0\mathbb{Z}$, *let* $\mathfrak{a}^\mu \in \mathfrak{A}_k(\delta_0\mu, \delta_0)$ *with* $\text{supp}_{s,\xi}\, \mathfrak{a}^\mu \subset I(s_0, \delta_*) \times \Gamma_k$. *Suppose Theorem 2.2 holds for* $L = N - 1$. *Then, if* $p \geq 4N - 2$, *for* $\epsilon > 0$ *there exist a constant* $C_\epsilon = C_\epsilon(B) \geq 2$ *and symbols* $\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_1\nu, \delta_1)$ *with* $\text{supp}_{s,\xi}\, \mathfrak{a}_\nu \subset I(s_0, \delta_*) \times \Gamma_k$, $\nu \in \cup_\mu \mathfrak{J}_0^\mu$, *such that*

$$\left(\sum_\mu \|\mathcal{T}[\mathfrak{a}^\mu]f\|_p^p \right)^{\frac{1}{p}} \leq C_\epsilon \left(\delta_1/\delta_0\right)^{\frac{2N}{p}-1-\epsilon} \left(\sum_\nu \|\mathcal{T}[\mathfrak{a}_\nu]f\|_p^p \right)^{\frac{1}{p}} + C_\epsilon \delta_0^{-\frac{2N}{p}+1+\epsilon} 2^{-\frac{2}{p}k+2\epsilon k} \|f\|_p.$$

Assuming Proposition 2.10, we prove Proposition 2.9.

2.4. Proof of Proposition 2.9

Let $\mathfrak{a} \in \mathfrak{A}_k(s_0, \delta_*)$. We may assume $s_0 = \delta_*\mu$ for some $\mu \in \mathbb{Z}$. To apply Proposition 2.10 iteratively, we need to choose an appropriate decreasing sequence of positive numbers since the decomposition is subject to the condition (2.31).

Let $\delta_0 = \delta_*$, so $(2^{7d}B^6)^N\delta_0 < 1$. Let J be the largest integer such that

$$(2^{7d}B^6)^{N(\frac{N+1}{N})^{J-1}-N}\delta_0^{(\frac{N+1}{N})^{J-1}} > 2^{-\frac{k}{N}}.$$

So, $J \leq C_1 \log k$ for a constant $C_1 \geq 1$. We set

$$\delta_J = 2^{-\frac{k}{N}}, \qquad \delta_j = (2^{7d}B^6)^{N(\frac{N+1}{N})^j-N}\delta_0^{(\frac{N+1}{N})^j} \tag{2.33}$$

for $j = J - 1, \ldots, 1$. Thus, it follows that

$$2^{7d} B^6 \delta_j^{(N+1)/N} \le \delta_{j+1} < \delta_j, \qquad j = 0, \ldots, J - 1. \tag{2.34}$$

For a given $\epsilon > 0$, let $\tilde{\epsilon} = \epsilon/4$. Since $\mathfrak{a} \in \mathfrak{A}_k(\delta_0 \mu, \delta_0)$ and (2.31) holds for δ_0 and δ_1, applying Proposition 2.10 to $\mathcal{T}[\mathfrak{a}]$, we have

$$\|\mathcal{T}[\mathfrak{a}]f\|_p \le C_{\tilde{\epsilon}} (\delta_1/\delta_0)^{\frac{2N}{p} - 1 - \tilde{\epsilon}} \left(\sum_{\nu_1} \|\mathcal{T}[\mathfrak{a}_{\nu_1}]f\|_p^p \right)^{\frac{1}{p}} + C_{\tilde{\epsilon}} \delta_0^{-\frac{2N}{p} + 1 + \tilde{\epsilon}} 2^{-\frac{2}{p}k + 2\tilde{\epsilon}k} \|f\|_p,$$

where $\mathfrak{a}_{\nu_1} \in \mathfrak{A}_k(\delta_1 \nu_1, \delta_1)$, $\nu_1 \in \mathfrak{J}_0^{\mu}$. Thanks to (2.34), we may again apply Proposition 2.10 to $\mathcal{T}[\mathfrak{a}_{\nu_1}]$ while δ_0, δ_1 replaced by δ_1, δ_2, respectively. Repeating this procedure up to J-th step yields symbols $\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_J \nu, \delta_J)$, $\delta_J \nu \in \delta_J \mathbb{Z} \cap I(\delta_0 \mu, \delta_0)$, such that

$$\|\mathcal{T}[\mathfrak{a}]f\|_p \le C_{\tilde{\epsilon}}^J \delta_J^{\frac{2N}{p} - 1 - \tilde{\epsilon}} \left(\sum_\nu \|\mathcal{T}[\mathfrak{a}_\nu]f\|_p^p \right)^{\frac{1}{p}} + \sum_{0 \le j \le J-1} C_{\tilde{\epsilon}}^{j+1} \delta_0^{-\frac{2N}{p} + 1 + \tilde{\epsilon}} 2^{-\frac{2}{p}k + 2\tilde{\epsilon}k} \|f\|_p$$

for $p \ge 4N - 2$. Now, assuming

$$\left(\sum_\nu \|\mathcal{T}[\mathfrak{a}_\nu]f\|_p^p \right)^{1/p} \lesssim_B 2^{-k/N} \|f\|_p, \qquad 2 \le p \le \infty \tag{2.35}$$

for the moment, we can finish the proof of Proposition 2.9. Since $C_{\tilde{\epsilon}} \ge 2$, combining the above inequalities, we get

$$\|\mathcal{T}[\mathfrak{a}]f\|_p \lesssim_B C_{\tilde{\epsilon}}^{J+1} \left(2^{-\frac{2}{p}k + \frac{\tilde{\epsilon}}{N}k} + 2^{-\frac{2}{p}k + 2\tilde{\epsilon}k} \right) \|f\|_p.$$

Since $J \le C_1 \log k$, $C_{\tilde{\epsilon}}^{J+1} \le C' 2^{\epsilon k/2}$ for some C' if k is sufficiently large. Therefore, the right-hand side is bounded by $C 2^{-2k/p + \epsilon k} \|f\|_p$.

It remains to show the estimate (2.35) for $2 \le p \le \infty$. By interpolation, it is enough to obtain (2.35) for $p = \infty$ and $p = 2$. The case $p = \infty$ follows by (2.15) since $\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_J \nu, \delta_J)$. So, we need only to prove the estimate (2.35) for $p = 2$. To do this, we first observe the following, which shows $\mathrm{supp}_\xi \, \mathfrak{a}_\nu$ are finitely overlapping.

Lemma 2.11. *For $b \ge 1$, $s \in I(s_0, \delta_*)$ and $0 < \delta \le \delta_*$, let us set*

$$\Lambda_k'(s, \delta, b) = \bigcap_{1 \le j \le N-1} \left\{ \xi \in \Gamma_k : |\langle \gamma^{(j)}(s), \xi \rangle| \le b 2^k \delta^{N-j} \right\}. \tag{2.36}$$

If $\Lambda_k'(s_1, \delta, b) \cap \Lambda_k'(s_2, \delta, b) \ne \emptyset$ for some $s_1, s_2 \in I(s_0, \delta_*)$, then there is a constant $C = C(B)$ such that $|s_1 - s_2| \le Cb\delta$.

Proof. Let $\xi \in \Lambda_k'(s_1, \delta, b) \cap \Lambda_k'(s_2, \delta, b)$. Since $|\gamma^{(N-1)}(s_j) \cdot \xi| \le b 2^k \delta$, $j = 1, 2$, by (2.29) and (2.28) we see $|s_j - \sigma(\xi)| \le 2^2 b B \delta$, $j = 1, 2$, using the mean value theorem. This implies $|s_1 - s_2| \le 2^3 b B \delta$. $\square$

We recall (2.13). Since (2.28) holds on $\mathrm{supp}_{s,\xi} \, \mathfrak{a}_\nu$, by van der Corput's lemma (e.g., see [32, p. 334]) we have

$$|m[\mathfrak{a}_\nu](\tau, \xi)| \lesssim 2^{-k/N} \left(\|\mathfrak{a}_\nu(\cdot, t, \tau, \xi)\|_\infty + \|\partial_s \mathfrak{a}_\nu(\cdot, t, \tau, \xi)\|_1 \right) \lesssim_B 2^{-k/N}.$$

The second inequality is clear since $\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_J \nu, \delta_J)$. From (2.14), note $\mathcal{F}(\mathcal{T}[\mathfrak{a}_\nu]f) = m[\mathfrak{a}_\nu]\widehat{f}$. Since $\operatorname{supp}\mathfrak{a}_\nu \subset \Lambda_k(\delta_J \nu, \delta_J, B)$, $\operatorname{supp}_\xi \mathfrak{a}_\nu \subset S_\nu := \Lambda'_k(\delta_J \nu, \delta_J, 2^5 B)$. So, $\operatorname{supp}_\xi \mathcal{F}(\mathcal{T}[\mathfrak{a}_\nu]f) \subset S_\nu$ (see (2.13)). By Lemma 2.11, it follows that the sets S_ν overlap at most $C = C(B)$ times. Therefore, Plancherel's theorem and the estimate above give

$$\Big\| \sum_\nu \mathcal{T}[\mathfrak{a}_\nu]f \Big\|_2^2 \lesssim_B 2^{-2k/N} \sum_\nu \int_{S_\nu} \int_{\{\tau : |\tau + \gamma(\delta_J \nu)\cdot\xi| \le 2^5 B\}} d\tau \, |\widehat{f}(\xi)|^2 \, d\xi,$$

by which we get (2.35) for $p = 2$.

2.5. Decoupling inequalities

We denote $\mathbf{r}_\circ^N(s) = (s, s^2/2!, \ldots, s^N/N!)$ and consider a collection of curves from I to $\mathbb{R}^N$ which are small perturbations of $\mathbf{r}_\circ^N$:

$$\mathfrak{C}(\epsilon_\circ; N) = \{\mathbf{r} \in C^{2N+1}(I) : \|\mathbf{r} - \mathbf{r}_\circ^N\|_{C^{2N+1}(I)} < \epsilon_\circ\}.$$

For $\mathbf{r} \in \mathfrak{C}(\epsilon_\circ; N)$ and $s \in I$, we define an anisotropic neighborhood by

$$\mathcal{N}_\mathbf{r}(s, \delta) = \Big\{\mathbf{r}(s) + \sum_{1 \le j \le N} u_j \mathbf{r}^{(j)}(s) : |u_j| \le \delta^j, \quad j = 1, \ldots, N\Big\}.$$

Let $s_1, \ldots, s_l \in I$ be δ-separated points, that is, $|s_n - s_j| \ge \delta$ if $n \ne j$ such that $\bigcup_{j=1}^l (s_j - \delta, s_j + \delta) \supset I$. Then, we set

$$\theta_j = \mathcal{N}_\mathbf{r}(s_j, \delta), \qquad 1 \le j \le l.$$

The following is due to Bourgain, Demeter and Guth [5] (also see [10]).

Theorem 2.12. *Let* $0 < \delta \ll 1$. *Suppose* $\mathbf{r} \in \mathfrak{C}(\epsilon_\circ; N)$ *for a small enough* $\epsilon_\circ > 0$. *Then, if* $2 \le p \le N(N+1)$, *for* $\epsilon > 0$ *we have*

$$\Big\| \sum_{1 \le j \le l} f_j \Big\|_{L^p(\mathbb{R}^N)} \le C_\epsilon \delta^{-\epsilon} \Big(\sum_{1 \le j \le l} \|f_j\|_{L^p(\mathbb{R}^N)}^2 \Big)^{1/2} \tag{2.37}$$

whenever $\operatorname{supp}\widehat{f_j} \subset \theta_j$ *for* $1 \le j \le l$.

The constant C_ϵ can be taken to be independent of particular choices of the δ-separated points $s_1, \ldots, s_l$. One can obtain a conical extension of the inequality (2.37) by modifying the argument in [4] which deduces the decoupling inequality for the cone from that for the paraboloid (see [2, Proposition 7.7]). Let us consider the conical sets

$$\bar{\theta}_j = \{(\eta, \rho) \in \mathbb{R}^N \times [1, 2] : \eta/\rho \in \theta_j\}, \quad 1 \le j \le l.$$

Corollary 2.13. *Let* $0 < \delta \le 1$, *and let* $\mathbf{r} \in \mathfrak{C}(\epsilon_\circ; N)$ *for a small enough* $\epsilon_\circ > 0$. *Then, if* $2 \le p \le N(N+1)$, *for* $\epsilon > 0$ *we have*

$$\Big\| \sum_{1 \le j \le l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \le C_\epsilon \delta^{-\epsilon} \Big(\sum_{1 \le j \le l} \|F_j\|_{L^p(\mathbb{R}^{N+1})}^2 \Big)^{1/2} \tag{2.38}$$

whenever $\operatorname{supp}\widehat{F_j} \subset \bar{\theta}_j$ *for* $1 \le j \le l$.

The inequality (2.38) does not fit with the symbols to appear when we decompose $\mathfrak{a}$ (see Section 3.1 and Section 4.2). As to be seen, those symbols are associated with the slabs of the following form.

Definition 2.14. Let $N \geq 2$ and $\tilde{\mathbf{r}} \in \mathfrak{C}(\epsilon_\circ; N + 1)$. For $s \in I$, we denote by $\mathbf{S}(s, \delta, \rho; \tilde{\mathbf{r}})$ the set of $(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^N$ which satisfies

$$\rho^{-1} \leq |\langle \tilde{\mathbf{r}}^{(N+1)}(s), (\tau, \eta) \rangle| \leq 2\rho,$$

$$|\langle \tilde{\mathbf{r}}^{(j)}(s), (\tau, \eta) \rangle| \leq \delta^{N+1-j}, \qquad j = N, \ldots, 1.$$

The same form of decoupling inequality remains valid for the slabs $\mathbf{S}(s_1, \delta, 1; \tilde{\mathbf{r}}), \ldots, \mathbf{S}(s_l, \delta, 1; \tilde{\mathbf{r}})$. Beltran et al. [2, Theroem 4.4] showed, using the Frenet–Serret formulas, that those slabs can be generated by conical extensions of the anisotropic neighborhoods given by a nondegenerate curve in $\mathbb{R}^N$. Therefore, the following is a consequence of Corollary 2.13 and a simple manipulation using decomposition and rescaling.

Corollary 2.15. *Let* $0 < \delta \leq 1$, $\rho \geq 1$ *and* $\tilde{\mathbf{r}} \in \mathfrak{C}(\epsilon_\circ; N + 1)$ *for a small enough* $\epsilon_\circ > 0$. *Denote* $\mathbf{S}_j = \mathbf{S}(s_j, \delta, \rho; \tilde{\mathbf{r}})$ *for* $1 \leq j \leq l$. *Then, if* $2 \leq p \leq N(N + 1)$, *for* $\epsilon > 0$ *there is a constant* $C_\epsilon = C_\epsilon(\rho)$ *such that*

$$\Big\| \sum_{1 \leq j \leq l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \leq C_\epsilon \delta^{-\epsilon} \Big(\sum_{1 \leq j \leq l} \|F_j\|_{L^p(\mathbb{R}^{N+1})}^2 \Big)^{1/2} \tag{2.39}$$

whenever $\operatorname{supp} \widehat{F_j} \subset \mathbf{S}_j$ *for* $1 \leq j \leq l$.

For our purpose of proving Proposition 2.10, we use a modified form. If $p_* \in [2, N(N + 1)]$, then we have

$$\Big\| \sum_{1 \leq j \leq l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \leq C_\epsilon \delta^{-1 + \frac{2+p_*}{2p} - \epsilon} \Big(\sum_{1 \leq j \leq l} \|F_j\|_{L^p(\mathbb{R}^{N+1})}^p \Big)^{1/p}$$

for $p \geq p_*$. The case $p = p_*$ follows by the inequality (2.39) and Hölder's inequality. Interpolation with the trivial $\ell^\infty L^\infty$–L^∞ estimate gives the estimate for $p > p_*$. One may choose different p_* for the particular purposes. In fact, for the local smoothing estimate we take $p_* = 4N - 2$ to get

$$\Big\| \sum_{1 \leq j \leq l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \leq C_\epsilon \delta^{-1 + \frac{2N}{p} - \epsilon} \Big(\sum_{1 \leq j \leq l} \|F_j\|_{L^p(\mathbb{R}^{N+1})}^p \Big)^{1/p} \tag{2.40}$$

for $p \geq 4N - 2$ (see Section 3.2). For the L^p Sobolev regularity estimate, we observe that

$$\Big\| \sum_{1 \leq j \leq l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \leq C_{\epsilon_0} \delta^{-1 + \frac{N+1}{p} + \epsilon_0} \Big(\sum_{1 \leq j \leq l} \|F_j\|_{L^p(\mathbb{R}^{N+1})}^p \Big)^{1/p} \tag{2.41}$$

holds for some $\epsilon_0 = \epsilon_0(p) > 0$ if $2N < p < \infty$. Indeed, we need only to take $p_* > 2N$ close enough to $2N$. The presence of ϵ_0 in (2.41) is crucial for proving the optimal Sobolev regularity estimate (see Proposition 4.5).

The inequalities (2.40) and (2.41) obviously extend to cylindrical forms via the Minkowski inequality. For example, set $\tilde{\mathbf{S}}_j = \left\{ (\xi, \eta) \in \mathbb{R}^{N+1} \times \mathbb{R}^M : \xi \in \mathbf{S}_j \right\}$ for $1 \le j \le l$. Using (2.40), we have

$$\Big\| \sum_{1 \le j \le l} G_j \Big\|_{L^p(\mathbb{R}^{N+M+1})} \le C_\epsilon \delta^{-1 + \frac{2N}{p} - \epsilon} \left(\sum_{1 \le j \le l} \|G_j\|_{L^p(\mathbb{R}^{N+M+1})}^2 \right)^{1/2} \tag{2.42}$$

whenever $\widehat{G}_j$ is supported in $\tilde{\mathbf{S}}_j$. Clearly, we also have a similar extension of the inequality (2.41).

3. Decomposition of the symbols

In this section, we prove Proposition 2.10 by applying the decoupling inequality. Meanwhile, the induction assumption (Theorem 2.2 with $L = N - 1$) plays an important role. We decompose a given symbol $\mathfrak{a}^\mu \in \mathfrak{A}_k(\delta_0 \mu, \delta_0)$ into the symbols with their s-supports contained in intervals of length about δ_1 while the consequent minor contribution is controlled within an acceptable bound. To achieve it up to δ_1 satisfying the condition (2.31), we approximate $\langle G(s), (\tau, \xi) \rangle$ in a local coordinate system near the set $\{(s, \xi) : \langle \gamma^{(N-1)}(s), \xi \rangle = 0\}$.

3.1. Decomposition of the symbol $\mathfrak{a}^\mu$

We begin by introducing some notations.

Fixing $\mu \in \mathbb{Z}$ such that $\delta_0 \mu \in I(s_0, \delta_*)$, we consider the linear maps

$$y_\mu^j(\tau, \xi) = \langle G^{(j)}(\delta_0 \mu), (\tau, \xi) \rangle, \qquad j = 0, 1, \dots, N.$$

In particular, $y_\mu^j(\tau, \xi) = \langle \gamma^{(j)}(\delta_0 \mu), \xi \rangle$ if $1 \le j \le N$. By (2.28), it follows that

$$|y_\mu^N(\tau, \xi)| \ge (2B)^{-1}|\xi|. \tag{3.1}$$

We denote

$$\omega_\mu(\xi) = \frac{y_\mu^{N-1}(\tau, \xi)}{y_\mu^N(\tau, \xi)},$$

which is close to $\delta_0 \mu - \sigma(\xi)$ (see (3.5) below). Then, we define $\mathfrak{g}_\mu^N, \mathfrak{g}_\mu^{N-1}, \dots, \mathfrak{g}_\mu^0$ recursively, by setting $\mathfrak{g}_\mu^N = y_\mu^N$, and

$$\mathfrak{g}_\mu^j(\tau, \xi) = y_\mu^j(\tau, \xi) - \sum_{\ell = j+1}^N \frac{\mathfrak{g}_\mu^\ell(\tau, \xi)}{(\ell - j)!} (\omega_\mu(\xi))^{\ell - j}, \quad j = N-1, \dots, 0. \tag{3.2}$$

Note that $\mathfrak{g}_\mu^{N-1} = 0$ and (3.2) can be rewritten as follows:

$$y_\mu^m(\tau, \xi) = \sum_{\ell = m}^N \frac{\mathfrak{g}_\mu^\ell(\tau, \xi)}{(\ell - m)!} (\omega_\mu(\xi))^{\ell - m}, \quad m = 0, \dots, N. \tag{3.3}$$

The identity continues to hold for $m = N$ since $\mathfrak{g}_\mu^N = y_\mu^N$. Apparently, $\mathfrak{g}_\mu^1, \dots, \mathfrak{g}_\mu^N$ are independent of τ since so are $y_\mu^1, \dots, y_\mu^N$.

For $j = 1, \dots, N$, set

$$\mathcal{E}_j(\xi) := (y_\mu^N(\tau, \xi))^{-1} \int_{\sigma(\xi)}^{\delta_0 \mu} \frac{\langle \gamma^{(N+1)}(r), \xi \rangle}{j!} (\sigma(\xi) - r)^j \, dr. \tag{3.4}$$

By (3.4) with $j = 1$ and integration by parts, we have

$$\mathcal{E}_1(\xi) = \sigma(\xi) - \delta_0\mu + \omega_\mu(\xi). \tag{3.5}$$

Lemma 3.1. *Let* $0 \le j \le N - 1$*. Then, we have*

$$\langle G^{(j)}(\sigma(\xi)), (\tau, \xi) \rangle = \sum_{\ell=j}^{N} \frac{\mathfrak{g}_\mu^\ell \, (\mathcal{E}_1)^{\ell-j}}{(\ell - j)!} - y_\mu^N \mathcal{E}_{N-j}. \tag{3.6}$$

Proof. When $j = N - 1$, the equation (3.6) is clear. To show (3.6) for $j = 0, 1, \dots, N - 2$, by Taylor's theorem with integral remainder we have

$$\langle G^{(j)}(\sigma(\xi)), (\tau, \xi) \rangle = \sum_{m=j}^{N} y_\mu^m(\tau, \xi) \frac{(\sigma(\xi) - \delta_0\mu)^{m-j}}{(m - j)!} - y_\mu^N(\tau, \xi)\mathcal{E}_{N-j}(\xi).$$

Using (3.3) and then changing the order of the sums, we see

$$\langle G^{(j)}(\sigma(\xi)), (\tau, \xi) \rangle = \sum_{\ell=j}^{N} \mathfrak{g}_\mu^\ell \left(\sum_{m=j}^{\ell} \frac{(\sigma(\xi) - \delta_0\mu)^{m-j}}{(\ell - m)!(m - j)!} (\omega_\mu)^{\ell-m} \right) - y_\mu^N \, \mathcal{E}_{N-j}.$$

The sum over m equals $(\sigma(\xi) - \delta_0\mu + \omega_\mu)^{\ell-j}/(\ell - j)!$. So, (3.6) follows by (3.5). □

We now decompose the symbol $\mathfrak{a}^\mu \in \mathfrak{A}_k(\delta_0\mu, \delta_0)$ by making use of $\mathfrak{g}_\mu^j$, $j = 0, \dots, N - 2$. We define

$$\mathfrak{G}_N^\mu(s, \tau, \xi) = \sum_{j=0}^{N-2} \left(2^{-k} \mathfrak{g}_\mu^j(\tau, \xi) \right)^{\frac{2N!}{N-j}} + (s - \sigma(\xi))^{2N!}.$$

Let $\beta_N = \beta_0 - \beta_0(2^{2N!}\cdot)$, so we have $\sum_{\ell \in \mathbb{Z}} \beta_N(2^{2N!\ell}\cdot) = 1$ on $\mathbb{R}_+$. We also take $\zeta \in C_c^\infty((-1, 1))$ such that $\sum_{\nu \in \mathbb{Z}} \zeta(\cdot - \nu) = 1$. For $n \ge 0$ and $\nu \in \mathfrak{J}_n^\mu$, we set

$$\mathfrak{a}_\nu^{\mu,n} = \mathfrak{a}^\mu \times \begin{cases} \beta_0\left(\delta_1^{-2N!} \mathfrak{G}_N^\mu\right) \zeta(\delta_1^{-1}s - \nu), & n = 0, \\ \beta_N\left((2^n\delta_1)^{-2N!} \mathfrak{G}_N^\mu\right) \zeta(2^{-n}\delta_1^{-1}s - \nu), & n \ge 1. \end{cases}$$

Then, it follows that

$$\mathfrak{a}^\mu = \sum_{n \ge 0} \sum_{\nu \in \mathfrak{J}_n^\mu} \mathfrak{a}_\nu^{\mu,n}. \tag{3.7}$$

Lemma 3.2. *There is a constant* $C = C(B)$ *such that* $C^{-1}\mathfrak{a}_\nu^{\mu,n} \in \mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$ *for* $n \ge 0$*,* μ *and* ν*.*

The proof of Lemma 3.2 is elementary though it is somewhat involved. We postpone the proof until Section 3.3.

We collect some elementary facts regarding $\mathfrak{a}_\nu^{\mu,n}$. First, we may assume

$$2^n\delta_1 \le 2^{10}B^3\delta_0 \tag{3.8}$$

since, otherwise, $\mathfrak{a}_\nu^{\mu,n} = 0$. To show this, we note $|\langle \gamma^{(N-1)}(\delta_0\mu), \xi \rangle| \le B2^{k+5}\delta_0$ if $\xi \in \text{supp}_\xi \, \mathfrak{a}^\mu$. Thus, (2.28), (2.29), and the mean value theorem show that

$$|\sigma(\xi) - \delta_0\mu| \le B^2 2^7 \delta_0 \tag{3.9}$$

for $\xi \in \mathrm{supp}_\xi \, \mathfrak{a}^\mu$. If $(\tau, \xi) \in \mathrm{supp}_{\tau, \xi} \, \mathfrak{a}^\mu \subset \Lambda_k(\delta_0 \mu, \delta_0, B)$, $|y_\mu^j(\tau, \xi)| \le B 2^{k+5} \delta_0^{N-j}$ for $0 \le j \le N - 1$. Note $|\omega_\mu| \le B^2 2^7 \delta_0$ and $|\mathfrak{g}_\mu^N| \le B 2^{k+1}$. A routine computation using (3.2) gives $|\mathfrak{g}_\mu^j| \le B 2^{k-1} (B^2 2^8 \delta_0)^{N-j}$ for $j = N - 2, \dots, 0$. Since $|s - \sigma(\xi)| \le (B^2 2^7 + 1) \delta_0$, we have $\mathfrak{G}_N^\mu \le 2(B^3 2^8)^{2N!} \delta_0^{2N!}$ on $\mathrm{supp} \, \mathfrak{a}_\nu^{\mu, n}$, and (3.8) follows.

Since $\mathfrak{G}_N^\mu \le (2^n \delta_1)^{2N!}$ on $\mathrm{supp} \, \mathfrak{a}_\nu^{\mu, n}$, the following hold:

$$|s - \sigma(\xi)| \le 2^n \delta_1, \tag{3.10}$$

$$2^{-k} |\mathfrak{g}_\mu^j(\tau, \xi)| \le (2^n \delta_1)^{N-j}, \qquad 0 \le j \le N - 1. \tag{3.11}$$

Obviously, (3.11) holds true for $j = N - 1$ since $\mathfrak{g}_\mu^{N-1} = 0$. We also have

$$|\mathcal{E}_j(\xi)| \le B^2 (B^2 2^7 \delta_0)^{j+1}, \tag{3.12}$$

$$|\sigma(\xi) - 2^n \delta_1 \nu| \le 2^{n+1} \delta_1 \tag{3.13}$$

on $\mathrm{supp}_\xi \, \mathfrak{a}_\nu^{\mu, n}$. By using (3.4), (3.9) and (3.1), it is easy to show (3.12). Since $|s - 2^n \delta_1 \nu| \le 2^n \delta_1$ on $\mathrm{supp}_s \, \mathfrak{a}_\nu^{\mu, n}$, (3.13) follows by (3.10).

3.2. Proof of Proposition 2.10

By (3.7) and the Minkowski inequality, we have

$$\left(\sum_\mu \| \mathcal{T}[\mathfrak{a}^\mu] f \|_p^p \right)^{1/p} \le \sum_{n \ge 0} \left(\sum_\mu \| \sum_{\nu \in \mathfrak{J}_n^\mu} \mathcal{T}[\mathfrak{a}_\nu^{\mu, n}] f \|_p^p \right)^{1/p}. \tag{3.14}$$

We use the inequality (2.40) for $\sum_{\nu \in \mathfrak{J}_n^\mu} \mathcal{T}[\mathfrak{a}_\nu^{\mu, n}] f$ after a suitable linear change of variables. The symbols $\mathfrak{a}_\nu^{\mu, 0}$ are to constitute the set $\{\mathfrak{a}_\nu\}$ while the operators associated to $\mathfrak{a}_\nu^{\mu, n}$, $n \ge 1$ are to be handled similarly as in Section 2.

Applying the inequality (2.40). To prove Proposition 2.10, we first show

$$\| \sum_{\nu \in \mathfrak{J}_n^\mu} \mathcal{T}[\mathfrak{a}_\nu^{\mu, n}] f \|_p \le C_\epsilon \left(2^n \delta_1 / \delta_0 \right)^{\frac{2N}{p} - 1 - \epsilon} \left(\sum_{\nu \in \mathfrak{J}_n^\mu} \| \mathcal{T}[\mathfrak{a}_\nu^{\mu, n}] f \|_p^p \right)^{1/p} \tag{3.15}$$

for $p \ge 4N - 2$. To use (2.40), we consider $\mathrm{supp}_{\tau, \xi} \, \mathfrak{a}_\nu^{\mu, n}$, which contains the Fourier support of $\mathcal{T}[\mathfrak{a}_\nu^{\mu, n}] f$ as is clear from (2.13) and (2.14).

We set

$$\mathbf{y}_\mu(\tau, \xi) = \left(y_\mu^0(\tau, \xi), \dots, y_\mu^N(\tau, \xi) \right).$$

Lemma 3.3. *Let* $\mathbf{r} = \mathbf{r}_\circ^{N+1}$ *and* $\mathcal{D}_\delta$ *denote the matrix* $(\delta^{-N} e_1, \delta^{1-N} e_2, \dots, \delta^0 e_{N+1})$*, where* e_j *denotes the j-th standard unit vector in* $\mathbb{R}^{N+1}$. *On* $\mathrm{supp}_{\tau, \xi} \, \mathfrak{a}_\nu^{\mu, n}$*, we have*

$$\left| \left\langle \mathcal{D}_{\delta_0} \mathbf{y}_\mu(\tau, \xi), \mathbf{r}^{(j)} \left(\frac{2^n \delta_1}{\delta_0} \nu - \mu \right) \right\rangle \right| \lesssim 2^k \left(\frac{2^n \delta_1}{\delta_0} \right)^{N+1-j}, \qquad 1 \le j \le N, \tag{3.16}$$

$$(2B)^{-1} 2^{k-1} \le \left| \left\langle \mathbf{y}_\mu(\tau, \xi), \mathbf{r}^{(N+1)} \right\rangle \right| \le B 2^{k+1}. \tag{3.17}$$

Proof. We write $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_{N+1})$. Note $\mathbf{r}_m^{(j)}(s) = s^{m-j}/(m-j)!$ for $m \geq j$. By the equation (3.3), we have

$$y_\mu^{m-1}\mathbf{r}_m^{(j)}(2^n\delta_1\nu - \delta_0\mu) = \sum_{\ell=m-1}^{N} \mathfrak{g}_\mu^\ell \frac{(2^n\delta_1\nu - \delta_0\mu)^{m-j}}{(\ell+1-m)!(m-j)!} \omega_\mu^{\ell+1-m}$$

for $m \geq j$. Since $\mathbf{r}_m^{(j)}(s) = 0$ for $j > m$, taking sum over m gives

$$\langle \mathbf{y}_\mu, \mathbf{r}^{(j)}(2^n\delta_1\nu - \delta_0\mu) \rangle = \sum_{\ell=j-1}^{N} \mathfrak{g}_\mu^\ell \frac{(2^n\delta_1\nu - \delta_0\mu + \omega_\mu)^{\ell+1-j}}{(\ell+1-j)!}.$$

From the equation (3.5), we note $2^n\delta_1\nu - \delta_0\mu + \omega_\mu = 2^n\delta_1\nu - \sigma(\xi) + \mathcal{E}_1$. Thus, (3.13), (3.12) with $j = 1$ and (2.32) with $j = 1$ show $|2^n\delta_1\nu - \delta_0\mu + \omega_\mu| \lesssim 2^n\delta_1$. By (3.11), we obtain

$$\left| \langle \mathbf{y}_\mu(\tau, \xi), \mathbf{r}^{(j)}(2^n\delta_1\nu - \delta_0\mu) \rangle \right| \lesssim 2^k (2^n\delta_1)^{N+1-j}, \qquad 1 \leq j \leq N.$$

By homogeneity, it follows that $\langle \eta, \mathbf{r}^{(j)}(\delta_0 s) \rangle = \delta_0^{N+1-j} \langle \mathcal{D}_{\delta_0}\eta, \mathbf{r}^{(j)}(s) \rangle$ for $\eta \in \mathbb{R}^{N+1}$. Therefore, we get (3.16). For the inequality (3.17), note that $\mathbf{r}^{(N+1)} = (0, \ldots, 0, 1)$. Thus, $\langle \mathbf{y}_\mu, \mathbf{r}^{(N+1)} \rangle = y_\mu^N$ and (3.17) follows by (3.1). □

Let $V = \text{span}\{\gamma'(\delta_0\mu), \ldots, \gamma^{(N)}(\delta_0\mu)\}$ and $\{v_{N+1}, \ldots, v_d\}$ be an orthonormal basis of $V^\perp$. Since γ satisfies $\mathfrak{B}(N, B)$, for each $\xi \in \mathbb{R}^d$ we can write

$$\xi = \overline{\xi} + \sum_{N+1 \leq j \leq d} y_j(\xi) v_j, \tag{3.18}$$

where $\overline{\xi} \in V$ and $y_j(\xi) \in \mathbb{R}$, $N+1 \leq j \leq d$. We define a linear map by

$$Y_\mu^{\delta_0}(\tau, \xi) = \left(2^{-k}\mathcal{D}_{\delta_0}\mathbf{y}_\mu(\tau, \xi), y_{N+1}(\xi), \ldots, y_d(\xi)\right).$$

From (3.16) and (3.17), we see

$$Y_\mu^{\delta_0}(\text{supp}_{\tau,\xi} \mathfrak{a}_\nu^{\mu,n}) \subset \mathbf{S}\left(\frac{2^n\delta_1}{\delta_0}\nu - \mu, C\frac{2^n\delta_1}{\delta_0}, 2^2 B; \mathbf{r}_\circ^{N+1}\right) \times \mathbb{R}^{d-N} \tag{3.19}$$

for some $C > 1$. We now have the inequality (2.40) for $\delta = C2^n\delta_1/\delta_0$ and the slabs $\mathbf{S}(2^n\delta_1\nu/\delta_0 - \mu, C2^n\delta_1/\delta_0, 2^2 B; \mathbf{r}_\circ^{N+1})$, $\nu \in \mathfrak{J}_n^\mu$. Therefore, by cylindrical extension in $y_{N+1}, \ldots, y_d$ (see (2.42)) and the change of variables $(\tau, \xi) \to Y_\mu^{\delta_0}(\tau, \xi)$, we obtain (3.15) since the decoupling inequality is not affected by an affine change of variables in the Fourier side.

Combining the inequalites (3.14) and (3.15), we obtain

$$\left(\sum_\mu \|\mathcal{T}[\mathfrak{a}^\mu]f\|_p^p\right)^{1/p} \leq \sum_{n \geq 0} \mathbf{E}_n$$

for $p \geq 4N - 2$, where

$$\mathbf{E}_n = C_\epsilon \left(2^n\delta_1/\delta_0\right)^{\frac{2N}{p}-1-\epsilon} \left(\sum_\mu \sum_{\nu \in \mathfrak{J}_n^\mu} \|\mathcal{T}[\mathfrak{a}_\nu^{\mu,n}]f\|_p^p\right)^{1/p}.$$

Since the intervals $I(\delta_0\mu, \delta_0)$ overlap, there are at most three nonzero $\mathfrak{a}_\nu^{\mu,0}$ for each ν. We take $\mathfrak{a}_\nu = \mathfrak{a}_\nu^{\mu,0}$ which maximizes $\|\mathcal{T}[\mathfrak{a}_\nu^{\mu,0}]f\|_p$. Then, it is clear that $\mathbf{E}_0 \leq$

$3^{1/p} C_\epsilon (\delta_1/\delta_0)^{\frac{2N}{p}-1-\epsilon} \left(\sum_\nu \|\mathcal{T}[\mathfrak{a}_\nu]f\|_p^p \right)^{1/p}$. By Lemma 3.2, $C^{-1}\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_1\nu, \delta_1)$ for a constant C. Thus, the proof of Proposition 2.10 is now reduced to showing

$$\sum_{n \geq 1} \mathbf{E}_n \lesssim_B \delta_0^{-\frac{2N}{p}+1+\epsilon} 2^{-\frac{2}{p}k+2\epsilon k} \|f\|_p, \quad p \geq 4N - 2. \tag{3.20}$$

Estimates for $\mathbf{E}_n$ when $n \geq 1$. To prove the estimate (3.20), we decompose $\mathfrak{a}_\nu^{\mu,n}$ so that the inequalites (3.25) or (3.26) (see Lemma 3.5 below) holds on the s, ξ-supports of the resulting symbols. If (3.25) holds, we use the assumption after rescaling, whereas we handle the other case using estimates for the kernels of the operators.

Let

$$\bar{\mathfrak{G}}_N^\mu(s, \xi) = \sum_{1 \leq j \leq N-2} \left(2^{-k}\mathfrak{g}_\mu^j \right)^{\frac{2N!}{N-j}} + \left(s - \sigma(\xi) \right)^{2N!}. \tag{3.21}$$

Note that the right-hand side is independent of τ since so are $\mathfrak{g}_\mu^j$, $1 \leq j \leq N-2$.

Let $C_0 = 2^{2d}B$. We set

$$\mathfrak{a}_{\nu,1}^{\mu,n} = \mathfrak{a}_\nu^{\mu,n} \beta_0 \left((2^{-k}\mathfrak{g}_\mu^0)^{2(N-1)!} / (C_0^{2N!}\bar{\mathfrak{G}}_N^\mu) \right), \quad n \geq 1, \tag{3.22}$$

and $\mathfrak{a}_{\nu,2}^{\mu,n} = \mathfrak{a}_\nu^{\mu,n} - \mathfrak{a}_{\nu,1}^{\mu,n}$, so we have $\mathfrak{a}_\nu^{\mu,n} = \mathfrak{a}_{\nu,1}^{\mu,n} + \mathfrak{a}_{\nu,2}^{\mu,n}$. Similarly as before, we have the following, which we prove in Section 3.4.

Lemma 3.4. *There exists a constant $C = C(B)$ such that $C^{-1}\mathfrak{a}_{\nu,1}^{\mu,n}$ and $C^{-1}\mathfrak{a}_{\nu,2}^{\mu,n}$ are contained in $\mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$ for $n \geq 1$.*

The estimate (3.20) follows if we show

$$\left(\sum_\mu \sum_{\nu \in \mathfrak{I}_n^\mu} \|\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\|_p^p \right)^{1/p} \leq C_\epsilon 2^{-\frac{2}{p}k+\epsilon k} (2^n\delta_1)^{-\frac{2N}{p}+1+\epsilon} \|f\|_p, \quad p \geq 4N - 6 \tag{3.23}$$

for any $\epsilon > 0$, and

$$\left(\sum_\mu \sum_{\nu \in \mathfrak{I}_n^\mu} \|\mathcal{T}[\mathfrak{a}_{\nu,2}^{\mu,n}]f\|_p^p \right)^{1/p} \lesssim_B 2^{-\frac{(N+2)k}{2N}} (2^n\delta_1)^{-\frac{N}{2}} \|f\|_p, \quad 2 \leq p \leq \infty \tag{3.24}$$

when $n \geq 1$. Thanks to the inequality (3.8), those estimates give

$$\sum_{n \geq 1} \mathbf{E}_n \leq C_\epsilon \delta_0^{-\frac{2N}{p}+1+\epsilon} \sum_{1 \leq n \leq \log_2(C\delta_0/\delta_1)} \left(2^{-\frac{2}{p}k+\epsilon k} + 2^{-\frac{(N+2)k}{2N}} (2^n\delta_1)^{\frac{2N}{p}-\frac{N+2}{2}-\epsilon} \right) \|f\|_p$$

for $p \geq 4N - 2$. Note $\log_2(\delta_0/\delta_1) \leq Ck$ from (2.31). So, the estimate (3.20) follows since $4N - 2 > 4N/(N+2)$ and $\delta_1 \geq 2^{-k/N}$.

In order to prove the estimates (3.23) and (3.24), we start with the next lemma.

Lemma 3.5. *Let $n \geq 1$. For a constant $C = C(B) > 0$, we have the following:*

$$\sum_{1 \leq j \leq N-1} (2^n\delta_1)^{-(N-j)} |\langle \gamma^{(j)}(s), \xi \rangle| \geq C2^k, \quad (s, \xi) \in \text{supp}_{s,\xi} \, \mathfrak{a}_{\nu,1}^{\mu,n}, \tag{3.25}$$

$$(2^n\delta_1)^{-N} |\tau + \langle \gamma(s), \xi \rangle| \geq C2^k, \quad (s, \xi) \in \text{supp}_{s,\xi} \, \mathfrak{a}_{\nu,2}^{\mu,n}. \tag{3.26}$$

Proof. We first prove (3.25). Since $\mathfrak{G}_N^\mu \geq 2^{-2N!-1}(2^n\delta_1)^{2N!}$ on $\mathrm{supp}_{s,\xi}\,\mathfrak{a}_\nu^{\mu,n}$, one of the following holds on $\mathrm{supp}\,\mathfrak{a}_{\nu,1}^{\mu,n}$:

$$|s - \sigma(\xi)| \geq (2^3 C_0 B)^{-1} 2^n \delta_1, \tag{3.27}$$

$$2^{-k}|\mathfrak{g}_\mu^j(\tau,\xi)| \geq (2^2 C_0)^{-(N-j)}(2^n\delta_1)^{N-j} \tag{3.28}$$

for some $1 \leq j \leq N-2$, where $C_0 = 2^{2d}B$ (see (3.22)). If (3.27) holds, by (2.28) and (2.29) it follows that $(2^n\delta_1)^{-1}|\langle \gamma^{(N-1)}(s), \xi\rangle| \gtrsim 2^k$. Thus, to show (3.25) we may assume the inequality (3.27) fails, that is, (3.28) holds for some $1 \leq j \leq N-2$. So, there is an integer $\ell \in [0, N-2]$ such that (3.28) fails for $\ell+1 \leq j \leq N-2$, whereas (3.28) holds for $j = \ell$. By (3.6) and (3.12), we have

$$|\langle G^{(\ell)}(\sigma(\xi)), (\tau,\xi)\rangle| \geq |\mathfrak{g}_\mu^\ell| - \sum_{j=\ell+1}^N |\mathfrak{g}_\mu^j| \frac{(B^6 2^{14}\delta_0^2)^{j-\ell}}{(j-\ell)!} - 2B^3 (B^2 2^7 \delta_0)^{N+1-\ell}|\xi|. \tag{3.29}$$

Thus, (2.32) gives $|\langle G^{(\ell)}(\sigma(\xi)), (\tau,\xi)\rangle| \geq (2^3 C_0)^{-(N-\ell)} 2^k (2^n\delta_1)^{N-\ell}$. Also, the equation (3.6) and our choice of ℓ yield $|\langle G^{(j)}(\sigma(\xi)), (\tau,\xi)\rangle| \leq (2C_0)^{-(N-j)} 2^k (2^n\delta_1)^{N-j}$ for $\ell+1 \leq j \leq N-2$. Combining this with $|s - \sigma(\xi)| < (2^3 C_0 B)^{-1} 2^n \delta_1$ and expanding $G^{(\ell)}$ in Taylor series at $\sigma(\xi)$, we see that $|\langle G^{(\ell)}(s), (\tau,\xi)\rangle| \geq C 2^k (2^n\delta_1)^{N-\ell}$ for some $C = C(B) > 0$. This proves (3.25).

We now show (3.26), which is easier. On $\mathrm{supp}\,\mathfrak{a}_{\nu,2}^{\mu,n}$, $2^{-k}|\mathfrak{g}_\mu^0| \geq 2^{-N-1}(2^n\delta_1)^N$ and $2^{-k}|\mathfrak{g}_\mu^j| \leq 2C_0^{-(N-j)}(2^n\delta_1)^{N-j}$ for $j = 1, \ldots, N-2$. Using (3.29) with $\ell = 0$, by (2.32) and (2.31) we get $(2^n\delta_1)^{-N}|\tau + \langle \gamma(\sigma(\xi)), \xi\rangle| \geq 2^{-N-2} 2^k$. We also note that $|s - \sigma(\xi)| \leq 2C_0^{-1} 2^n \delta_1$ and $|\langle G^{(j)}(\sigma(\xi)), (\tau,\xi)\rangle| \leq C_0^{-1} 2^k (2^n\delta_1)^{N-j}$ for $1 \leq j \leq N-2$ on $\mathrm{supp}\,\mathfrak{a}_{\nu,2}^{\mu,n}$. Since $|\langle G^{(N)}(s), (\tau,\xi)\rangle| \leq B 2^{k+1}$, using Taylor series expansion at $\sigma(\xi)$ as above, we see (3.26) holds true for some $C = C(B) > 0$. $\qquad\square$

Additionally, we make use of disjointness of $\mathrm{supp}_\xi\,\mathfrak{a}_\nu^{\mu,n}$ by combining Lemma 2.11 and the next.

Lemma 3.6. *There is a positive constant $C = C(B)$ such that*

$$|(\widetilde{\mathcal{L}}_s^\delta)^{-1}\xi| \leq Cb 2^k \tag{3.30}$$

whenever $\xi \in \Lambda_k'(s,\delta,b)$ (see (2.36)). If $\xi \in \Gamma_k$ and (3.30) holds with $C = 1$, then $\xi \in \Lambda_k'(s,\delta,C_1 b)$ for some $C_1 = C_1(B) > 0$.

Proof. Let $\eta \in \mathbb{R}^d$ and $\{v_N, \ldots, v_d\}$ be an orthonormal basis of $(\mathrm{span}\{\gamma^{(j)}(s) : 1 \leq j \leq N-1\})^\perp$. We write $\eta = \sum_{j=1}^{N-1} \mathbf{c}_j \gamma^{(j)}(s) + \sum_{j=N}^d \mathbf{c}_j v_j$. Since $\mathfrak{B}(N, B)$ holds for γ, $|\eta| \sim |(\mathbf{c}_1, \cdots, \mathbf{c}_d)|$. Let $\xi \in \Lambda_k'(s,\delta,b)$. Then, (2.4) gives

$$\langle \eta, (\widetilde{\mathcal{L}}_s^\delta)^{-1}\xi\rangle = \langle (\widetilde{\mathcal{L}}_s^\delta)^{-\top}\eta, \xi\rangle = \sum_{j=1}^{N-1} \delta^{j-N} \mathbf{c}_j \langle \gamma^{(j)}(s), \xi\rangle + \sum_{j=N}^d \mathbf{c}_j \langle v_j, \xi\rangle.$$

Thus, by (2.36) we get $|\langle \eta, (\widetilde{\mathcal{L}}_s^\delta)^{-1}\xi\rangle| \leq Cb|\eta|2^k$, which shows (3.30).

By (2.4), $\langle \gamma^{(j)}(s), \xi\rangle = \delta^{N-j}\langle \gamma^{(j)}(s), (\widetilde{\mathcal{L}}_s^\delta)^{-1}\xi\rangle$ for $1 \leq j \leq N-1$. Therefore, (3.30) with $C = 1$ gives $|\langle \gamma^{(j)}(s), \xi\rangle| \leq C_1 b \delta^{N-j} 2^k$ for a constant $C_1 > 0$ when $1 \leq j \leq N-1$. This proves the second statement. $\qquad\square$

Now, we are ready to prove the estimates (3.23) and (3.24).

Proof of (3.23). By Lemma 3.4, $C^{-1}\mathfrak{a}_{\nu,1}^{\mu,n} \in \mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$ for some $C > 0$. Besides, (3.25) holds on $\text{supp}_{s,\xi}\,\mathfrak{a}_{\nu,1}^{\mu,n}$, and we note $2^n\delta_1 < \delta'$ from (3.8), (2.26), and (2.31). Thus, taking $\delta = 2^n\delta_1$ and $s_0 = 2^n\delta_1\nu$, we may use Lemma 2.8 for $\tilde{\chi}\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f$ to get

$$\big\|\tilde{\chi}\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\big\|_{L^p(\mathbb{R}^{d+1})} \leq C \sum_{1\leq l\leq C} \delta\big\|\mathcal{A}_t[\gamma_{s_0}^\delta, a_l]\tilde{f}_l\big\|_{L^p(\mathbb{R}^{d+1})},$$

where $\|\tilde{f}_l\|_p = \|f\|_p$, a_l are of type $(j, N-1, B')$ relative to $\gamma_{s_0}^\delta$ for some $B' > 0$, and $2^j \sim (2^n\delta_1)^N 2^k$. As seen before, $\gamma = \gamma_{s_0}^\delta$ satisfies $\mathfrak{B}(N, 3B)$ and (2.1) with B replaced by $3B$ for $\delta \leq \delta_*$. So, $\mathfrak{B}(N-1, B')$ with a large B' holds for $\gamma = \gamma_{s_0}^\delta$.

Therefore, we may apply the assumption (Theorem 2.2 with $L = N-1$) to $\mathcal{A}_t[\gamma_{s_0}^\delta, a_l]$, which gives $\|\mathcal{A}_t[\gamma_{s_0}^\delta, a_l]f\|_p \leq C_\epsilon \big(2^k(2^n\delta_1)^N\big)^{-\frac{2}{p}+\epsilon}\|f\|_p$ for a constant $C_\epsilon = C_\epsilon(B')$. Consequently, we obtain

$$\big\|\tilde{\chi}\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\big\|_p \leq C_\epsilon 2^{-\frac{2}{p}k+\epsilon k}(2^n\delta_1)^{1-\frac{2N}{p}+\epsilon}\|f\|_p$$

for $p \geq 4(N-1) - 2$. Besides, since $C^{-1}\mathfrak{a}_{\nu,1}^{\mu,n} \in \mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$, by (2.16) we have $\|(1-\tilde{\chi})\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\|_{L^p(\mathbb{R}^{d+1})} \lesssim_B 2^{-k}(2^n\delta_1)^{1-N}\|f\|_{L^p(\mathbb{R}^d)}$ for $p > 1$. Note $2^n\delta_1 \gtrsim 2^{-k/N}$. Combining those two estimates yields

$$\|\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\|_p \leq C_\epsilon 2^{-\frac{2}{p}k+\epsilon k}(2^n\delta_1)^{1-\frac{2N}{p}+\epsilon}\|f\|_p. \tag{3.31}$$

To exploit disjointness of $\text{supp}_\xi\,\mathfrak{a}_{\nu,1}^{\mu,n}$, we define a multiplier operator by

$$\mathcal{F}(P_s^\delta f)(\xi) = \beta_0\big(|(\widetilde{\mathcal{L}}_s^\delta)^{-1}\xi|/(C_1 2^k)\big)\widehat{f}(\xi)$$

for a constant $C_1 > 0$. Since $\text{supp}_\xi\,\mathfrak{a}_{\nu,1}^{\mu,n} \subset \Lambda_k'(2^n\delta_1\nu, 2^n\delta_1, 2^5B)$, by Lemma 3.6 we may choose C_1 large enough so that $\beta_0\big(|(\widetilde{\mathcal{L}}_{2^n\delta_1\nu}^{2^n\delta_1})^{-1}\cdot|/(C_1 2^k)\big) = 1$ on $\text{supp}_\xi\,\mathfrak{a}_{\nu,1}^{\mu,n}$. Thus, $\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f = \mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]P_{2^n\delta_1\nu}^{2^n\delta_1}f$. Combining this and (3.31), we obtain

$$\left(\sum_\mu \sum_{\nu\in\mathfrak{I}_n^\mu} \|\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\|_p^p\right)^{1/p} \leq C_\epsilon 2^{-\frac{2}{p}k+\epsilon k}(2^n\delta_1)^{1-\frac{2N}{p}+\epsilon}\left(\sum_\mu \sum_{\nu\in\mathfrak{I}_n^\mu} \|P_{2^n\delta_1\nu}^{2^n\delta_1}f\|_p^p\right)^{1/p}$$

for a constant $C_\epsilon = C_\epsilon(B)$ if $p \geq 4N - 6$. Therefore, the estimate (3.23) follows if we show

$$\left(\sum_\mu \sum_{\nu\in\mathfrak{I}_n^\mu} \|P_{2^n\delta_1\nu}^{2^n\delta_1}f\|_p^p\right)^{1/p} \lesssim_B \|f\|_p, \qquad 2 \leq p \leq \infty. \tag{3.32}$$

By interpolation, it suffices to obtain (3.32) for $p = 2, \infty$. The case $p = \infty$ is trivial since $\|P_{2^n\delta_1\nu}^{2^n\delta_1}f\|_\infty \lesssim \|f\|_\infty$. For $p = 2$, (3.32) follows by Plancherel's theorem since $\text{supp}\,\beta_0\big(|(\widetilde{\mathcal{L}}_{2^n\delta_1\nu}^{2^n\delta_1})^{-1}\cdot|/(C_1 2^k)\big)\widehat{f}$, $\nu\in\mathfrak{I}_n^\mu$ are finitely overlapping. Indeed, by Lemma 3.6 we have $\text{supp}\,\beta_0\big(|(\widetilde{\mathcal{L}}_{2^n\delta_1\nu}^{2^n\delta_1})^{-1}\cdot|/(C_1 2^k)\big)\widehat{f} \subset \Lambda_k'(2^n\delta_1\nu, 2^n\delta_1, CB)$ for a constant C. It is clear from Lemma 2.11 that $\Lambda_k'(2^n\delta_1\nu, 2^n\delta_1, CB)$, $\nu\in\mathfrak{I}_n^\mu$ overlap at most $C(B)$ times. $\square$

The proof of the estimate (3.24) is much easier since we have a favorable estimate for the kernel of $\mathcal{T}[\mathfrak{a}_{\nu,2}^{\mu,n}]$ thanks to the inequality (3.26).

Proof of (3.24). Let

$$\mathfrak{b}(s, t, \tau, \xi) = i^{-1}(\tau + \langle \gamma(s), \xi \rangle)^{-1} \partial_t \mathfrak{a}_{\nu,2}^{\mu,n}(s, t, \tau, \xi).$$

Then, integration by parts in t shows $m[\mathfrak{a}_{\nu,2}^{\mu,n}] = m[\mathfrak{b}]$. Note that (3.26) holds and $C^{-1}\mathfrak{a}_{\nu,2}^{\mu,n} \in \mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$ for a constant $C \geq 1$. Thus, $\mathfrak{a} := C^{-1}2^k(2^n\delta_1)^N\mathfrak{b}$ satisfies, with $\delta = 2^n\delta_1$ and $s_0 = 2^n\delta_1\nu$, (2.6) and (2.7) for $0 \leq j \leq 1, 0 \leq l \leq 2N - 1$, and $|\alpha| \leq d + N + 2$. Applying the estimate (2.15), we obtain $\|\mathcal{T}[\mathfrak{a}_{\nu,2}^{\mu,n}]f\|_\infty \lesssim_B 2^{-k}(2^n\delta_1)^{1-N}\|f\|_\infty$. Since $\delta_1 \geq 2^{-k/N}$, this gives

$$\|\mathcal{T}[\mathfrak{a}_{\nu,2}^{\mu,n}]f\|_\infty \lesssim_B 2^{-\frac{(N+2)k}{2N}}(2^n\delta_1)^{-\frac{N}{2}}\|f\|_\infty.$$

By interpolation it is sufficient to show (3.24) for $p = 2$. Note that $\|b(\cdot, t, \tau, \xi)\|_\infty + \|\partial_s b(\cdot, t, \tau, \xi)\|_1 \lesssim 2^{-k}(2^n\delta_1)^{-N}$. Thus, (2.28) and using van der Corput's lemma in s give $|m[\mathfrak{a}_{\nu,2}^{\mu,n}](\tau, \xi)| \lesssim 2^{-k(1+N)/N}(2^n\delta_1)^{-N}$. Since $\text{supp}_\xi \mathfrak{a}_{\nu,2}^{\mu,n} \subset \Lambda'_k(2^n\delta_1\nu, 2^n\delta_1, 2^5B)$, as before, we have $\mathcal{T}[\mathfrak{a}_{2,\nu}^{\mu,n}]f = \mathcal{T}[\mathfrak{a}_{2,\nu}^{\mu,n}]P_{2^n\delta_1\nu}^{2^n\delta_1}f$ with a positive constant C_1 large enough. Thus, by Plancherel's theorem

$$\|\mathcal{T}[\mathfrak{a}_{\nu,2}^{\mu,n}]f\|_{L^2}^2 \lesssim_B 2^{-\frac{2(1+N)}{N}k}(2^n\delta_1)^{-2N}\iint_{\{\tau:|\mathfrak{g}_\mu^0(\tau,\xi)|\leq 2^{k+1}(2^n\delta_1)^N\}} d\tau \, |\mathcal{F}(P_{2^n\delta_1\nu}^{2^n\delta_1}f)(\xi)|^2 \, d\xi.$$

Combining this and (3.32) yields (3.24) for $p = 2$. $\qquad\square$

3.3. Proof of Lemma 3.2

To simplify notations, we denote

$$\delta_\star = 2^n\delta_1, \qquad s_\star = 2^n\delta_1\nu$$

for the rest of this section. To prove Lemma 3.2, we verify the conditions (2.6) and (2.7) with $\mathfrak{a} = \mathfrak{a}_\nu^{\mu,n}$, $\delta = \delta_\star$, and $s_0 = s_\star$. The first is easy. In fact, since $\mathfrak{a}^\mu \in \mathfrak{A}_k(\delta_0\mu, \delta_0)$ and $\text{supp}_s \mathfrak{a}_\nu^{\mu,n} \subset I(s_\star, \delta_\star)$, we only need to show

$$|\langle G^{(j)}(s_\star), (\tau, \xi) \rangle| \leq B2^{k+5}\delta_\star^{N-j}, \quad j = 0, \ldots, N - 1 \tag{3.33}$$

on $\text{supp}_{\tau,\xi} \mathfrak{a}_\nu^{\mu,n}$. Using (3.6) and (3.11) together with (2.32) and (3.12), one can easily obtain

$$|\langle G^{(j)}(\sigma(\xi)), (\tau, \xi) \rangle| \leq 2^{k+1}\delta_\star^{N-j}, \quad j = 0, \ldots, N - 1 \tag{3.34}$$

on $\text{supp}_{\tau,\xi} \mathfrak{a}_\nu^{\mu,n}$. Expanding $\langle G^{(j)}(s), (\tau, \xi) \rangle$ in Taylor's series at $\sigma(\xi)$ gives (3.33) since (3.13) holds.

We now proceed to show (2.7) with $\mathfrak{a} = \mathfrak{a}_\nu^{\mu,n}$, $\delta = \delta_\star$, and $s_0 = s_\star$. Since $\mathfrak{a}_\nu^{\mu,n}$ consists of three factors $\mathfrak{a}^\mu, \beta_N(\delta_\star^{-2N!}\mathfrak{G}_N^\mu)$, and $\zeta(\delta_\star^{-1}s - \nu)$, by Leibniz's rule it is sufficient to consider the derivatives of each of them. The bounds on the derivatives of $\zeta(\delta_\star^{-1}s - \nu)$ are clear. So, it is enough to show (2.7) for

$$\mathfrak{a} = \mathfrak{a}^\mu, \ \beta_N(\delta_\star^{-2N!}\mathfrak{G}_N^\mu)$$

with $\delta = \delta_\star$ and $s_0 = s_\star$ whenever $(\tau, \xi) \in \text{supp}\,\mathfrak{a}_\nu^{\mu,n}(s, t, \mathcal{L}_{s_\star}^{\delta_\star}\cdot)$.

We handle $\mathfrak{a}^\mu$ first. That is to say, we show

$$\left|\partial_s^j\partial_t^l\partial_{\tau,\xi}^\alpha\left(\mathfrak{a}^\mu(s, t, \mathcal{L}_{s_\star}^{\delta_\star}(\tau, \xi))\right)\right| \lesssim_B \delta_\star^{-j}|(\tau, \xi)|^{-|\alpha|}, \qquad (j, l, \alpha) \in \mathcal{I}_N, \tag{3.35}$$

for $(\tau, \xi) \in \text{supp} \, \mathfrak{a}_\nu^{\mu,n}(s, t, \mathcal{L}_{s_\star}^{\delta_\star} \cdot)$. Since $\mathfrak{a}^\mu \in \mathfrak{A}_k(\delta_0 \mu, \delta_0)$ and $|s_\star - \delta_0 \mu| \le \delta_0$, we have

$$|\partial_s^j \partial_t^l \partial_{\tau, \xi}^\alpha \big(\mathfrak{a}^\mu(s, t, \mathcal{L}_{s_\star}^{\delta_0}(\tau, \xi))\big)| \lesssim_B \delta_0^{-j} |(\tau, \xi)|^{-|\alpha|}, \qquad (j, l, \alpha) \in \mathcal{I}_N. \tag{3.36}$$

One can show this using (2.11). We consider $\mathcal{U} := (\mathcal{L}_{s_\star}^{\delta_0})^{-1} \mathcal{L}_{s_\star}^{\delta_\star}$. By (2.8), we have $|\mathcal{U}^\intercal z| \lesssim_B |z|$ because $|\delta_0^{-1} 2^n \delta_1| \lesssim_B 1$. Thus, (3.36) gives

$$|\partial_s^j \partial_t^l \partial_{\tau, \xi}^\alpha \big(\mathfrak{a}^\mu(s, t, \mathcal{L}_{s_\star}^{\delta_0} \mathcal{U}(\tau, \xi))\big)| \lesssim_B \delta_0^{-j} |\mathcal{U}(\tau, \xi)|^{-|\alpha|}$$

for $(\tau, \xi) \in \text{supp} \, \mathfrak{a}_\nu^{\mu,n}(s, t, \mathcal{L}_{s_\star}^{\delta_\star} \cdot)$.

Let $(\tau, \xi) \in \text{supp} \, \mathfrak{a}_\nu^{\mu,n}(s, t, \mathcal{L}_{s_\star}^{\delta_\star} \cdot)$. Then, $\mathcal{L}_{s_\star}^{\delta_0} \mathcal{U}(\tau, \xi) = \mathcal{L}_{s_\star}^{\delta_\star}(\tau, \xi) \in \Lambda_k(s_\star, \delta_\star, B)$, so $|\widetilde{\mathcal{L}}_{s_\star}^{\delta_\star} \xi| \sim |(\tau, \xi)|$ by Lemma 2.5. This and (2.9) give

$$|(\tau, \xi)| \sim |\widetilde{\mathcal{L}}_{s_\star}^{\delta_\star} \xi| \le |\mathcal{L}_{s_\star}^{\delta_\star}(\tau, \xi)| \le |\mathcal{U}(\tau, \xi)|$$

for $(\tau, \xi) \in \text{supp} \, \mathfrak{a}_\nu^{\mu,n}(s, t, \mathcal{L}_{s_\star}^{\delta_\star} \cdot)$. Therefore, we obtain (3.35) since $\delta_\star \lesssim \delta_0$.

We continue to show (2.7) for $\mathfrak{a} = \beta_N(\delta_\star^{-2N!} \mathfrak{G}_N^\mu)$. Note that $\delta_\star^{-2N!} \mathfrak{G}_N^\mu$ is a sum of $(\delta_\star^{-1}(s - \sigma(\xi)))^{2N!}$ and $(\delta_\star^{-(N-j)} 2^{-k} \mathfrak{g}_\mu^j)^{2N!/(N-j)}$, $0 \le j \le N - 2$. Since the exponents $2N!/(N-j)$ are even integers, for the desired bounds on $\partial_{\tau, \xi}^\alpha (\beta_N(\delta_\star^{-2N!} \mathfrak{G}_N^\mu))$ it suffices to show the same bounds on the derivatives of

$$\delta_\star^{-1}(s - \sigma(\xi)), \qquad \delta_\star^{-(N-j)} 2^{-k} \mathfrak{g}_\mu^j, \quad 0 \le j \le N - 2.$$

The bound on $\partial_\xi^\alpha \delta_\star^{-1}(s - \sigma)$ is a consequence of (2.10) and the following lemma. For simplicity, we denote

$$\Xi = \mathcal{L}_{s_\star}^{\delta_\star}(\tau, \xi), \qquad \tilde{\Xi} = \widetilde{\mathcal{L}}_{s_\star}^{\delta_\star} \xi.$$

Lemma 3.7. *If $\Xi \in \text{supp}_{\tau, \xi} \, \mathfrak{a}_\nu^{\mu,n}$, then we have*

$$|\delta_\star^{-1} \partial_\xi^\alpha (\sigma(\tilde{\Xi}))| \lesssim_B |\xi|^{-|\alpha|}, \qquad 1 \le |\alpha| \le 2d + 2. \tag{3.37}$$

Proof. By (2.29), $\gamma^{(N-1)}(\sigma(\tilde{\Xi})) \cdot \tilde{\Xi} = 0$. Differentiation gives

$$\gamma^{(N)}(\sigma(\tilde{\Xi})) \cdot \tilde{\Xi} \, \nabla_\xi(\sigma(\tilde{\Xi})) + (\widetilde{\mathcal{L}}_{s_\star}^{\delta_\star})^\intercal \gamma^{(N-1)}(\sigma(\tilde{\Xi})) = 0. \tag{3.38}$$

Denote $s = \sigma(\tilde{\Xi})$. By (2.4), $(\widetilde{\mathcal{L}}_{s_\star}^{\delta_\star})^\intercal \gamma^{(N-1)}(s) = \delta_\star (\widetilde{\mathcal{L}}_{s_\star}^{\delta_\star})^\intercal (\widetilde{\mathcal{L}}_s^{\delta_\star})^{-\intercal} \gamma^{(N-1)}(s)$. Since $|s_\star - s| \le \delta_\star$, that is, (3.10), by Lemma 2.6 we have $|(\widetilde{\mathcal{L}}_{s_\star}^{\delta_\star})^\intercal \gamma^{(N-1)}(\sigma(\tilde{\Xi}))| \lesssim_B \delta_\star$. Besides, $|\gamma^{(N)}(\sigma(\tilde{\Xi})) \cdot \tilde{\Xi}| \gtrsim |\tilde{\Xi}| \sim 2^k$ (see (2.28)). Thus, (3.38) and (2.10) give

$$|\nabla_\xi(\sigma(\tilde{\Xi}))| \lesssim_B \delta_\star |\xi|^{-1},$$

which proves (3.37) with $|\alpha| = 1$.

We show the bounds on the derivatives of higher orders by induction. Assume that (3.37) holds true for $|\alpha| \le L$. Let α' be a multi-index such that $|\alpha'| = L + 1$. Then, differentiating the equation (3.38) and using the induction assumption, one can easily see $\gamma^{(N)}(\sigma(\tilde{\Xi})) \cdot \tilde{\Xi} \, \partial_\xi^{\alpha'}(\sigma(\tilde{\Xi})) = O(\delta_\star |\xi|^{-L})$, by which we get (3.37) for $|\alpha| = L + 1$. Since $\sigma \in C^{2d+2}$, one can continue this as far as $L \le 2d + 1$. $\qquad \square$

The proof of Lemma 3.2 is now completed if we show

$$\left| 2^{-k} \partial_{\tau,\xi}^{\alpha} \left(\mathfrak{g}_{\mu}^{\ell}(\Xi) \right) \right| \lesssim_B \delta_{\star}^{N-\ell} 2^{-k|\alpha|}, \qquad |\alpha| \leq d + N + 2 \tag{3.39}$$

for $0 \leq \ell \leq N - 2$ whenever $\Xi \in \operatorname{supp} \mathfrak{a}_{\nu}^{\mu,n}(s, t, \cdot)$. To this end, we use the following.

Lemma 3.8. *For $j = 0, \ldots, N$, set*

$$A_j = \delta_{\star}^{-(N-j)} 2^{-k} \left\langle G^{(j)}(\sigma(\tilde{\Xi})), \Xi \right\rangle.$$

If $(\tau, \xi) \in \operatorname{supp} \mathfrak{a}_{\nu}^{\mu,n}(s, t, \mathcal{L}_{s_{\star}}^{\delta_{\star}} \cdot)$, then for $j = 0, \ldots, N$ we have

$$|\partial_{\tau,\xi}^{\alpha} A_j| \lesssim_B |(\tau, \xi)|^{-|\alpha|}, \qquad 1 \leq |\alpha| \leq 2d + 2. \tag{3.40}$$

Proof. When $j = N$, the estimate (3.40) follows by Lemma 3.7 and (2.10). So, we may assume $j \leq N-1$. Differentiating A_j, we have

$$\nabla_{\tau,\xi} A_j = B_j + D_j,$$

where

$$B_j = \delta_{\star}^{-1} \left(0, \nabla_{\xi}(\sigma(\tilde{\Xi})) \right) A_{j+1}, \qquad D_j = \delta_{\star}^{-(N-j)} 2^{-k} (\mathcal{L}_{s_{\star}}^{\delta_{\star}})^{\top} G^{(j)}(\sigma(\tilde{\Xi})).$$

Note that $(\mathcal{L}_{s_{\star}}^{\delta_{\star}})^{\top} G^{(j)}(s_{\star}) = \delta_{\star}^{N-j} G^{(j)}(s_{\star})$ for $0 \leq j \leq N - 1$. Since $|s_{\star} - \sigma(\tilde{\Xi})| \lesssim \delta_{\star}$, similarly as before, Lemma 2.6 and (2.8) give

$$|(\mathcal{L}_{s_{\star}}^{\delta_{\star}})^{\top} G^{(j)}(\sigma(\tilde{\Xi}))| \lesssim_B \delta_{\star}^{N-j}, \qquad 0 \leq j \leq N - 1. \tag{3.41}$$

By Lemma 3.7 and (3.34), $|B_j| \lesssim |\xi|^{-1}$. Thus, for $\Xi \in \Lambda_k(s_{\star}, \delta_{\star}, B)$, we have

$$|\nabla_{\tau,\xi} A_j| \lesssim_B |\xi|^{-1} + 2^{-k} \lesssim_B |(\tau, \xi)|^{-1}, \qquad j = 0, \ldots, N - 1.$$

For the second inequality we use (2.10). This gives the inequality (3.40) when $|\alpha| = 1$.

To show (3.40) for $2 \leq |\alpha| \leq 2d + 2$, we use backward induction. By (2.29), we note $A_{N-1} = 0$, so (3.40) trivially holds when $j = N - 1$. We now assume that (3.40) holds true if $j_0 + 1 \leq j \leq N - 1$ for some $j_0 \leq N - 2$. Lemma 3.7, (2.10) and the induction assumption show $\partial_{\tau,\xi}^{\alpha'} B_{j_0} = O(|(\tau, \xi)|^{-1-|\alpha'|})$ for $1 \leq |\alpha'| \leq 2d + 1$. Concerning D_{j_0}, observe that $\partial_{\xi}^{\alpha'}(G^{(j_0)}(\sigma(\tilde{\Xi})))$ is given by a sum of the terms

$$G^{(j)}(\sigma(\tilde{\Xi})) \prod_{n=1}^{j-j_0} \partial_{\xi}^{\alpha'_n}(\sigma(\tilde{\Xi})),$$

where $j \geq j_0$ and $\alpha'_1 + \cdots + \alpha'_{j-j_0} = \alpha'$. Hence, Lemma 3.7, (3.41) and (2.10) give $\partial_{\xi}^{\alpha'} D_{j_0} = O(|(\tau, \xi)|^{-1-|\alpha'|})$ for $1 \leq |\alpha'| \leq 2d + 1$. Therefore, combining the estimates for B_{j_0} and D_{j_0}, we get $\partial_{\tau,\xi}^{\alpha'} \nabla_{\tau,\xi} A_{j_0} = O(|(\tau, \xi)|^{-1-|\alpha'|})$. This proves (3.40) for $j = j_0$. $\square$

Before proving (3.39), we first note that

$$|\partial_{\xi}^{\alpha}(\mathcal{E}_j(\tilde{\Xi}))| \lesssim_B \delta_{\star}^j |\xi|^{-|\alpha|}, \qquad |\alpha| \leq 2d + 2 \tag{3.42}$$

for $j = 1, \ldots, N$. This can be shown by a routine computation. Indeed, differentiating (3.4) and using Lemma 3.7 and (2.32), one can easily see (3.42) holds since $|\sigma(\tilde{\Xi}) - \delta_0 \mu| \lesssim \delta_0$.

To show (3.39) for $0 \le \ell \le N - 2$, we again use backward induction. Observe that (3.39) holds for $\ell = N, N - 1$. Then, we assume that (3.39) holds for $j + 1 \le \ell \le N$ for some $j \le N - 2$. By (3.6), we have

$$2^{-k} \mathfrak{g}_\mu^j = \delta_\star^{N-j} A_j - \sum_{j+1 \le \ell \le N} (2^{-k} \mathfrak{g}_\mu^\ell)(\mathcal{E}_1)^{\ell-j}/(\ell - j)! + 2^{-k} y_\mu^N \mathcal{E}_{N-j}.$$

Thus, by Lemma 3.8 and (3.42), we get (3.39) with $\ell = j$. This completes the proof of Lemma 3.2.

3.4. Proof of Lemma 3.4

Lemma 3.4 can be shown in the same manner as Lemma 3.2. So, we shall be brief.

By Lemma 3.2, we have $C^{-1} \mathfrak{a}_\nu^{\mu,n} \in \mathfrak{A}_k(s_\star, \delta_\star)$ for a constant $C \ge 1$, so it suffices to show $C^{-1} \mathfrak{a}_{\nu,1}^{\mu,n} \in \mathfrak{A}_k(s_\star, \delta_\star)$ for some $C \ge 1$. The support condition (2.6) is obvious, so we need only to show (2.7) with $\mathfrak{a} = \mathfrak{a}_{\nu,1}^{\mu,n}$, $\delta = \delta_\star$, and $s_0 = s_\star$. Moreover, by recalling (3.22), it is enough to consider the additional factor only, that is, to show

$$\left| \partial_{\tau,\xi}^\alpha \left(\beta_0 \left(\frac{(\delta_\star^{-N} 2^{-k} \mathfrak{g}_\mu^0 (\mathcal{L}_{s_\star}^{\delta_\star}(\tau,\xi)))^{2(N-1)!}}{C_0^{2N!} \delta_\star^{-2N!} \bar{\mathfrak{G}}_N^\mu(s, \widetilde{\mathcal{L}}_{s_\star}^{\delta_\star} \xi)} \right) \right) \right| \lesssim |(\tau,\xi)|^{-|\alpha|}$$

for $(\tau,\xi) \in \text{supp} \, \mathfrak{a}_{\nu,1}^{\mu,n}(s,t,\mathcal{L}_{s_\star}^{\delta_\star}\cdot)$. Since $\delta_\star^{-2N!} \bar{\mathfrak{G}}_N^\mu \gtrsim 1$ on $\text{supp}_{s,\xi} \, \mathfrak{a}_{\nu,1}^{\mu,n}$, one can obtain the estimate in the same way as in the proof of Lemma 3.2.

3.5. Sharpness of Theorem 1.3

Before closing this section, we show optimality of the regularity exponent α in Theorem 1.3.

Proposition 3.9. *Suppose the estimate (1.4) holds for ψ satisfying $\psi(0) \ne 0$. Then $\alpha \le 2/p$.*

Proof. We write $\gamma = (\gamma_1, \dots, \gamma_d)$. Via an affine change of variables, we may assume $\gamma_1(0) = 0$ and $\gamma_1'(s) \ne 0$ on an interval $J = [-\delta_0, \delta_0]$ for $0 < \delta_0 \ll 1$. Since $\psi(0) \ne 0$, we may also assume $\psi \ge 1$ on J.

We choose $\zeta_0 \in \mathcal{S}(\mathbb{R})$ such that $\text{supp} \, \widehat{\zeta_0} \subset [-1, 1]$ and $\zeta_0 \ge 1$ on $[-r_1, r_1]$, where $r_1 = 1 + 2 \max\{|\gamma(s)| : s \in J\}$. Denoting $\bar{x} = (x_1, \dots, x_{d-1})$ and $\bar{\gamma}(t) = (\gamma_1(t), \dots, \gamma_{d-1}(t))$, we define

$$\bar{\mathcal{A}}_t h(x) = \int e^{it\lambda\gamma_d(s)} \zeta_0(x_d - t\gamma_d(s)) h(\bar{x} - t\bar{\gamma}(s)) \psi(s) \, ds.$$

Let $\zeta \in C_c^\infty((-2,2))$ be a positive function such that $\zeta = 1$ on $[-1, 1]$. For a positive constant $c \ll \delta_0$, let $g_1(\bar{x}) = \sum_{\nu \in \lambda^{-1}\mathbb{Z} \cap [-c,c]} \zeta(\lambda|\bar{x} + \bar{\gamma}(\nu)|)$. We consider

$$g(\bar{x}) = e^{-i\lambda\varphi(x_1)} g_1(\bar{x}),$$

where $\varphi(s) = \gamma_d \circ (-\gamma_1)^{-1}(s)$. We claim that, if c is small enough,

$$|\bar{\mathcal{A}}_t g(x)| \gtrsim 1, \qquad (x,t) \in S_c, \tag{3.43}$$

where $S_c = \{(x,t) : |\bar{x}| \le c\lambda^{-1}, |x_d| \le c, |t - 1| \le c\lambda^{-1}\}$. To show this, note

$$\bar{\mathcal{A}}_t g(x) = \int e^{i\lambda(t\gamma_d(s) - \varphi(x_1 - t\gamma_1(s)))} \zeta_0(x_d - t\gamma_d(s)) g_1(\bar{x} - t\bar{\gamma}(s)) \psi(s) \, ds.$$

Let $(x, t) \in S_c$. Then, $\operatorname{supp} g_1(\bar{x} - t\bar{\gamma}(\cdot)) \subset [-C_1 c, C_1 c]$ for some $C_1 > 0$. Since $\varphi(s) = \gamma_d \circ (-\gamma_1)^{-1}(s)$, by the mean value theorem we see $|\varphi(x_1 - t\gamma_1(s)) - \gamma_d(s)| \leq 2r_0 c\lambda^{-1}$, where $r_0 = 10r_1 \max\{|\partial_s \varphi(s)| : s \in (-\gamma_1)(J_*)\}$ and $J_* = [-(C_1 + 1)c, (C_1 + 1)c]$. Thus, we have

$$|t\gamma_d(s) - \varphi(x_1 - t\gamma_1(s))| \leq 3r_0 c\lambda^{-1}. \tag{3.44}$$

Besides, if λ is sufficiently large, $g_1(\bar{x} - t\bar{\gamma}(s)) = \sum_{\nu \in \lambda^{-1}\mathbb{Z} \cap [-c,c]} \zeta(\lambda|\bar{x} - (t-1)\bar{\gamma}(s) + \bar{\gamma}(\nu) - \bar{\gamma}(s)|) \gtrsim 1$ if $s \in [-c/2, c/2]$. Since $\operatorname{supp} g_1(\bar{x} - t\bar{\gamma}(\cdot)) \subset J$ with c small enough and $\zeta_0(x_d - t\gamma_d(s)) \geq 1$, we get $\int \zeta_0(x_d - t\gamma_d(s)) g_1(\bar{x} - t\bar{\gamma}(s)) \psi(s)\, ds \gtrsim 1$. Therefore, (3.43) follows by (3.44) if c is small enough, that is, $c \ll 1/(3r_0)$.

We set $f(x) = e^{-i\lambda x_d} \zeta_0(x_d) g(\bar{x})$. Then, $\chi(t)\mathcal{A}_t f(x) = e^{-i\lambda x_d} \chi(t)\bar{\mathcal{A}}_t g(x)$. By our choice of ζ_0, $\operatorname{supp} \hat{f} \subset \{\xi : |\xi_d + \lambda| \leq 1\}$, so $\operatorname{supp} \mathcal{F}(\chi(t)\mathcal{A}_t f) \subset \{(\tau, \xi) : |\xi_d + \lambda| \leq 1\}$. This gives

$$\lambda^\alpha \|\chi(t)\mathcal{A}_t f\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|\chi(t)\mathcal{A}_t f\|_{L^p_\alpha(\mathbb{R}^{d+1})}. \tag{3.45}$$

Indeed, $\lambda^\alpha \|\chi(t)\mathcal{A}_t f\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|\chi(t)\mathcal{A}_t f\|_{L^p(\mathbb{R}_{t,\bar{x}}; L^p_\alpha(\mathbb{R}_{x_d}))}$ by Mihlin's multiplier theorem in x_d. Similarly, one also sees $\|F\|_{L^p(\mathbb{R}_{t,\bar{x}}; L^p_\alpha(\mathbb{R}_{x_d}))} \leq C\|F\|_{L^p_\alpha(\mathbb{R}^{d+1})}$ for $\alpha \geq 0$ and any F. Combining those inequalities gives (3.45).

From (3.43), we have $\|\chi(t)\mathcal{A}_t f\|_p = \|\chi(t)\bar{\mathcal{A}}_t g\|_p \geq C\lambda^{-d/p}$. Note that $\operatorname{supp} g$ is contained in a $O(\lambda^{-1})$-neighborhood of $-\bar{\gamma}$, so it follows that $\|f\|_p \lesssim \lambda^{-(d-2)/p}$. Therefore, by (3.45) the inequality (1.4) implies $\lambda^\alpha \lambda^{-d/p} \lesssim \lambda^{-(d-2)/p}$. Taking $\lambda \to \infty$ gives $\alpha \leq 2/p$. $\qquad\square$

4. L^p Sobolev regularity

In this section, we prove Theorem 1.1, whose proof proceeds in a similar way as that of Theorem 1.3. However, we provide some details to make it clear how the optimal bounds are achieved. There are no t, τ variables appearing in the symbols, so the proof is consequently simpler but some modifications are necessary.

For a large $B \geq 1$, we assume

$$\max_{0 \leq j \leq 2d} |\gamma^{(j)}(s)| \leq B, \qquad s \in I. \tag{4.1}$$

Let $2 \leq L \leq d$. For γ satisfying $\mathfrak{B}(L, B)$, we say $\bar{a} \in C^{d+1}(\mathbb{R}^{d+1})$ is *a symbol of type* (k, L, B) *relative to* γ if $\operatorname{supp} \bar{a} \subset I \times \mathbb{A}_k$, $\mathfrak{R}(L, B)$ holds for γ on $\operatorname{supp} \bar{a}$, and

$$|\partial_s^j \partial_\xi^\alpha \bar{a}(s, \xi)| \leq B|\xi|^{-|\alpha|} \tag{4.2}$$

for $0 \leq j \leq 1$ and $|\alpha| \leq d + 1$. As before, Theorem 1.1 is a straightforward consequence of the following. We denote $\mathcal{A}[\gamma, \bar{a}] = \mathcal{A}_1[\gamma, \bar{a}]$.

Theorem 4.1. *Suppose* $\gamma \in C^{2d}(I)$ *satisfies* (4.1) *and* $\mathfrak{B}(L, B)$ *for some* $B \geq 1$. *Suppose* $\bar{a}$ *is a symbol of type* (k, L, B) *relative to* γ *for some* $B \geq 1$. *Then, if* $p > 2(L - 1)$, *for a constant* $C = C(B)$

$$\|\mathcal{A}[\gamma, \bar{a}]f\|_{L^p(\mathbb{R}^d)} \leq C 2^{-k/p} \|f\|_{L^p(\mathbb{R}^d)}. \tag{4.3}$$

In order to prove Theorem 1.1, we consider $\bar{a}_k(s, \xi) := \psi(s)\beta(2^{-k}|\xi|)$, where $\beta \in C_c^\infty((1/2, 4))$. By the condition (1.1), $\bar{a}_k$ is a symbol of type (k, d, B) relative to γ for some B, thus Theorem 4.1 gives (4.3) for $p > 2(d - 1)$. The estimate (4.3) for each dyadic pieces can be put together by the result in [22]. So, we get the estimate (1.2) for $\alpha = \alpha(p)$ if $p > 2(d - 1)$ (e.g., see [2]).

Interpolation with $\|\mathcal{A}[\gamma, \bar{a}_k]f\|_2 \lesssim 2^{-k/d}\|f\|_2$ which follows from (1.3) gives $\|\mathcal{A}[\gamma, \bar{a}_k]f\|_p \lesssim_B 2^{-\alpha k}\|f\|_p$ for $\alpha \leq \alpha(p)$ with strict inequality when $p \in (2, 2(d - 1)]$. Using those estimates, we

can prove Corollary 1.2. Indeed, if γ is a curve of maximal type $\ell > d$, a typical anisotropic scaling argument gives $\|\mathcal{A}[\gamma, \bar{a}_k] f\|_p \lesssim_B 2^{-\min(\alpha(p), 1/\ell)k} \|f\|_p$ for $p \neq \ell$ when $\ell \geq 2d - 2$, and for $p \in [2, 2\ell/(2d - \ell)) \cup (2d - 2, \infty)$ when $d < \ell < 2d - 2$. As in the above, one can combine those estimates to get the estimate (1.2) ([22]).

4.1. Proof of Theorem 4.1

The case $L = 2$ is easy. Since $\bar{a}$ is a symbol of type $(k, 2, B)$ relative to γ, van der Corput's lemma and Plancherel's theorem give (4.3) for $p = 2$. Interpolation with the L^∞ estimate shows (4.3) for $p \geq 2$. When $L \geq 3$, we have the following, which immediately yields Theorem 4.1.

Proposition 4.2. *Let $3 \leq N \leq d$. Suppose Theorem 4.1 holds for $L = N - 1$. Then, Theorem 4.1 holds true with $L = N$.*

To prove the proposition, we fix $N \in [3, d]$ and γ satisfying $\mathfrak{B}(N, B)$, and $\bar{a}$ of type (k, N, B) relative to γ. For s_0 and $\delta > 0$ such that $I(s_0, \delta) \subset I$, let

$$\bar{\Lambda}_k(s_0, \delta, B) = \bigcap_{1 \leq j \leq N-1} \left\{ \xi \in \mathbb{A}_k : |\langle \gamma^{(j)}(s_0), \xi \rangle| \leq B 2^{k+5} \delta^{N-j} \right\}.$$

By $\bar{\mathfrak{A}}_k(s_0, \delta)$ we denote the collection of $\bar{a} \in C^{d+1}(\mathbb{R}^{d+1})$ satisfying $\operatorname{supp} \bar{a} \subset I(s_0, \delta) \times \bar{\Lambda}_k(s_0, \delta, B)$ and $|\partial_s^j \partial_\xi^\alpha \bar{a}(s, \widetilde{\mathcal{L}}_{s_0}^\delta \xi)| \leq B \delta^{-j} 2^{-k|\alpha|}$ for $0 \leq j \leq 1$, $|\alpha| \leq d + 1$.

The next lemma which plays the same role as Lemma 2.8 can be shown by routinely following the proof of Lemma 2.8.

Lemma 4.3. *Let $\bar{a} \in \bar{\mathfrak{A}}_k(s_0, \delta)$ and $j_* = \log(2^k \delta^N)$. Suppose (2.20) holds on $\operatorname{supp} \bar{a}$. Then, there exist constants $C, l_*, \tilde{B} \geq 1$, and $\delta' > 0$ depending on B, and symbols $\bar{a}_1, \ldots, \bar{a}_{l_*}$ of type $(j, N-1, \tilde{B})$ relative to $\gamma_{s_0}^\delta$ such that*

$$\|\mathcal{A}[\gamma, \bar{a}] f\|_{L^p(\mathbb{R}^d)} \leq C\delta \sum_{1 \leq l \leq l_*} \|\mathcal{A}[\gamma_{s_0}^\delta, \bar{a}_l] \tilde{f}_l\|_{L^p(\mathbb{R}^d)},$$

$\|\tilde{f}_l\|_p = \|f\|_p$, *and $j \in [j_* - C, j_* + C]$ as long as $0 < \delta < \delta'$.*

The required regularity order for γ is reduced thanks to the fact that $\bar{a}$ is independent of τ, t. Actually, one may take $\tilde{a}(s, \xi) = \bar{a}(\delta s + s_0, \delta^{-N} \widetilde{\mathcal{L}}_{s_0}^\delta \xi)$ when following the Proof of Lemma 2.8, since (4.2) clearly holds for $\bar{a} = \tilde{a}$.

Using η_N (see (2.27)), we break

$$\mathcal{A}[\gamma, \bar{a}] = \mathcal{A}[\gamma, \bar{a}\eta_N] + \mathcal{A}[\gamma, \bar{a}(1 - \eta_N)].$$

Note that $C^{-1}\bar{a}(1 - \eta_N)$ is of type $(k, N-1, B')$ relative to γ for some large constants B' and C, so we may apply the assumption to $\mathcal{A}[\gamma, \bar{a}(1 - \eta_N)] f$. Consequently, we have the estimate (4.3) for $\bar{a} = \bar{a}(1 - \eta_N)$ if $p > 2N - 4$.

To handle $\mathcal{A}[\gamma, \bar{a}\eta_N]$, as before, we may assume that $\operatorname{supp} \bar{a}\eta_N \subset I(s_0, \delta_*) \times \bar{\Gamma}_k$ for some s_0 and a small δ_*. Here, $\bar{\Gamma}_k$ is defined in the same way as Γ_k by replacing $a\eta_N$ by $\bar{a}\eta_N$ (see Section 2.3). Since (2.28) holds on $\operatorname{supp}(\bar{a}\eta_N)$, we may work under the same **Basic assumption** as in Section 2.3. That is to say, we have σ on $\bar{\Gamma}_k$ satisfying (2.29) and $\sigma(\xi) \in I(s_0, \delta_*)$ for $\xi \in \bar{\Gamma}_k$. Moreover, $\sigma \in C^{d+1}$ since $\gamma \in C^{2d}(I)$, and (2.30) holds for $\xi \in \bar{\Gamma}_k$ and $|\alpha| \leq d + 1$. Thus, (4.2) remains valid for the symbols to be given by decomposing $\bar{a}$ with cutoff functions associated with σ and $\bar{\mathfrak{G}}_N^\mu$.

Apparently, $C^{-1}\bar{a}\eta_N \in \bar{\mathfrak{A}}_k(s_0, \delta_*)$ for a constant $C = C(B, \delta_*)$. Therefore, the proof of Proposition 4.2 is completed if we show the following.

Proposition 4.4. *Let* $3 \le N \le d$ *and* $\bar{\mathfrak{a}} \in \bar{\mathfrak{A}}_k(s_0, \delta_*)$ *with* $\mathrm{supp}_\xi\, \bar{\mathfrak{a}} \subset \bar{\Gamma}_k$. *Suppose Theorem 4.1 holds for* $L = N - 1$. *Then, if* $p > 2(N - 1)$, *we have the estimate* (4.3).

We prove Proposition 4.4 using the next, which corresponds to Proposition 2.10. In what follows, we denote $\mathcal{A}[\bar{\mathfrak{a}}] = \mathcal{A}[\gamma, \bar{\mathfrak{a}}]$.

Proposition 4.5. *Let* δ_0 *and* δ_1 *satisfy* (2.31). *For* μ *such that* $\delta_0\mu \in I(s_0, \delta_*) \cap \delta_0\mathbb{Z}$, *let* $\bar{\mathfrak{a}}^\mu \in \bar{\mathfrak{A}}_k(\delta_0\mu, \delta_0)$ *with* $\mathrm{supp}\, \bar{\mathfrak{a}}^\mu \subset I(s_0, \delta_*) \times \bar{\Gamma}_k$. *Suppose Theorem 4.1 holds for* $L = N - 1$. *Then, if* $p \in (2N - 2, \infty)$, *there are constants* $\epsilon_0 > 0$, $C_0 = C_0(\epsilon_0, B) \ge 2$, *and symbols* $\bar{\mathfrak{a}}_\nu \in \bar{\mathfrak{A}}_k(\delta_1\nu, \delta_1)$ *with* $\mathrm{supp}\, \bar{\mathfrak{a}}_\nu \subset I(s_0, \delta_*) \times \bar{\Gamma}_k$, $\nu \in \cup_\mu \mathfrak{J}_0^\mu$ *such that*

$$\left(\sum_\mu \| \mathcal{A}[\bar{\mathfrak{a}}^\mu] f \|_p^p \right)^{\frac{1}{p}} \le C_0 (\delta_1/\delta_0)^{\frac{N}{p} - 1 + \epsilon_0} \left(\sum_\nu \| \mathcal{A}[\bar{\mathfrak{a}}_\nu] f \|_p^p \right)^{\frac{1}{p}} + C_0 \delta_0^{-\frac{N}{p} + 1} 2^{-\frac{k}{p}} \| f \|_p.$$

Let δ' be the number given in Lemma 4.3. We choose a positive constant $\delta_\circ$ (cf. (2.26)) such that

$$\delta_\circ < \min\{ 2^{-10} B^{-3} \delta', \ (2^{7d} B^6)^{-N} C_0^{-2N/\epsilon_0} \}. \tag{4.4}$$

Proof of Proposition 4.4. Set $\delta_0 = \delta_\circ$, and let $\delta_1, \ldots, \delta_J$ be given by (2.33). Then, applying Proposition 4.5 iteratively up to J-th step (cf. Section 2.4), we have symbols $\bar{\mathfrak{a}}_\nu \in \bar{\mathfrak{A}}_k(\delta_J\nu, \delta_J)$, $\delta_J\nu \in I(s_0, \delta_0)$, such that

$$\left\| \mathcal{A}[\bar{\mathfrak{a}}] f \right\|_p \le C_0^J \delta_J^{\frac{N}{p} - 1 + \epsilon_0} \left(\sum_\nu \| \mathcal{A}[\bar{\mathfrak{a}}_\nu] f \|_p^p \right)^{1/p} + 2^{-\frac{k}{p}} \delta_0^{-\frac{N}{p} + 1 - \epsilon_0} \sum_{0 \le j \le J-1} C_0^{j+1} \delta_j^{\epsilon_0} \| f \|_p.$$

By (4.4) and (2.33), $\delta_j \le C_0^{-2((N+1)/N)^j N/\epsilon_0}$ for $0 \le j \le J - 1$. So, $\sum_{j=0}^{J-1} C_0^{j+1} \delta_j^{\epsilon_0} \le C_1$ for a constant C_1, and $C_0^J \delta_J^{\epsilon_0} \le C_1$. Thus, the matter is now reduced to showing

$$\left(\sum_\nu \| \mathcal{A}[\bar{\mathfrak{a}}_\nu] f \|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \lesssim_B 2^{-\frac{k}{N}} \| f \|_{L^p(\mathbb{R}^d)}, \qquad 2 \le p \le \infty,$$

which corresponds to the estimate (2.35). The case $p = \infty$ follows from the estimate $\| \mathcal{A}[\bar{\mathfrak{a}}] f \|_{L^\infty} \le C\delta \| f \|_{L^\infty}$ when $\bar{\mathfrak{a}} \in \bar{\mathfrak{A}}_k(s_0, \delta)$ for some s_0, δ (cf. (2.15)). One can obtain this in the same manner as in the proof of Lemma 2.7. The case $p = 2$ can be handled similarly as before, using Plancherel's theorem and van der Corput's lemma combined with Lemma 2.11 and (2.28). $\qquad \square$

The proof of Proposition 4.5 is similar to that of Proposition 2.10. Instead of (2.40), we use the estimate (2.41), in which the exponent is adjusted to the sharp Sobolev regularity estimate. However, a similar approach breaks down if one tries to obtain the local smoothing estimate (1.4) with the optimal regularity $\alpha = 2/p$. To do so, we need the inequality (2.39) for $4N - 2 < p \le N(N + 1)$. However, there is no such estimate available when $N = 2$.

4.2. Proof of Proposition 4.5

Let $\bar{\mathfrak{a}}^\mu \in \bar{\mathfrak{A}}_k(\delta_0\mu, \delta_0)$. For $\nu \in \mathfrak{J}_n^\mu$, set

$$\bar{\mathfrak{a}}_\nu^{\mu,n} = \bar{\mathfrak{a}}^\mu \times \begin{cases} \beta_0 \big(\delta_1^{-2N!} \bar{\mathfrak{G}}_N^\mu \big) \zeta(\delta_1^{-1} s - \nu), & n = 0, \\ \beta_N \big((2^n \delta_1)^{-2N!} \bar{\mathfrak{G}}_N^\mu \big) \zeta(2^{-n} \delta_1^{-1} s - \nu), & n \ge 1. \end{cases}$$

(See (3.21)). Let $\bar{\mathbf{y}}_\mu = (y_\mu^1, \ldots, y_\mu^N)$, and let $\bar{\mathcal{D}}_\delta$ denote the $N \times N$ matrix $(\delta^{1-N}\bar{e}_1, \delta^{2-N}\bar{e}_2, \ldots, \delta^0\bar{e}_N)$ where $\bar{e}_j$ is the j-th standard unit vector in $\mathbb{R}^N$. Recalling (3.18), we consider a linear map

$$\bar{Y}_\mu^{\delta_0}(\xi) = \big(2^{-k}\bar{\mathcal{D}}_{\delta_0}\bar{\mathbf{y}}_\mu, \, y_{N+1}, \ldots, y_d\big).$$

Let $\mathbf{r}$ denote the curve $\mathbf{r}_\circ^N$. Note that (3.10) and (3.11) hold on $\mathrm{supp}\,\bar{\mathfrak{a}}_\nu^{\mu,n}$. Similarly as in *Proof of Lemma 3.3*, we see $|\langle\bar{\mathbf{y}}_\mu, \mathbf{r}^{(j)}((2^n\delta_1/\delta_0)\nu - \mu)\rangle| \lesssim 2^k(2^n\delta_1/\delta_0)^{N-j}$ for $1 \leq j \leq N-1$ and $2^{k-2}/B \leq |\langle\bar{\mathbf{y}}_\mu, \mathbf{r}^{(N)}\rangle| \leq CB2^k$ on $\mathrm{supp}_\xi\,\bar{\mathfrak{a}}_\nu^{\mu,n}$. Thus, as before (cf. (3.19)), we have

$$\bar{Y}_\mu^{\delta_0}(\mathrm{supp}_\xi\,\bar{\mathfrak{a}}_\nu^{\mu,n}) \subset \mathbf{S}\Big(\frac{2^n\delta_1}{\delta_0}\nu - \mu, \, C\frac{2^n\delta_1}{\delta_0}, \, CB; \, \mathbf{r}_\circ^N\Big) \times \mathbb{R}^{d-N}$$

for some $C > 0$. Note $\mathrm{supp}\,\mathcal{F}(\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f) \subset \mathrm{supp}_\xi\,\bar{\mathfrak{a}}_\nu^{\mu,n}$. Therefore, using the change of variables $\xi \to \bar{Y}_\mu^{\delta_0}(\xi)$, (2.41) with N replaced by $N-1$ and its cylindrical extension (e.g., (2.42)), we get

$$\Big\| \sum_{\nu \in \mathfrak{I}_n^\mu} \mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f \Big\|_p \leq C_0(2^n\delta_1/\delta_0)^{\frac{N}{p}-1+\epsilon_0}\Big(\sum_{\nu \in \mathfrak{I}_n^\mu} \big\|\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f\big\|_p^p\Big)^{1/p} \tag{4.5}$$

for $2N-2 < p < \infty$ (cf. (3.15)). Since $\mathcal{A}[\bar{\mathfrak{a}}^\mu]f = \sum_n \sum_{\nu \in \mathfrak{I}_n^\mu} \mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f$, by Minkowski's inequality and (4.5), we have $(\sum_\mu \|\mathcal{A}[\bar{\mathfrak{a}}^\mu]f\|_p^p)^{1/p}$ bounded by

$$\sum_{n \geq 0} \bar{\mathbf{E}}_n := C_0 \sum_{n \geq 0} (2^n\delta_1/\delta_0)^{\frac{N}{p}-1+\epsilon_0}\Big(\sum_\mu \sum_{\nu \in \mathfrak{I}_n^\mu} \big\|\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f\big\|_p^p\Big)^{1/p}.$$

The proof of Lemma 3.2 also shows $C^{-1}\bar{\mathfrak{a}}_\nu^{\mu,n} \in \bar{\mathfrak{A}}_k(2^n\delta_1\nu, 2^n\delta_1)$ for a positive constant C. The matter is reduced to obtaining

$$\Big(\sum_\mu \sum_{\nu \in \mathfrak{I}_n^\mu} \big\|\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f\big\|_{L^p(\mathbb{R}^d)}^p\Big)^{1/p} \lesssim_B (2^n\delta_1)^{1-\frac{N}{p}}2^{-\frac{k}{p}}\|f\|_{L^p(\mathbb{R}^d)}, \qquad n \geq 1 \tag{4.6}$$

for $p > 2(N-2)$. This gives $\sum_{n \geq 1} \bar{\mathbf{E}}_n \lesssim_B \delta_0^{-N/p+1}2^{-k/p}\|f\|_p$ since $2^n\delta_1 \leq C\delta_0$.

The proof of (4.6) is similar with that of (3.23). Since $C^{-1}\bar{\mathfrak{a}}_\nu^{\mu,n} \in \bar{\mathfrak{A}}_k(2^n\delta_1\nu, 2^n\delta_1)$, we have $\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f = \mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]P_{2^n\delta_1\nu}^{2^n\delta_1}f$. Besides, (3.27) or (3.28) for some $1 \leq j \leq N-2$ holds on $\mathrm{supp}\,\bar{\mathfrak{a}}_\nu^{\mu,n}$. Thus, we have (2.20) with $\delta = 2^n\delta_1$ for some B' on $\mathrm{supp}\,\bar{\mathfrak{a}}_\nu^{\mu,n}$ for $n \geq 1$ (see *Proof of Lemma 3.5*). Therefore, applying Lemma 4.3 to $\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f$ and then the assumption (Theorem 4.1 with $L = N-1$), we obtain

$$\|\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f\|_{L^p} \lesssim_B (2^n\delta_1)^{1-\frac{N}{p}}2^{-\frac{k}{p}}\|P_{2^n\delta_1\nu}^{2^n\delta_1}f\|_p.$$

This combined with (3.32) gives (4.6) as desired.

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References

[1] D. Beltran, S. Guo, J. Hickman and A. Seeger, 'Sharp L^p bounds for the helical maximal function,' *Amer. J. of Math.* (to appear). arXiv:2102.08272.

[2] D. Beltran, S. Guo, J. Hickman and A. Seeger, 'Sobolev improving for averages over curves in $\mathbb{R}^4$', *Adv. Math.* **393** (2021) 108089.

[3] J. Bourgain, 'Averages in the plane over convex curves and maximal operators', *J. Analyse. Math.* **47** (1986), 69–85.

[4] J. Bourgain and C. Demeter, 'The proof of the ℓ^2 decoupling conjecture', *Ann. of Math.* **182** (2015), 351–389.

[5] J. Bourgain, C. Demeter and L. Guth, 'Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three', *Ann. of Math.* **184** (2016), 633–682.

[6] M. Christ, Failure of an endpoint estimate for integrals along curves, Fourier analysis and partial differential equations (Miraflores de la Sierra, 1992), Stud. Adv. Math. (CRC, Boca Raton, FL, 1995), 163–168.

[7] M. Christ, 'Convolution, curvature, and combinatorics. A case study', *Int. Math. Res. Not.* **19** (1998), 1033–1048.

[8] M. Christ, A. Nagel, E. M. Stein and S. Wainger, 'Singular and maximal Radon transforms: analysis and geometry', *Ann. of Math.* **150**(2) (1999), 489–577.

[9] P. Constantin and J.-C. Saut, 'Local smoothing properties of dispersive equations', *J. Amer. Math. Soc.* **1** (1988), 413–439.

[10] S. Guo, Z. Li, P.-L. Yung and P. Zorin-Kranich, 'A short proof of ℓ^2 decoupling for the moment curve', *Amer. J. Math.* **143** (2021), 1983–1998.

[11] L. Guth, H. Wang and R. Zhang, 'A sharp square function estimate for the cone in $\mathbb{R}^3$', *Ann. of Math.* **192** (2020), 551–581.

[12] S. Ham and S. Lee, 'Restriction estimates for space curves with respect to general measure', *Adv. Math.* **254** (2014), 251–279.

[13] I. Ikromov, M. Kempe and D. Müller, 'Estimates for maximal functions associated with hypersurfaces in $\mathbb{R}^3$ and related problems of harmonic analysis', *Acta Math.* **204** (2010), 151–271.

[14] H. Ko, S. Lee and S. Oh, 'Maximal estimates for averages over space curves', *Invent. Math.* **228** (2022), 991–1035.

[15] S. Lee, 'Endpoint estimates for the circular maximal function', *Proc. Amer. Math. Soc.* **131** (2003), 1433–1442.

[16] S. Lee and A. Vargas, 'On the cone multiplier in $\mathbb{R}^3$', *J. Funct. Anal.* **263** (2012), 925–940.

[17] J. M. Marstrand, 'Packing circles in the plane', *Proc. London Math. Soc.* **55** (1987), 37–58.

[18] T. Mitsis, 'On a problem related to sphere and circle packing', *J. London Math. Soc.* **60** (1999), 501–516.

[19] G. Mockenhaupt, A. Seeger and C. D. Sogge, 'Wave front sets, local smoothing and Bourgain's circular maximal theorem', *Ann. of Math.* **136** (1992), 207–218.

[20] D. Oberlin, H. Smith and C. D. Sogge, 'Averages over curves with torsion', *Math. Res. Lett.* **5** (1998), 535–539.

[21] D. H. Phong and E. M. Stein, 'Radon transforms and torsion', *Internat. Math. Res. Notices* **1991**(4) (1991), 49–60.

[22] M. Pramanik, K. M. Rogers and A. Seeger, 'A Calderón–Zygmund estimate with applications to generalized radon transforms and Fourier integral operators', *Stud. Math.* **202** (2011), 1–15.

[23] M. Pramanik and A. Seeger, 'L^p Sobolev regularity of a restricted X-ray transform in $\mathbb{R}^3$', in *Harmonic Analysis and Its Applications* (Yokohama Publ., Yokohama 2006), 47–64.

[24] M. Pramanik and A. Seeger, 'L^p regularity of averages over curves and bounds for associated maximal operators', *Amer. J. Math.* **129** (2007), 61–103.

[25] M. Pramanik and A. Seeger, 'L^p-Sobolev estimates for a class of integral operators with folding canonical relations', *J. Geom. Anal.* **31** (2021), 6725–6765.

[26] W. Schlag, 'A generalization of Bourgain's circular maximal theorem', *J. Amer. Math. Soc.* **10** (1997), 103–122.

[27] W. Schlag and C. D. Sogge, 'Local smoothing estimates related to the circular maximal theorem', *Math. Res. Let.* **4** (1997), 1–15.

[28] A. Seeger and J. Wright, *Problems on Averages and Lacunary Maximal Functions, Marcinkiewicz Centenary Volume, Banach Center Publ.*, vol. 95, (Polish Acad. Sci. Inst. Math., Warsaw, 2011), 235–250.

[29] P. Sjölin, 'Regularity of solutions to the Schrödinger equation', *Duke Math. J.* **55** (1987), 699–715.

[30] C. D. Sogge, 'Propagation of singularities and maximal functions in the plane', *Invent. Math.* **104** (1991), 349–376.

[31] E. M. Stein, 'Maximal functions: spherical means', *Proc. Nat. Acad. Sci. USA* **73** (1976), 2174–2175.

[32] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals* (Princeton Univ. Press, Princeton, NJ, 1993).

[33] E. M. Stein and S. Wainger, 'Problems in harmonic analysis related to curvature', *Bull. Amer. Math. Soc.* **84** (1978), 1239–1295.

[34] B. Stovall, 'Endpoint bounds for a generalized Radon transform', *J. Lond. Math. Soc.* **80** (2009), 357–374.

[35] T. Tao and A. Vargas, 'A bilinear approach to cone multipliers. II. Applications', *Geom. Funct. Anal.* **10** (2000), 216–258.

[36] T. Tao and J. Wright, 'L^p improving bounds for averages along curves', *J. Amer. Math. Soc.* **16** (2003), 605–638.

[37] T. Wolff, 'A Kakeya type problem for circles', *Amer. J. Math.* **119** (1997), 985–1026.

[38] T. Wolff, 'Local smoothing type estimates on L^p for large p', *Geom. Funct. Anal.* **10** (2000), 1237–1288.