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The local index and the Swan conductor

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Abstract. In this paper we study local indices of systems of p-adic linearly differential equations which arise from p-adic representations of the absolute Galois group of local field of characteristic p with finite monodromy. We show the induction formula of the local index of p-adic differential equations and prove the equality between the local index of differential equations and the Swan conductor of p-adic Galois representations by inductive methods.

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1. Introduction

Let p be a prime. In [TN2] we showed that the category of p-adic representations with finite monodromy on a local field of positive characteristic p is equivalent to that of overconvergent etale φ-∇-modules, which are differential modules with etale Frobenius structures. In this paper we show that the Swan conductor of a p-adic representation with finite monodromy coincides with the irregularity of the corresponding p-adic differential module. Here the irregularity is a generalization of that of Robba which is defined using local indices [Ro2]. In the case of rank one, Matsuda showed the equality for an odd prime p [Ma].

Let F be a complete discrete valuation field of positive characteristic p with a perfect residue class field k and denote by G_F the absolute Galois group of F. Let K be the field of fraction of the Witt vector ring with k-coefficients and denote by $|\ |$ an absolute value of K. Put

$$\mathcal{E}^\dagger = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \mid \begin{array}{l} a_n \in K, |a_n| \text{ is bounded,} \\ |a_n|\rho^n \to 0 \ (n \to -\infty) \text{ for some } 0 < \rho < 1 \end{array} \right\},$$

then $\mathcal{E}^\dagger$ is a henselian discrete valuation field with residue class field F. In [Tn2] we showed an equivalence of categories

$$\begin{pmatrix} p\text{-adic reprsentations of } G_F \\ \text{with finite local monodromy} \end{pmatrix} \xrightarrow{\mathcal{D}^\dagger} \begin{pmatrix} \text{overconvergent etale} \\ \varphi\text{-}\nabla\text{-modules} \end{pmatrix}.$$

Here finite local monodromy means that the inertia subgroup of G_F acts through a finite quotient and an overconvergent etale φ-∇-module is an $\mathcal{E}^\dagger$-module with

Frobenius φ, all of whose slopes are 0, and with a connection ∇. For any overconvergent φ-∇-module M of rank one, there always exists a base such that the differential operator which is associated to the base has a coefficient in $K(x)$ ([Ma, 5.2, 5.3] or (8.2.1)). So one can define an irregularity of M by the irregularity of Robba [Ro2, 10.1]. In the case of general rank, however, it is not known that there exists a good basis such that the coefficients of the corresponding differential operator are matrices of $K(x)$-coefficients, so that we can not define the irregularity of overconvergent etale φ-∇-modules directly.

Let us explain how to define the irregularity. Assume that k is algebraically closed. An overconvergent etale φ-module M is π-unipotent if and only if the action of Frobenius φ is unipotent modulo the maximal ideal (3.2). We show that M is π-unipotent if and only if the corresponding p-adic representation V is totally wild ramified (3.5.1). In this case there is a basis such that the coefficients of the corresponding differential operator L are matrices of $\mathcal{H}^\dagger = \{\sum_{n=0}^{-\infty} a_n x^n \in \mathcal{E}^\dagger\}$-coefficients and all morphisms of overconvergent etale φ-modules comes from those as $\mathcal{H}^\dagger$-modules (3.3). We define the irregularity of M on K by

$$i(M) = -\dim_K \ker(L\colon (\mathcal{H}^\dagger)^r \to (\mathcal{H}^\dagger)^r) + \dim_K \operatorname{coker}(L\colon (\mathcal{H}^\dagger)^r \to (\mathcal{H}^\dagger)^r),$$

where r is the rank of M. $i(M)$ is independent of the choice of such a basis. In general case, first we pull back to the case that k is algebraically closed. Then, for any overconvergent etale φ-∇-module M, there is a positive integer N $(p \nmid N)$ such that the pull back $[N]^*M$ by the map $[N]\colon \mathcal{E}^\dagger \to \mathcal{E}^\dagger$ $(x \mapsto x^N)$ is π-unipotent. Define the irregularity by $\frac{1}{N} i([N]^*M)$ (7.1.1). Our irregularity coincides with that of Robba if L has coefficients in $K(x)$.

One of our main theorems is as follows;

THEOREM (7.2.2) *Let V be a p-adic representation of G_F with finite local monodromy and put $M = \mathcal{D}^\dagger(V)$. Then, the irregularity $i(M)$ is finite and*

$$i(M) = \mathrm{Swan}(V).$$

Here $\mathrm{Swan}(V)$ *is the Swan conductor of* V [Og, II.].

To prove the theorem, we use an inductive method. If $f\colon F \to F'$ is a finite separable extension, we denote by $(\mathcal{E}^\dagger)'$ the corresponding finite unramified extension over $\mathcal{E}^\dagger$. For an overconvergent φ-∇-modules M over $(\mathcal{E}^\dagger)'$, we define a direct image f_*M (5.1). The functor of direct image commutes with the functor $\mathrm{D}^\dagger$, that is, if V is a representation on F', then $\mathrm{D}^\dagger(f_*V) \cong f_*\mathrm{D}^\dagger(V)$ (5.2.1).

Assume that k is algebraically closed. If the extension F'/F is an Artin–Schreier extension of degree p and if M is a π-unipotent overconvergent φ-module over $(\mathcal{E}^\dagger)'$ with a basis such that the corresponding differential operator L has coefficients in rational functions, then f_*M also has such a good basis (5.6). Moreover, if $i(M)$ is finite, then $i(f_*M)$ is also finite and the inductive formula

$$i(f_*M) = \mathrm{rank}(M)\mathrm{length}_{O_{F'}} \omega_{O_{F'}/O_F} + i(M),$$

holds (8.1.2). Here $\omega_{O_{F'}/O_F}$ is the logarithmic differential module of $O_{F'}$ over O_F [KK, (1.7)]. The formula above corresponds to the formula for Swan conductor as follows. Let V be a potentially unipotent representation of $G_{F'}$, then

$$\mathrm{Swan}(f_*V) = \mathrm{rank}(V)\mathrm{length}_{O_{F'}}\omega_{O_{F'}/O_F} + \mathrm{Swan}(V).$$

Therefore, the formula 'irregularity $=$ Swan' holds for the pair (f_*M, f_*V) if it holds for the pair (M, V) by the commutativity of f_* and $\mathrm{D}^\dagger$.

To finish the proof, we use Brauer induction. In the case of rank one we give a new proof which in cludes the case of $p = 2$ (8.2.2). If k is algebraically closed, and if V is absolutely irreducible and totally wild ramified, V is an induced representation of a representation of rank one. Then we can calculate the irregularity of M using the formula (1.0.1) inductively and show that it coincides with Swan conductor of V. In general case, we can reduce to the preceding case. We calculate a local index not only in the case when the field of coefficients of differential structures is the p-adic completion $\widehat{K}^{\mathrm{alg}}$ of an algebraically closure of K, but also in the case when it is K. Because it is useful for applying to the Euler characteristic of overconvergent unit-root F-isocrystals on a curve. (See [Be] [Ga].)

Our method of calculation of local indices relies on the finite monodromy theorem for overconvergent etale φ-∇-modules [TN2]. If we consider irregularities for overconvergent φ-∇-modules with arbitrary slopes, we need such a type of monodromy theorem. (See [Cr, 4.16.2.].)

The category of p-adic representations on a local field of residue characteristic p is large. In this paper we treat only those with finite local monodromy. It is a future problem that what kinds of phenomena happen on the side of differential structures which correspond to infinitely ramified representations.

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2. Notations

In this section we fix notations.

(2.1) Let p be a prime number. For Λ and k such that

Λ is a finite extension over the field $\mathbf{Q}_p$ of p-adic numbers;
k is a perfect field of characteristic p; (2.1.1)
the residue class field $\mathbf{F}_q$ of Λ is included in k,

we define a complete discrete valuation field $K = K(\Lambda, k)$ by $\Lambda \otimes_{W(\mathbf{F}_q)} W(k)$. Here $W(A)$ is the ring of Witt vectors of a ring A.

For a field Ω and for an indeterminant x, denote by $F_{x,\Omega}$ the field $\Omega((x))$ of formal power series. We use the notation F_x for $F_{x,\Omega}$ if there is no ambiguity.

For any field Ω, we denote by Ω^{sep} a separable closure of Ω in a fixed algebraically closed field Ω^{alg} and by G_Ω the absolute Galois group $\mathrm{Gal}(\Omega^{\mathrm{sep}}/\Omega)$.

Let Ω be a valuation field. Denote by O_Ω (resp. Ω^{un}, resp. $\widehat{\Omega}$) the valuation ring (resp. the maximal unramified extension in Ω^{sep}, resp. the completion under the valuation) of Ω. We denote by $\underline{\mathbf{Rep}}_\Lambda(G_\Omega)$ the category of continuous representations of finite dimension with coefficients in Λ of the absolute Galois group of Ω and by $\underline{\mathbf{Rep}}_\Lambda^{\mathrm{fin}}(G_\Omega)$ the full subcategory of $\underline{\mathbf{Rep}}^\Lambda(G_\Omega)$ which consists of representations with finite local monodromy, that is, the inertia subgroup of G_Ω acts via a finite quotient.

(2.2) Denote by $\mathbf{R}_{\geqslant 0}$ the set of non-negative real numbers. For a complete field Ω under a non-Archimedean absolute value $|\ |\colon \Omega \to \mathbf{R}_{\geqslant 0}$ and for an indeterminant x, we define several Ω-algebras as follows:

$$\begin{aligned}
S_{x,\Omega} &= O_\Omega[[x]],\\
\mathcal{A}_{x,\Omega} &= \left\{\sum_{n=0}^{\infty} a_n x^n \mid a_n \in \Omega, |a_n|\rho^n \to 0\ (n \to \infty) \text{ for any } 0 < \rho < 1\right\},\\
\mathcal{H}_{x,\Omega} &= \left\{\sum_{n=0}^{-\infty} a_n x^n \mid a_n \in \Omega, |a_n| \to 0\ (n \to -\infty)\right\},\\
\mathcal{H}^\dagger_{x,\Omega} &= \left\{\sum_{n=0}^{-\infty} a_n x^n \mid a_n \in \Omega, |a_n|\rho^n \to 0\ (n \to -\infty) \text{ for some } 0 < \rho < 1\right\},\\
\mathcal{E}_{x,\Omega} &= \left\{\sum_{n=-\infty}^{\infty} a_n x^n \mid a_n \in \Omega, \sup_n |a_n| < \infty, \sum_{n=0}^{-\infty} a_n x^n \in \mathcal{H}_{x,\Omega}\right\},\\
\mathcal{E}^\dagger_{x,\Omega} &= \left\{\sum_{n=-\infty}^{\infty} a_n x^n \in \mathcal{E}_{x,\Omega} \mid \sum_{n=0}^{-\infty} a_n x^n \in \mathcal{H}^\dagger_{x,\Omega}\right\},\\
\mathcal{R}_{x,\Omega} &= \left\{\sum_{n=-\infty}^{\infty} a_n x^n \mid \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}_{x,\Omega}, \sum_{n=0}^{-\infty} a_n x^n \in \mathcal{H}^\dagger_{x,\Omega}\right\},\\
\mathcal{L}_{x,\Omega} &= \left\{\frac{b(x)}{a(x)} \mid a(x) \in O_\Omega[x], a(x) \neq 0, b(x) \in S_{x,\Omega}\right\}.
\end{aligned}$$

Here we regard $\mathcal{L}_{x,\Omega}$ as a subring of $\mathcal{E}^\dagger_{x,\Omega}$ since a nonzero element of $O_\Omega[x]$ is a unit of $\mathcal{E}^\dagger_{x,\Omega}$. We have several sequences of Ω-algebras;

$$S_{x,\Omega} \subset \mathcal{A}_{x,\Omega} \subset \mathcal{E}^\dagger_{x,\Omega} \subset \mathcal{R}_{x,\Omega},$$

and so on, via the natural inclusion. The ring $S_{x,\Omega}, \mathcal{A}_{x,\Omega}, \ldots$ is functorial in Ω. We use the notations $S_x, \ldots$ instead of $S_{x,\Omega}, \ldots$ if there is no ambiguity.

(2.2.1) REMARK. Our $\mathcal{A}_{x,\Omega}$ (resp. $\mathcal{H}^\dagger_{x,\Omega}$, resp. $\mathcal{R}_{x,\Omega}$) coincides with $\mathcal{A}_0(1)$ (resp. $\mathcal{H}^\dagger_0(1)$, resp. $\mathcal{R}_0(1)$) in [Ro2, 2].

(2.2.2) LEMMA. *We have* $\mathcal{L}_x \cap \mathcal{H}_x \subset \Omega(x)$ *in* $\mathcal{E}_x$.

For formal Laurent power series $a = \sum a_n x^n$, we define $|a|_G \in \mathbf{R}_{\geqslant 0} \cup \{\infty\}$ by $\sup_n |a_n|$. Denote by O_{R_x} the subalgebra of R_x which consists of elements $a \in R_x$ such that $|a|_G \leqslant 1$ for $R_x = \mathcal{A}_x, \ldots$.

If Ω is a complete discrete valuation field, then $\mathcal{E}_x$ (resp. $\mathcal{E}_x^\dagger$) is a complete discrete valuation field (resp. a henselian discrete valuation field) under the absolute value $|\ |_G$ and a uniformizer of Ω is also that of $\mathcal{E}_x$ (resp. $\mathcal{E}_x^\dagger$). The residue class field of $\mathcal{E}_x$ (resp. $\mathcal{E}_x^\dagger$) is F_{x,k_Ω} by the natural projection, where k_Ω is the residue class field of Ω. (See [Cr, 4.2] [Ma 3.2].) By the Weierstrass preparation theorem we have

(2.2.3) LEMMA. *If* Ω *is a discrete valuation field, then the field* $\mathcal{L}_x$ *coincides with the quotient field Frac*(S_x) *of* S_x *in* $\mathcal{E}_x^\dagger$.

(2.2.4) REMARK. If Ω is not a discrete valuation field, $\mathcal{E}_x$ (resp. $\mathcal{E}_x^\dagger$) is not a field. Let $\{a_n\}_{n\geqslant 1}$ be a sequence in m_Ω such that $|a_n| \to 1$. Consider an element

$$a = \sum_{n=1}^{\infty} a_n x^n \in S_{x,\Omega},$$

then a has infinitely many zeros in the disk $\mathbf{B}(0, 1^-) = \{\xi \in \widehat{\Omega}^{\text{alg}} \mid |\xi| < 1\}$ and there is a sequence ξ_i of zeros of a such that $|\xi_i| \to 0$ $(i \to \infty)$. Therefore, $\xi(x)^{-1}$ is not contained in $\mathcal{E}_{x,\widehat{\Omega}^{\text{alg}}}$. (See [DGS, II.2, 3].)

(2.3) For formal Laurent power series $\sum a_n x^n$ of indeterminant x, we define an additive map $\delta_x = x(\mathrm{d}/\mathrm{d}x)$ by

$$\delta_x\left(\sum a_n x^n\right) = \sum n a_n x^n.$$

Then δ_x is an Ω-derivation on $S_{x,\Omega}, \mathcal{A}_{x,\Omega}, \ldots$.

Let R_x be one of $S_x, \mathcal{A}_x, \mathcal{H}_x, \ldots$. We define a free R_x-module ω_{R_x} of rank one by

$$\omega_{R_x} = R_x \frac{\mathrm{d}x}{x}.$$

We define an additive map $d\colon R_x \to \omega_{R_x}$ by $d(a) = \delta_x(a)(\mathrm{d}x/x)$ for $a \in R_x$. Then d is an Ω-derivation on R_x.

(2.3.1) DEFINITION We say that a free R_x-module M of finite rank with an additive map $\nabla\colon M \to \omega_{R_x} \otimes_{R_x} M$ is a ∇-module over R_x if and only if ∇ is an Ω-connection, that is, $\nabla(am) = da \otimes m + a\nabla(m)$ for $a \in R_x$ and $m \in M$. A morphism of ∇-modules over R_x is an R_x-homomorphism which commutes with connections. We denote the category of ∇-modules over R_x by $\underline{\mathbf{M}}_{R_x}^{\nabla}$.

For a ∇-module M of rank r and for a basis $\{e_i\}$ of M, we define a matrix $C_{M,\boldsymbol{e}} \in M_r(R_x)$ ($\boldsymbol{e} = (e_1, e_2, \ldots, e_r)$) by

$$\nabla(\boldsymbol{e}) = \frac{\mathrm{d}x}{x} \otimes \boldsymbol{e}C_{M,\boldsymbol{e}}.$$

We also define a differential operator $L_{M,\boldsymbol{e}} \in M_r(R_x[\delta_x])$ by $L_{M,\boldsymbol{e}} = \delta_x + C_{M,\boldsymbol{e}}$.

(2.4) Fix Λ and k which satisfy the condition (2.1.1). Denote by π a uniformizer of Λ. Put $K = K(\Lambda, k)$ and fix the algebraic closure k^{alg} (resp. K^{alg}) such that the residue class field of K^{alg} is k^{alg}. Define an endomorphism σ on K by $\sigma = \mathrm{id}_\Lambda \otimes \mathrm{Frob}^f$, where id_Λ is an identity map on Λ, Frob is the usual Frobenius on the ring of Witt vectors and $q = p^f$ is the cardinal of the residue class field of Λ. By the condition (2.1.1), σ is well-defined and the fixed subring of σ is Λ. We call σ Frobenius. It is easily seen that σ extends uniquely on any unramified extension L in K^{un} of K and on its p-adic completion $\widehat{L}$. We use the same notation σ for this extension.

Let L be an unramified extension of K or its p-adic completion in $\widehat{K}^{\mathrm{alg}}$. We say a ring endomorphism σ on $\mathcal{E}_{x,L}$ is a Frobenius if and only if

$$\sigma(a) \equiv a^q \pmod{\pi},$$

for $a \in O_{\mathcal{E}}^{\times}$ and the Frobenius σ on L. We say that σ is a Frobenius on $\mathcal{E}_x^\dagger$ if and only if σ is a Frobenius on $\mathcal{E}_x$ with $\sigma(\mathcal{E}_x^\dagger) \subset \mathcal{E}_x^\dagger$. A Frobenius σ on $\mathcal{E}_{x,L}$ is a Frobenius on $\mathcal{E}_x^\dagger$ if and only if $\sigma(x) \in \mathcal{E}_x^\dagger$.

For a Frobenius σ on $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$), define an element $\mu(x, \sigma)$ (simply, $\mu(x)$ or μ) in $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) by $\mu = \delta_x(\sigma(x))/\sigma(x)$. Then $|\mu|_G < 1$ and we have an equality $\delta_x(\sigma(a)) = \mu\sigma(\delta_x(a))$, equivalently, $d(\sigma(a)) = \sigma(d(a))$ in $\omega_{\mathcal{E}}$ (resp. $\omega_{\mathcal{E}^\dagger}$) for any $a \in \mathcal{E}$ (resp. $\mathcal{E}^\dagger$). Here $\sigma(a(\mathrm{d}x/x)) = \mu\sigma(a)(\mathrm{d}x/x)$. Later, we often use Frobenius σ with

$$\frac{\sigma(x)}{x^q} \in \mathcal{H}^\dagger,$$

or especially $\sigma(x) = x^q$. Then, $\mu \in \mathcal{H}^\dagger$, or $\mu = q$.

Fix an algebraic closure F^{alg} of $F_{x,k}$ such that k^{alg} is the residue class field of F^{alg}. Put $\widetilde{\mathcal{E}} = \Lambda \otimes_{W(\mathbf{F}_q)} W(F^{\mathrm{alg}})$. For a Frobenius σ on $\mathcal{E}$, there is an embedding $i_\sigma\colon \mathcal{E} \to \widetilde{\mathcal{E}}$ such that (i) $|a|_G = |i_\sigma(a)|$ for $a \in \mathcal{E}$, where $|\ |$ is the unique valuation on $\widetilde{\mathcal{E}}$ which is the extension of that on K, (ii) the map on residue class field induced by i_σ is the injection $F \subset F^{\mathrm{alg}}$ and (iii) $i_\sigma(\sigma(a)) = (\mathrm{id}_\Lambda \otimes \mathrm{Frob}^f)(i_\sigma(a))$ [TN2, 2.5.1]. By [Ma, 3.3], for any finite separable extension $F_{y,l}$ over $F_{x,k}$ in F^{alg}, there is a unique finite unramified extension over $\mathcal{E}_{x,K}$ (resp. $\mathcal{E}^\dagger_{x,K}$) which is isomorphic to $\mathcal{E}_{y,L}$ (resp. $\mathcal{E}^\dagger_{y,L}$) in $\widetilde{\mathcal{E}}$. Here L is the finite unramified extension of L over K whose residue class field is l. Then the derivation δ_x and a Frobenius

σ on $\mathcal{E}_{x,K}$ (resp. $\mathcal{E}^{\dagger}_{x,K}$) are uniquely extended to that on $\mathcal{E}_{y,L}$ (resp. $\mathcal{E}^{\dagger}_{y,L}$) and the relation $\delta_x(\sigma(a)) = \mu(x)\sigma(\delta_x(a))$ is preserved on it. If $x = a(y)$, then we have the following relations

$$\begin{aligned}\delta_x &= \frac{a(y)}{\delta_y(a(y))}\delta_y,\\ \mu(x) &= \left(\sigma\left(\frac{\delta_y(a(y))}{a(y)}\right)\Big/\frac{\delta_y(a(y))}{a(y)}\right)\mu(y).\end{aligned}\tag{2.4.1}$$

Of course, $\sigma(y)$ is not always contained in $\mathcal{H}_y$ (resp. $\mathcal{H}^{\dagger}_y$) even if $\sigma(x)$ is contained in $\mathcal{H}_x$ (resp. $\mathcal{H}^{\dagger}_x$). If $F_{y,l}$ is Galois over $F_{x,k}$, then $\mathrm{Gal}(F_{y,l}/F_{x,k})$ acts on $\mathcal{E}_{y,L}$ (resp. $\mathcal{E}^{\dagger}_{y,L}$) via the canonical identification $\mathrm{Gal}(\mathcal{E}_{y,L}/\mathcal{E}_{x,K}) \cong \mathrm{Gal}(\mathcal{E}^{\dagger}_{y,L}/\mathcal{E}^{\dagger}_{x,K}) \cong \mathrm{Gal}(F_{y,l}/F_{x,k})$ and δ_x (resp. σ) commutes with the Galois action.

(2.5) For a matrix (a_{ij}) and for an application f, we define $f((a_{ij})) = (f(a_{ij}))$.

3. π-unipotent etale φ-∇-modules

In this section we define the notion of (strictly) π-unipotent φ-∇-modules and show some properties of them.

Fix Λ and k as in (2.1.1) and put $K = K(\Lambda, k)$. Let π be a uniformizer of Λ. Denote by $\mathcal{E}^{(\dagger)}$ either $\mathcal{E}_{x,K}$ or $\mathcal{E}^{\dagger}_{x,K}$ and let σ be a Frobenius on $\mathcal{E}^{(\dagger)}$. We use the notation $\mathcal{H}^{(\dagger)}$ for $\mathcal{H}_{x,K}$ or $\mathcal{H}^{\dagger}_{x,K}$ which respects to $\mathcal{E}^{(\dagger)}$.

(3.1) First we recall the definition of φ-∇-modules and some properties of them [TN1] [TN2].

(3.1.1) DEFINITION. We say an $\mathcal{E}^{(\dagger)}$-vector space M of finite dimension with a morphism φ is a φ-module over $\mathcal{E}^{(\dagger)}$ with respect to σ if and only if they satisfy the condition

(1) the application $\varphi\colon M \to M$ is a σ-linear endomorphism such that $\varphi(M)$ generates M over $\mathcal{E}^{(\dagger)}$. The map φ is called Frobenius.

A morphism of φ-modules is an $\mathcal{E}^{(\dagger)}$-homomorphism which commutes with Frobenius φ. For a φ-module M of rank r and for a basis $\{e_i\}$ of M, we define a matrix $A_{M,\boldsymbol{e}} \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ ($\boldsymbol{e} = (e_1, e_2, \ldots, e_r)$) by

$$\varphi(\boldsymbol{e}) = \boldsymbol{e}A_{M,\boldsymbol{e}}.$$

We denote the category of φ-modules over $\mathcal{E}^{(\dagger)}$ with respect to σ by $\underline{\mathbf{M\Phi}}_{\mathcal{E}^{(\dagger)},\sigma}$.

(3.1.2) DEFINITION. We say an $\mathcal{E}^{(\dagger)}$-vector space M with a Frobenius φ and a connection ∇ is a φ-∇-module over $\mathcal{E}^{(\dagger)}$ with respect to σ if and only if they satisfy the conditions

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(1) (M, φ) is a φ-module and (M, ∇) is a ∇-module;
(2) $\nabla \circ \varphi = (\sigma \otimes \varphi) \circ \nabla$.

A morphism of φ-∇-modules is an $\mathcal{E}^{(\dagger)}$-homomorphism which commutes with Frobenius φ and connection ∇. We denote the category of φ-∇-modules over $\mathcal{E}^{(\dagger)}$ with respect to σ by $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^{(\dagger)},\sigma}$.

(3.1.3) DEFINITION. (1) The slope of a φ-module (resp. φ-∇-module) M is the slope of $\widetilde{\mathcal{E}} \otimes_{\mathcal{E}^{(\dagger)}} M$ as a usual sense of F-space over $\widetilde{\mathcal{E}}$, where $\widetilde{\mathcal{E}}$ is defined in (2.4).
(2) A φ-module (resp. a φ-∇-module) M is etale if and only if all slopes of M are 0.

We denote the category of etale φ-modules (resp. etale φ-∇-modules) over $\mathcal{E}^{(\dagger)}$ with respect to σ by $\underline{\mathbf{M\Phi}}^{\mathrm{et}}_{\mathcal{E}^{(\dagger)},\sigma}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^{(\dagger)},\sigma}$). It is a full subcategory of $\underline{\mathbf{M\Phi}}_{\mathcal{E}^{(\dagger)},\sigma}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^{(\dagger)},\sigma}$).

(3.1.4) REMARK. In [TN1] [TN2], we use the terminology 'overconvergent φ-∇-module' (resp. 'φ-∇-module') for φ-∇-module over $\mathcal{E}^{\dagger}$ with respect to σ (resp. φ-∇-module over $\mathcal{E}$ with respect to σ). We will use both in this paper.

All categories $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^{(\dagger)}}$, $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^{(\dagger)},\sigma}$ $\ldots$ are abelian and have tensor products and duals. The natural functor ν from the category over $\mathcal{E}^{\dagger}$ to that over $\mathcal{E}$ commutes with tensor products and duals.

Let $A \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ and $C \in M_r(\mathcal{E}^{(\dagger)})$ which satisfy the relation

$$\delta_x(A) + CA = \mu(x, \sigma) A \sigma(C). \tag{3.1.5}$$

We define a φ-∇-module $M_{A,C}$ over $\mathcal{E}^{(\dagger)}$ with respect to σ by

$$\varphi(e_1, e_2, \ldots, e_r) = (e_1, e_2, \ldots, e_r)A,$$
$$\nabla(e_1, e_2, \ldots, e_r) = \frac{\mathrm{d}x}{x} \otimes (e_1, e_2, \ldots, e_r)C,$$

where $\{e_i\}$ is a basis of $M_{A,C}$. We say A is etale if and only if the φ-module whose Frobenius is defined by A is etale.

Let Λ' be a finite extension of Λ and let k' be an extension of k such that the pair Λ' and k' satisfy the condition (2.1.1). Denote by K' (resp. by $\mathcal{E}^{(\dagger)'}$) $K(\Lambda', k')$ (resp. $\mathcal{E}^{(\dagger)}_{x,K'}$). Define the Frobenius σ' on $\mathcal{E}^{(\dagger)'}$ by

$$\sigma' = \sigma^{f(\Lambda'/\Lambda)},$$

where $f(\Lambda'/\Lambda)$ is the degree of extension of the residue class field of Λ' over that of Λ. We define a functor

$$\begin{aligned} \iota_{\Lambda'/\Lambda}&: \underline{\mathbf{M\Phi}}_{\mathcal{E}^{(\dagger)},\sigma} \to \underline{\mathbf{M\Phi}}_{\mathcal{E}^{(\dagger)'},\sigma'} \\ (\text{resp. } \iota_{\Lambda'/\Lambda}&: \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^{(\dagger)},\sigma} \to \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^{(\dagger)'},\sigma'}), \end{aligned} \tag{3.1.6}$$

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by $(M, \varphi, (\nabla)) \mapsto (\mathcal{E}^{(\dagger)'} \otimes_{\mathcal{E}^{(\dagger)}} M, \sigma' \otimes \varphi^{f(\Lambda'/\Lambda)}, (\nabla))$. For $A \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ and $C \in M_r(\mathcal{E}^{(\dagger)})$ which satisfy the relation (3.1.5), if we put $A' = A\sigma(A) \ldots \sigma^{f(\Lambda'/\Lambda)-1}(A)$, then one can easily see that the equation

$$\delta_x(A') + CA' = \mu(x, \sigma')A'\sigma'(C),$$

holds. Hence, if we put $A = A_{M,\boldsymbol{e}}$ for a basis of M, then we have $\iota_{\Lambda'/\Lambda}(M) = M_{A',C}$. The functor $\iota_{\Lambda'/\Lambda}$ is exact, and commutes with ν, tensor products and duals. Moreover, if M is etale, then $\iota_{\Lambda'/\Lambda}(M)$ is also etale.

(3.2) We define the notion of π-unipotent φ-∇-modules. Put $S = S_{x,K}$.

(3.2.1) DEFINITION. (1) We say that a matrix $A \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ is π-unipotent if and only if A belongs to $\mathrm{GL}_r(O_{\mathcal{E}}^{(\dagger)})$ and

$$A \equiv \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \pmod{\pi}.$$

(2) We say that a φ-module M over $\mathcal{E}^{(\dagger)}$ with respect to σ is π-unipotent if and only if there exists a basis $\{e_i\}$ of M such that the matrix $A_{M,\boldsymbol{e}}$ is π-unipotent. We say such a basis is π-unipotent.

(3.2.2) DEFINITION. (1) We say that a matrix $A \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ is strictly π-unipotent if and only if A is π-unipotent and A is contained in $\mathrm{GL}_r(\mathcal{H}^{(\dagger)})$.

(2) We say that a φ-module M over $\mathcal{E}^{(\dagger)}$ with respect to σ is strictly π-unipotent if and only if there exists a basis $\{e_i\}$ of M such that the matrix $A_{M,\boldsymbol{e}}$ is strictly π-unipotent. We say such a basis is strictly π-unipotent.

If a matrix A is π-unipotent, then A is etale.

(3.2.3) PROPOSITION. *If a matrix $A \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ is π-unipotent, then there is a matrix $Q \in \mathrm{GL}_r(S)$ such that Q is π-unipotent and $Q^{-1}A\sigma(Q)$ is strictly π-unipotent.*

(3.2.4) COROLLARY. *If M is a π-unipotent φ-module over $\mathcal{E}^{(\dagger)}$ with respect to σ, then M is strictly π-unipotent.*

(3.2.5) REMARK. If k is algebraically closed, then there is a matrix $Q \in \mathrm{GL}_r(S)$ such that Q is π-unipotent and $Q^{-1}A\sigma(Q) \in 1 + \pi x^{-1}M_r(O_{\mathcal{H}}^{(\dagger)})$ for any π-unipotent matrix A.

Before proving (3.2.3), we prove two lemmas.

(3.2.6) LEMMA. *Let A be a matrix in $\mathrm{GL}_r(k((x)))$ such that $A = \begin{pmatrix} 1 & \ddots & * \\ 0 & & 1 \end{pmatrix}$. For any element $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in k((x))^r$, there is a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_r) \in k[[x]]^r$ such that*

$$A\sigma_q({}^t\xi) - {}^t\xi + {}^t\alpha \in k[x^{-1}]^r,$$
$$(\text{resp. } \sigma_q(\xi) - \xi A + \alpha \in k[x^{-1}]^r).$$

Here σ_q is a qth power map and ${}^t\xi$ is the transpose of ξ.

Proof. In the case $r = 1$, put $\alpha = \alpha_{(+)} + \alpha_{(-)}$ ($\alpha_{(+)} \in xk[[x]], \alpha_{(-)} \in k[x^{-1}]$). Set $\xi = \sum_{i=0}^{\infty} \sigma_q^i(\alpha_{(+)})$. Since $\alpha_{(+)} \in xk[[x]]$, the right-hand side is convergent and $\sigma_q(\xi) - \xi + \alpha \in k[x^{-1}]$. The rest is easy by induction on r. □

(3.2.7) LEMMA. *If a matrix $A \in \mathrm{GL}_r(O_K/(\pi^{n+1})[[x]][x^{-1}])$ satisfies the conditions that $A \equiv \begin{pmatrix} 1 & \ddots & * \\ 0 & & 1 \end{pmatrix} \pmod{\pi}$ and that $A \pmod{\pi^n}$ belongs to $M_r(O_K/(\pi^n)[x^{-1}])$ for a non-negative integer n (in the case that $n = 0$, we assume only π-unipotent for A), then there is a matrix $Q_n \in \mathrm{GL}_r(O_K/(\pi^{n+1})[[x]])$ such that $Q_n \equiv 1 \pmod{\pi^n}$ (in the case that $n = 0$, $Q_0 = \begin{pmatrix} 1 & \ddots & * \\ 0 & & 1 \end{pmatrix}$) and that ${Q_n}^{-1} A\sigma(Q_n)$ is contained in $M_r(O_K/(\pi^{n+1})[x^{-1}])$.*

Proof. The assertion is easily seen from (3.2.6) in the case when $r = 1$. We use induction on r. Decompose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, A_{11} of size $(r-1) \times (r-1)$, A_{12} of size $(r-1) \times 1$, A_{21} of size $1 \times (r-1)$, and A_{22} of size 1×1. We may assume that $A_{21} \in (O_K/(\pi^{n+1})[x^{-1}])^{r-1}$. In fact, by (3.2.6), there exists $\xi \in k[[x]]^{r-1}$ such that the (2, 1)-component of $\begin{pmatrix} 1 & 0 \\ \pi^n\xi & 1 \end{pmatrix}^{-1} A\sigma \begin{pmatrix} 1 & 0 \\ \pi^n\xi & 1 \end{pmatrix}$ is contained in $(O_K/(\pi^{n+1})[x^{-1}])^{r-1}$. Since $A_{21} \equiv 0 \pmod{\pi}$, the $(2,1)$-component does not change after $A \to (Q')^{-1}A\sigma(Q')$ for $Q' = 1 + \pi^n \begin{pmatrix} Q'_{11} & Q'_{12} \\ 0 & Q'_{22} \end{pmatrix}$, $Q'_{ij} \in M_{**}(k[[x]])$. By the assumption of induction, we may assume that all coefficients of A_{11} and A_{22} belong to $O_K/(\pi^{n+1})[x^{-1}]$ modulo π^{n+1}. Then we have the assertion by (3.2.6). □

Proof of (3.2.3). By (3.2.7) there is a sequence of $A_n \in \mathrm{GL}_r(O_{\mathcal{E}}^{(\dagger)}), Q_n \in \mathrm{GL}_r(S)$ such that

$$A_0 = A, \quad A_n = Q_{n-1}^{-1} A_{n-1} \sigma(Q_{n-1}),$$
$$A_n (\mathrm{mod}\, \pi^n) \in M_r(O_K/\pi^n[x^{-1}]),$$
$$Q_0 \equiv \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \pmod{\pi},$$
$$Q_n \equiv 1 \pmod{\pi^n} \quad (n \geqslant 1).$$

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Then the infinite product $Q_0Q_1Q_2\ldots$ is convergent in $\mathrm{GL}_r(S)$ and this product is the desired Q. □

We show some properties of π-unipotent φ-modules.

(3.2.8) PROPOSITION. *Let M_1, M_2 be π-unipotent φ-modules. The direct sum $M_1 \oplus M_2$, the tensor product $M_1 \otimes M_2$, and the dual $M_1^\vee$ are also π-unipotent.*

(3.2.9) PROPOSITION. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of φ-modules. M_2 is π-unipotent if and only if both M_1 and M_3 are π-unipotent.

Proof. Assume that M_2 is π-unipotent. Let $\{e_i\}$ be a π-unipotent basis of M_2. Let L_3 be an $O_{\mathcal{E}}^{(\dagger)}$-submodule in M_3 which is generated by the images of $\{e_i\}$. Then L_3 is a lattice of M_3, that is, $\mathcal{E} \otimes_{O_{\mathcal{E}}^{(\dagger)}} L_3 \cong M_3$, and L_3 is stable under the action of Frobenius φ. Put $\overline{L_3} = L_3/\pi L_3$ and denote by $\overline{e}_i$ the image of e_i in $\overline{L}_3$. Take a subset T of $\{e_1, e_2, \ldots, e_r\}$ such that e_i belongs to T if and only if $\overline{e}_i$ and $\langle \overline{e}_{i+1}, \ldots, \overline{e}_r\rangle$ are linearly independent over $O_{\mathcal{E}}^{(\dagger)}/(\pi)$. Then T generates L_3 over $O_{\mathcal{E}}^{(\dagger)}$ by Nakayama's Lemma and the representation matrix of the Frobenius φ with respect to the basis is π-unipotent. Therefore, M_3 is π-unipotent. Considering duals, M_1 is π-unipotent by (3.2.9). The converse is easy. □

Let Λ' be a finite extension of Λ and let k' be an extension of k in k^{alg} such that the pair (Λ', k') satisfies the condition (2.1.1).

(3.2.10) PROPOSITION. *Under the notation as in* (3.1.6), *if $\{e_i\}$ be a π-unipotent basis of φ-module M over $\mathcal{E}^{(\dagger)}$, then $1 \otimes e_i$ is a π-unipotent basis of $\iota_{\Lambda'/\Lambda}(M)$.*

(3.2.11) REMARK. If k is algebraically closed, then the converse is also true. (Use (3.5.1).)

Let k' be a perfect field over k and put $K' = K(\Lambda, k')$ and $\mathcal{E}^{(\dagger)'} = \mathcal{E}_{y,K'}^{(\dagger)}$. Let $f\colon \mathcal{E}^{(\dagger)} \to \mathcal{E}^{(\dagger)'}$ be a K-algebra homomorphism such that the absolute value $|\ |_G$ is preserved and that the Frobenius σ extends to $\mathcal{E}^{(\dagger)'}$. Then, for a φ-module M over $\mathcal{E}^{(\dagger)}$, the pull back $f^*M = \mathcal{E}^{(\dagger)'} \otimes_{\mathcal{E}^{(\dagger)}} M$ is also a φ-module. Moreover, M is etale if and only if f^*M is so. (See [TN2, (3.2)].)

(3.2.12) PROPOSITION. *Under the situation as above, let $\{e_i\}$ be a π-unipotent basis of φ-module M over $\mathcal{E}^{(\dagger)}$. Then $1 \otimes e_i$ is a π-unipotent basis of f^*M.*

(3.2.13) REMARK. We obtain such an extension $\mathcal{E}^{(\dagger)'}/\mathcal{E}^{(\dagger)}$ as in (3.2.12) by the corresponding finite unramified extension to a finite separable extension $F_{y,k'}/F_{x,k}$. Assume that k is algebraically closed and the degree of the extension $F_{y,k}/F_{x,k}$ is a power of p. Then, the converse of (3.2.12) is also true by (3.5.1).

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(3.2.14) LEMMA. *Assume that the Frobenius σ on $\mathcal{E}^{(\dagger)}$ satisfies the condition $\sigma(x)/x^q \in \mathcal{H}^{(\dagger)}$. Let M be a φ-∇-module over $\mathcal{E}^{(\dagger)}$ with respect to σ and let $\{e_i\}$ be a basis of M. If $A_{M,\boldsymbol{e}}$ is a strictly π-unipotent, then $C_{M,\boldsymbol{e}} \in \pi x^{-1} M_r(O_{\mathcal{H}}^{(\dagger)})$.*

Proof. Note that the condition that $\sigma(x)/x^q \in \mathcal{H}^{(\dagger)}$ is equivalent to that $\mu = \mu(x,\sigma) \in \mathcal{H}^{(\dagger)}$. Define a linear map $\psi: M_r(\mathcal{E}) \to M_r(\mathcal{E})$ by $\psi(Q) = \mu A\sigma(Q)A^{-1}$. Then ψ is a contraction in the p-adic topology and $\psi(M_r(O_{\mathcal{H}})) \subset \pi M_r(O_{\mathcal{H}})$ for $|\mu|_G < 1$. Since $\delta_x(A_{M,\boldsymbol{e}}) \in \pi M_r(O_{\mathcal{H}}^{(\dagger)})$, we have

$$\begin{aligned} C_{M,\boldsymbol{e}} &= -(1-\psi)^{-1}(\delta_x(A_{M,\boldsymbol{e}})A_{M,\boldsymbol{e}}^{-1}) \in \pi x^{-1} M_r(O_{\mathcal{H}}) \cap \mathcal{E}^{(\dagger)} \\ &= \pi x^{-1} M_r(O_{\mathcal{H}}^{(\dagger)}), \end{aligned}$$

by the relation (3.1.5). □

(3.3) Assume that the Frobenius σ satisfies the condition that $\sigma(x)^{-1} \in \mathcal{H}^{(\dagger)}$ in (3.3.1) and (3.3.2).

(3.3.1) LEMMA. *Let $A \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ and $B \in \mathrm{GL}_s(\mathcal{E}^{(\dagger)})$ be strictly π-unipotent. If a matrix $Q \in M_{s\times r}(\mathcal{E}^{(\dagger)})$ satisfies the relation $QA = B\sigma(Q)$, then Q is contained in $M_{s\times r}(\mathcal{H}^{(\dagger)})$.*

Proof. We may assume that Q belongs to $M_{s\times r}(O_{\mathcal{E}}^{(\dagger)})$. Assume that there is a non-negative integer n such that $Q \pmod{\pi^n} \in M_{s\times r}(O_K/(\pi^n)[x^{-1}])$ but $Q \pmod{\pi^{n+1}} \notin M_{s\times r}(O_K/(\pi^{n+1})[x^{-1}])$. Order the component $Q = (q_{ij})$ by

$$q_{s1}, q_{s2}, \ldots, q_{sr}, q_{(s-1)1}, \ldots, q_{1r}.$$

and let q_{ij} is the first component of Q such that $q_{ij} \pmod{\pi^{n+1}} \notin O_K/(\pi^{n+1})[x^{-1}]$. Since A and B are strictly π-unipotent, the only terms of positive power of x modulo π^{n+1} which appear in the both sides of equation

$$\sum_k q_{ik}a_{kj} = \sum_k b_{ik}\sigma(q_{kj}),$$

are the positive power of x in $q_{ij} \pmod{\pi^{n+1}}$ in the left-hand side and the positive power of x in $\sigma(q_{ij}) \pmod{\pi^{n+1}}$. The minimal positive orders of x in both sides are different. Therefore, we have the assertion. □

(3.3.2) LEMMA. *Let $A \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ and $B \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ be strictly π-unipotent. If a matrix $Q \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ satisfies the relation $QA = B\sigma(Q)$, then Q is contained in $\mathrm{GL}_r(\mathcal{H}^{(\dagger)})$.*

Proof. One knows $Q \in M_r(\mathcal{H}^{(\dagger)})$ by (3.3.1) and the rest is to check $\det(Q) \in (\mathcal{H}^{(\dagger)})^\times$. Hence we have only to check the assertion in the case where $r = 1$. Multiplying a suitable power of π, we may assume that Q is contained in $O_{\mathcal{E}}^{(\dagger)}$ and $\not\equiv 0 \pmod{\pi}$. Comparing the valuation of both sides in $k((x))$, $Q \pmod{\pi}$ must belong to k by (3.3.1). Therefore, Q is a unit in $\mathcal{H}^{(\dagger)}$. □

From now until the end of (3.3) the Frobenius σ satisfies the condition that $\sigma(x)/x^q \in \mathcal{H}^{(\dagger)}$.

For a space M over $\mathcal{E}^{(\dagger)}$ and for a basis $\{e_i\}$ of M, define a $\mathcal{H}^{(\dagger)}$-submodule $U_{M,\boldsymbol{e}}$ of M by the $\mathcal{H}^{(\dagger)}$-module which is generated by e_i's.

Let M be a π-unipotent φ-module (resp. φ-∇-module) over $\mathcal{E}^{(\dagger)}$ with respect to σ and let $\{e_i\}$ be a strictly π-unipotent basis. Then the Frobenius φ (resp. the connection ∇) preserves $U_{M,\boldsymbol{e}}$ by (3.2.14). $U_{M,\boldsymbol{e}}$ is independent of the choice of a basis $\{e_i\}$ by (3.3.2) and we use the notation U_M for it.

(3.3.3) DEFINITION. For a π-unipotent φ-module (resp. a φ-∇-module M), we call U_M a lattice over $\mathcal{H}^{(\dagger)}$ with respect to σ.

By (3.3.1) we have

(3.3.4) PROPOSITION. *For a morphism $\eta\colon M_1 \to M_2$ of π-unipotent φ-modules (resp. φ-∇-modules), there exists a uniquely $\mathcal{H}^{(\dagger)}$-linear homomorphism $\eta_U\colon U_{M_1} \to U_{M_2}$ such that η_U commutes with the Frobenius φ* (resp. *the connection ∇*) *and that the scalar extension* $\mathrm{id}_{\mathcal{E}^{(\dagger)}} \otimes \eta_U$ *coincides with η.*

(3.3.5) PROPOSITION. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

be an exact sequence of π-unipotent φ-modules (resp. *φ-∇-modules*). *Then the induced sequence* (3.3.4) *of $\mathcal{H}^{(\dagger)}$-modules*

$$0 \to U_{M_1} \to U_{M_2} \to U_{M_3} \to 0,$$

is exact.

Proof. Let $e_1, e_2, \ldots, e_r$ (resp. $f_1, f_2, \ldots, f_s$) be a strictly π-unipotent basis of M_1 (resp. a system of elements of M_2 such that the image in M_3 is a strictly π-unipotent basis). Then one can easily check that $(e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_s)$ $\begin{pmatrix} 1_r & 0 \\ 0 & \pi^n 1_s \end{pmatrix}$ is a π-unipotent basis of M_2 for a suitable non-negative integer n. By the argument as in (3.3.2) there is a matrix $Q = \begin{pmatrix} 1_r & * \\ 0 & 1_s \end{pmatrix}$ in $\mathrm{GL}_{s+r}(S)$ such that $(e_1, e_2, \ldots, e_r, \pi^n f_1, \pi^n f_2, \ldots, \pi^n f_s)Q$ is a strictly π-unipotent basis of M_2. □

(3.4) Let F be a discrete valuation field of characteristic p with a perfect residue class field k. Fix a uniformizer x of F. Then $F \cong F_{x,k}$, and F is the residue class field of $\mathcal{E}^{(\dagger)} = \mathcal{E}^{(\dagger)}_{x,K}$ by the natural projection. By [Fo2, A.1.2.] [TN2, Sect. 4] there are canonical equivalences of categories

$$\begin{aligned} D_\sigma &\colon \underline{\mathbf{Rep}}_\Lambda(G_F) \to \underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E},\sigma}, \\ D^\dagger_\sigma &\colon \underline{\mathbf{Rep}}^{\mathrm{fin}}_\Lambda(G_F) \to \underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger,\sigma}. \end{aligned} \tag{3.4.1}$$

The functors depend on the choice of the Frobenius σ, but the structure of connections does not rely on the choice of the Frobenius σ. We will now explain why the structure of connection does not rely on the choice of Frobenius.

Fix a Frobenius σ on $\mathcal{E}$. Then the embedding $\mathcal{E} \subset \widetilde{\mathcal{E}}$ is determined as in (2.4). Let $\mathcal{E}^{un}$ be the maximal unramified extension of $\mathcal{E}$ in $\widetilde{\mathcal{E}}$ and let $\widehat{\mathcal{E}}^{un}$ be the p-adic completion of $\mathcal{E}^{un}$. Then δ_x extends uniquely on $\widehat{\mathcal{E}}^{un}$ and commutes with the action of G_F on $\widehat{\mathcal{E}}^{un}$ since the extension of δ_x on $\mathcal{E}^{un}$ is continuous by (2.4.1). If σ_1 is a Frobenius on $\mathcal{E}$, then σ_1 can also extend on $\widehat{\mathcal{E}}^{un}$ and commutes with the action of G_F on $\widehat{\mathcal{E}}^{un}$. Moreover, the relation that

$$\delta_x(\sigma_1(a)) = \mu(x, \sigma_1)\sigma_1(\delta_x(a)),$$

holds for any $a \in \widehat{\mathcal{E}}^{\mathrm{un}}$.

Let $\rho: G_F \to \mathrm{GL}_r(\Lambda)$ be a continuous representation. By [Fo2, 1.2.4.] there exists a matrix $X \in \mathrm{GL}_r(\widehat{\mathcal{E}}^{un})$ such that

$$\rho(\tau) = X^{-1}\tau(X),$$

for all $\tau \in G_F$. For a Frobenius σ_1 on $\mathcal{E}$, put $A_1 = X\sigma_1(X)^{-1}$ and $C = -\delta(X)X^{-1}$. We have

(3.4.2) LEMMA. A_1 (resp. C) *is included in* $\mathrm{GL}_r(\mathcal{E})$ (resp. $M_r(\mathcal{E})$) *and the relation*

$$\delta_x(A_1) + CA_1 = \mu(x, \sigma_1)A_1\sigma_1(C),$$

holds.

Let $M_{A_1,C}$ be a φ-∇-module over $\mathcal{E}$ with respect to σ_1 which corresponds to the pairs A_1 and C. Since $A_1 = X\sigma_1(X)^{-1}$, $M_{A_1,C}$ is etale.

(3.4.3) PROPOSITION. *Under the equivalence*

$$D_{\sigma_1}: \underline{\mathbf{Rep}}_\Lambda(G_F) \to \underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E},\sigma_1},$$

of categories, there is an isomorphism $D_{\sigma_1}(\rho) \cong M_{A_1,C}$ *in* $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E},\sigma_1}$.

Proof. Let V_ρ (resp. $\{v_i\}$) be a corresponding representation (resp. a basis of V_ρ). Define an $\widehat{\mathcal{E}}^{un}$-linear map

$$\widehat{\mathcal{E}}^{un} \bigotimes_{\Lambda} V_\rho \to \widehat{\mathcal{E}}^{un} \bigotimes_{\mathcal{E}} M_{A_1,C},$$

by $(v_1, v_2, \ldots, v_r) = (e_1, e_2, \ldots, e_r)X$, where $\{e_i\}$ is the canonical basis of $M_{A_1,C}$. Then the map above is an isomorphism which is equivariant to the action of Galois group G_F and Frobenius. Therefore, $\mathrm{D}_{\sigma_1}(V_\rho) \cong M_{A_1,C}$ by definition. □

In the case of φ-∇-modules over $\mathcal{E}^\dagger$, we have only to replace $\widehat{\mathcal{E}}^{un}$ into

$$\widetilde{\mathcal{E}^\dagger} = \varinjlim_{E} (\mathcal{E}^\dagger_E)^u.$$

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Here E runs through all finite separable extensions of F and $(\mathcal{E}_E^\dagger)^u$ is the field of composition of $\mathcal{E}_E^\dagger$ and the p-adic completion $\widehat{K^{un}}$ of the maximal unramified extension of K in $\widetilde{\mathcal{E}}$. Then the same assertions (3.4.2) and (3.4.3) hold for the functor $\mathrm{D}^\dagger$.

We point out that to find X for a representation V is independent of the choice of the Frobenius. Hence, the connection is independent of the choice of the Frobenius.

(3.4.4) COROLLARY. *Let* $A \in \mathrm{GL}_r(\mathcal{E})$ (resp. $\mathrm{GL}_r(\mathcal{E}^\dagger)$) *and* $C \in M_r(\mathcal{E})$ (resp. $M_r(\mathcal{E}^\dagger)$) *which satisfy the relation* (3.1.5) *and the condition that* A *is etale for a Frobenius* σ. *Then, for any Frobenius* σ_1 (resp. *for any Frobenius* σ_1 *such that* $\sigma_1(x) \in \mathcal{E}^\dagger$), *there exists* $A_1 \in \mathrm{GL}_r(\mathcal{E})$ (resp. $\mathrm{GL}_r(\mathcal{E}^\dagger)$) *such that* A_1 *is etale and*

$$\delta_x(A_1) + CA_1 = \mu(x, \sigma_1)A_1\sigma_1(C).$$

Moreover, if A *is* π*-unipotent, then we can choose* A_1 *which is* π*-unipotent.*

Proof. The former part follows from (3.4.3). If A is π-unipotent, then we can choose a matrix $X \in \mathrm{GL}_r(O_{\widehat{\mathcal{E}}^{un}})$ (resp. $X \in \mathrm{GL}_r(O_{\widetilde{\mathcal{E}}^\dagger})$) of $A\sigma(X) = X$ such that X is π-unipotent by the proof of [Fo2, 1.2.4.] [TN2, (4.2)]. Therefore, we have $A_1 = X\sigma_1(X)^{-1}$ is π-unipotent. □

(3.4.5) COROLLARY. *Let* M *be a* φ-∇*-module of rank* r *over* $\mathcal{E}^{(\dagger)}$ *with a* π*-unipotent basis* $\{e_i\}$. *Then, there is a matrix* $Q \in \mathrm{GL}_r(S)$ *such that* $\boldsymbol{e}Q$ *is* π*-unipotent and the matrix* $C_{M,\boldsymbol{e}Q}$ *belongs to* $\pi x^{-1}M_r(O_{\mathcal{H}}^{(\dagger)})$.

Proof. The assertion follows (3.2.3), (3.2.14) and (3.4.4). □

We mention the extension of coefficients of representations. We do not treat it in [TN2]. Let Λ' be a finite extension of Λ and let k' be a separable extension of k in k^{alg} such that the pair (Λ', k') satisfies the condition (2.1.1). Put $K' = K(\Lambda', k')$, $F' = F_{x,k'}$ and $\mathcal{E}^{(\dagger)'} = \mathcal{E}_{x,K'}^{(\dagger)}$. There is a natural $G_{F'}$-injection $\Lambda' \otimes_\Lambda \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}'$, where $G_{F'}$ acts naturally on the left-hand side and via the natural map $G_{F'} \to G_F$ on the right-hand side. For a Frobenius on $\mathcal{E}^{(\dagger)}$, put $\sigma' = \sigma^{f(\Lambda'/\Lambda)}$ as in (3.1.6). We have

(3.4.6) PROPOSITION. *The following diagram is commutative*.

$$\begin{tikzcd}
\underline{\mathbf{Rep}}_\Lambda^{(\mathrm{fin})}(G_F) \arrow[r, "D_\sigma^{(\dagger)}"] \arrow[d, "\Lambda'\otimes"'] & \underline{\mathbf{M\Phi}}_{\mathcal{E}^{(\dagger)},\sigma}^{\nabla\mathrm{et}} \arrow[d, "\iota_{\Lambda'/\Lambda}"] \\
\underline{\mathbf{Rep}}_\Lambda^{(\mathrm{fin})}(G_{F'}) \arrow[r, "D_{\sigma'}^{(\dagger)}"'] & \underline{\mathbf{M\Phi}}_{\mathcal{E}^{(\dagger)'},\sigma'}^{\nabla\mathrm{et}}
\end{tikzcd}.$$

(3.5) Keep the notations as in (3.4).

(3.5.1) THEOREM. *Assume that the field k is algebraically closed. Let V be an object in* $\underline{\mathbf{Rep}}_\Lambda(G_F)$ (resp. $\underline{\mathbf{Rep}}_\Lambda^{\mathrm{fin}}(G_F)$) *and denote by M the φ-∇-module over* $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) *which corresponds to V through the equivalence* (3.4.1). *Then M is π-unipotent if and only if the image of $G_F \to \mathrm{GL}(V)$ is a pro-p group.*

Proof. Assume that M is π-unipotent. Choose a π-unipotent basis $\{e_i\}$. Let $X \in \mathrm{GL}_r(O_{\widehat{\mathcal{E}}^{un}})$ (resp. $X \in \mathrm{GL}_r(O_{\widetilde{\mathcal{E}}^\dagger})$) be a solution of the equation $A_{M,\boldsymbol{e}}\sigma(X) = X$ such that X is π-unipotent. Then the representation V is given by

$$\tau \in G_F \mapsto X^{-1}\tau(X) \in \mathrm{GL}_r(\Lambda). \tag{3.5.2}$$

The image of the map (3.5.2) is included in the pro-p group

$$\begin{pmatrix} 1+\pi\Lambda & & \lambda \\ & \ddots & \\ \pi\Lambda & & 1+\pi\Lambda \end{pmatrix}$$

(multiplicative). By the continuity the image of pro-finite group G_F is closed, so that the image is pro-p.

Assume that the image of $G_F \to \mathrm{GL}(V)$ is pro-p. Let T be an O_Λ-lattice of V. Since the image of $G_F \to \mathrm{GL}(V)$ is pro-p and $T/\pi T$ is a finite p-group, there is a basis $v_1, v_2, \ldots, v_r$ of T such that $\tau \in G_F$ acts on T by the matrix $\left(\begin{smallmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{smallmatrix}\right)$ modulo π with respect to the basis $\{v_i\}$. Then we can choose a π-unipotent matrix X in $\mathrm{GL}_r(O_{\widehat{\mathcal{E}}^{un}})$ (resp. $\mathrm{GL}_r(O_{\widetilde{\mathcal{E}}^\dagger})$) as in (3.4) which is determined by the representation V. Hence, $\mathrm{D}_\sigma(V)$ (resp. $D_\sigma^\dagger(V)$) is π-unipotent. □

For a positive integer N, denote by $[N]\colon \mathcal{E}_x^{(\dagger)} \to \mathcal{E}_y^{(\dagger)}$ the endomorphism which is defined by $x \mapsto y^N$.

(3.5.3) COROLLARY *Assume that k is algebraically closed. Let M be a φ-∇-module over $\mathcal{E}_x^{(\dagger)}$. Then, there is a positive integer N such that $p \nmid N$ and that the pull back $[N]^*M$ is π-unipotent over $\mathcal{E}_y^\dagger$.*

Proof. Let V be the corresponding representation of M. Since Λ is a finite extension of $\mathbf{Q}_p$ and V is a continuous representation, there is a finite Galois and tamely ramified extension F' of degree N over F such that V is totally wild ramified as a representation of $G_{F'}$. □

4. Criterion of connection with coefficients of rational functions

In this section we give a criterion for a connection to be defined over a field of rational functions.

(4.1) Let Ω be a complete field of characteristic 0 under a non-Archimedean absolute value $|\ |\colon \Omega \to \mathbf{R}_{\geqslant 0}$ and denote by $S, \mathcal{R}, \mathcal{H}$, and $\mathcal{L}$ the ring $S_{x,\Omega}, \mathcal{R}_{x,\Omega}, \mathcal{H}_{x,\Omega}$, and $\mathcal{L}_{x,\Omega}$, respectively.

(4.1.1) DEFINITION. Let M be a ∇-module of rank r over $\mathcal{R}$.

(1) We say that a basis $\{e_i\}$ of M is ∇-rational if and only if the associated differential operator $L_{M,\boldsymbol{e}}$ (2.3) belongs to $M_r(\Omega(x)[\delta_x])$. We say M has a connection with rational coefficients if and only if there exists a ∇-rational basis $\{e_i\}$ of M.

(2) We say that a basis e_i of M is ∇-formally rational if and only if the associated differential operator $L_{M,\boldsymbol{e}}$ belongs to $M_r(\mathcal{L}[\delta_x])$. We say M has a connection with formally rational coefficients if and only if there exists a ∇-formally rational basis $\{e_i\}$ of M.

Now we give a criterion of connection with rational coefficients.

(4.1.2) LEMMA. *Let L be a differential operator in $M_r(\mathcal{L}[\delta_x])$. If there is a matrix $P \in \mathrm{GL}_r(\mathcal{L})$ such that $P^{-1}LP$ is contained in $M_r(\mathcal{H}[\delta_x])$, then $P^{-1}LP$ is contained in $M_r(\Omega(x)[\delta_x])$.*

Proof. Since $P^{-1}LP$ belongs to both $M_r(\mathcal{L}[\delta_x])$ and $M_r(\mathcal{H}[\delta_x])$, the assertion follows from (2.2.2). $\square$

(4.2) We apply (4.1.2) to π-unipotent φ-∇-modules. The notations are as in Section 3.

(4.2.1) PROPOSITION. *Assume that the Frobenius σ satisfies the condition that $\sigma(x)/x^q \in \mathcal{H}^{(\dagger)}$. Let M be a π-unipotent φ-∇-module of rank r over $\mathcal{E}^{(\dagger)}$ with respect to σ and let $\{e_i\}$ be a π-unipotent basis of M. If $\{e_i\}$ is ∇-formally rational, then there is a matrix $Q \in \mathrm{GL}_r(S)$ such that Q is π-unipotent and that $\boldsymbol{e}Q$ is a strictly π-unipotent and ∇-rational basis of M. In particular, if $\{e_i\}$ is a strictly π-unipotent and ∇-formally rational basis, then $\{e_i\}$ is ∇-rational.*

Proof. By (3.2.3) there is a matrix $Q \in \mathrm{GL}_r(S)$ such that Q is π-unipotent and $\boldsymbol{e}Q$ is a strictly π-unipotent basis. By (3.2.14), $L_{M,\boldsymbol{e}Q} = Q^{-1}LQ$ is contained in $M_r(\mathcal{H}^{(\dagger)}[\delta_x])$. Therefore, the assertion follows (4.1.2). $\square$

Now fix a Frobenius σ on $\mathcal{E}^{(\dagger)}$ arbitrarily.

(4.2.2) COROLLARY. *Let M be a π-unipotent φ-∇-module of rank r over $\mathcal{E}^{(\dagger)}$ with respect to σ and let $\{e_i\}$ be a π-unipotent basis of M. If $\{e_i\}$ is ∇-formally rational, then there is a matrix $Q \in \mathrm{GL}_r(S)$ such that Q is π-unipotent and that $\boldsymbol{e}Q$ is a π-unipotent and ∇-rational basis of M.*

Proof. Choose a Frobenius σ_1 such that $\sigma_1(x)/x^q \in \mathcal{H}^{(\dagger)}$. By (3.4.4) there is a matrix $A_1 \in \mathrm{GL}_r(\mathcal{E}^{(\dagger)})$ such that

$$\delta_x(A_1) + C_{M,\boldsymbol{e}}A_1 = \mu(x,\sigma_1)A_1\sigma_1(C_{M,\boldsymbol{e}}),$$

and A_1 is π-unipotent. Define a Frobenius structure φ_1 on M by $\varphi(\boldsymbol{e}) = \boldsymbol{e}A_1$. Then the triple (M, φ_1, ∇) satisfies the assumption of (4.2.1). $\square$

5. Direct image of φ-∇-modules

In this section we define direct images of φ-∇-modules and show some properties of direct images.

(5.1) Fix a pair (Λ, k) as in (2.1.1). Denote by $\mathcal{E}_x^{(\dagger)}$ (resp. ω_x) one of the fields $\mathcal{E}_{x,K}$ and $\mathcal{E}_{x,K}^{\dagger}$ (resp. $\omega_{\mathcal{E}_x^{(\dagger)}}$). The residue class field of $\mathcal{E}_x^{(\dagger)}$ is $F_x = F_{x,k}$. Let $f\colon F_x \to F_y = F_{y,k'}$ be a finite separable extension in F_x^{alg}. Then the unique finite unramified extension over $\mathcal{E}_x^{(\dagger)}$ in $\widetilde{\mathcal{E}}$ (2.4), which corresponds to F_y, is isomorphic to $\mathcal{E}_y^{(\dagger)} = \mathcal{E}_{y,K'}^{(\dagger)}$ for some element y and for $K' = K(\Lambda, k')$. We also denote by f the inclusion $\mathcal{E}_x^{(\dagger)} \subset \mathcal{E}_y^{(\dagger)}$. The Frobenius σ (resp. δ_x) on $\mathcal{E}_x^{(\dagger)}$ extends uniquely on $\mathcal{E}_y^{(\dagger)}$. We have an isomorphism $\omega_y = \omega_{\mathcal{E}_y^{(\dagger)}} \cong \omega_x \otimes_{\mathcal{E}_x^{(\dagger)}} \mathcal{E}_y^{(\dagger)}$ by $\mathrm{d}y/y \mapsto \mathrm{d}x/x \otimes x/\delta_y(x)$.

Let M be a φ-module (resp. a ∇-module, resp. a φ-∇-module) over $\mathcal{E}_y^{(\dagger)}$. Denote by M_x the $\mathcal{E}_x^{(\dagger)}$-module M via the inclusion f. Define a Frobenius φ on M_x by φ itself (resp. a connection $\nabla\colon M_x \to \omega_x \otimes_{\mathcal{E}_x^{(\dagger)}} M_x$ by $\nabla(m) = \mathrm{d}x/x \otimes (x/\delta_y(x))\nabla(\delta_y)(m)$, where $\nabla(m) = \mathrm{d}y/y \otimes \nabla(\delta_y)(m)$). One can easily see that, for a φ-∇-module M over $\mathcal{E}_y^{(\dagger)}$, the diagram

$$\begin{tikzcd} M_x \arrow[r, "\nabla"] \arrow[d, "\varphi"'] & \omega_x \otimes_{\mathcal{E}_x^{(\dagger)}} M_x \arrow[d, "\sigma\otimes\varphi"] \\ M_x \arrow[r, "\nabla"'] & \omega_x \otimes_{\mathcal{E}_x^{(\dagger)}} M_x \end{tikzcd},$$

is commutative.

(5.1.1) PROPOSITION. *For a φ-module M over $\mathcal{E}_y^{(\dagger)}$, the pair (M_x, φ) is a φ-module over $\mathcal{E}_x^{(\dagger)}$.*

Proof. To prove that the induced map $\sigma^*(M_x) \to M_x$ by φ is bijective, it is enough to show that the natural map $\sigma^*(M_x) \to \sigma^* M$ is bijective. Here $\sigma^* M_x$ (resp. $\sigma^* M$) is the scalar extension of M_x (resp. M) by σ on $\mathcal{E}_x^{(\dagger)}$ (resp. $\mathcal{E}_y^{(\dagger)}$). One can easily see that the lemma below implies the assertion.

(5.1.2) LEMMA. *The $\mathcal{O}_{\mathcal{E}_x}^{(\dagger)}$-homomorphism $\mathrm{id}_{\mathcal{O}_{\mathcal{E}_x}^{(\dagger)}} \otimes \sigma\colon \sigma^*((\mathcal{O}_{\mathcal{E}_y}^{(\dagger)})_x) \to \mathcal{O}_{\mathcal{E}_y}^{(\dagger)}$ is bijective. Here we denote by $(\mathcal{O}_{\mathcal{E}_y}^{(\dagger)})_x$ the natural $\mathcal{O}_{\mathcal{E}_x}^{(\dagger)}$-module $\mathcal{O}_{\mathcal{E}_y}^{(\dagger)}$.*

Proof. Denote by σ_q the qth power map. Consider the perfection of F_x and F_y and dimensions over F_x, $\sigma_q^*(F_y)_x \to F_y$ is injective, hence bijective. The assertion holds by Nakayama's Lemma. □

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We know that the φ-module M over $\mathcal{E}^{(\dagger)}$ is etale if and only if there is a sub $\mathcal{O}_{\mathcal{E}}^{(\dagger)}$-module L of M such that $\varphi(L)$ is included in L and generates L over $\mathcal{O}_{\mathcal{E}}^{(\dagger)}$.

(5.1.3) PROPOSITION. *If M is an etale φ-module M on $\mathcal{E}_y^{(\dagger)}$, then (M_x, φ) is also etale.*

Proof. Denote by L_x the $\mathcal{O}_{\mathcal{E}_x}^{(\dagger)}$-module L via the inclusion f. Let L be a sub $\mathcal{O}_{\mathcal{E}}^{(\dagger)}$-module of M such that the induced map $\mathrm{id} \otimes \varphi \colon \sigma^* L \to L$ is isomorphic. It is easy that we have only to show the natural map $\sigma^* L_x \to \sigma^* L$ is bijective, where $\sigma^* L_x$ (resp. $\sigma^* L$) is the scalar extension of L_x (resp. L) by σ on $\mathcal{O}_{\mathcal{E}_x}^{(\dagger)}$ (resp. $\mathcal{O}_{\mathcal{E}_y}^{(\dagger)}$). This follows from (5.1.2). $\square$

(5.1.4) REMARK. The converse of (5.1.3) is also true since M is naturally embedded into the etale φ-module $f^* M_x$. (See (5.1.8).)

By (5.1.1) we define direct image functors

$$f_* \underline{\mathbf{M\Phi}}_{\mathcal{E}_y^{(\dagger)}, \sigma} \to \underline{\mathbf{M\Phi}}_{\mathcal{E}_x^{(\dagger)}, \sigma},$$
$$f_* \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}_y^{(\dagger)}, \sigma} \to \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}_x^{(\dagger)}, \sigma},$$

for the morphism f by $f_* M = (M_x, \varphi, (\nabla))$ as above. If we restrict f_* to the etale object, we get

$$f_* \underline{\mathbf{M\Phi}}^{\mathrm{et}}_{\mathcal{E}_y^{(\dagger)}, \sigma} \to \underline{\mathbf{M\Phi}}^{\mathrm{et}}_{\mathcal{E}_x^{(\dagger)}, \sigma},$$
$$f_* \underline{\mathbf{M\Phi}}^{\nabla \mathrm{et}}_{\mathcal{E}_y^{(\dagger)}, \sigma} \to \underline{\mathbf{M\Phi}}^{\nabla \mathrm{et}}_{\mathcal{E}_x^{(\dagger)}, \sigma},$$

by (5.1.3).

We denote by ν the natural functor $\underline{\mathbf{M\Phi}}_{\mathcal{E}^\dagger} \to \underline{\mathbf{M}}\mathbf{\Phi}_{\mathcal{E}}$. Since $\mathcal{E}_y \cong \mathcal{E}_x \otimes_{\mathcal{E}_x^\dagger} \mathcal{E}_y^\dagger$, we have

(5.1.5) PROPOSITION. *The functor f_* commutes with ν.*

Let $g \colon F_y \to F_z$ be a finite separable extension in F_x^{alg} and denote by $\mathcal{E}_z^{(\dagger)}$ the finite unramified extension in $\widetilde{\mathcal{E}}$ with the residue class field F_z. By our definition we have

(5.1.6) PROPOSITION. $(gf)_* = f_* g_*$.

Denote by $\mathcal{C}_?$ one of the categories $\underline{\mathbf{M\Phi}}_{\mathcal{E}_?^{(\dagger)}}$, $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}_?^{(\dagger)}}$, $\underline{\mathbf{M\Phi}}_{\mathcal{E}_?^{(\dagger)}}$, and $\underline{\mathbf{M\Phi}}^{\nabla \mathrm{et}}_{\mathcal{E}_?^{(\dagger)}}$ for $? = x, y$.

(5.1.7) PROPOSITION. *Let M be an object in $\mathcal{C}_y$. Then we have*

$$f_*(M^\vee) \cong (f_* M)^\vee.$$

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Proof. Define a map

$$f_*(M^\vee) \bigotimes_{\mathcal{E}_x^{(\dagger)}} f_*M \to \mathcal{E}_x^{(\dagger)},$$

by $m_1 \otimes m_2 \mapsto \mathrm{trace}_{\mathcal{E}_y^{(\dagger)}/\mathcal{E}_x^{(\dagger)}}((m_1, m_2)_{\mathcal{C}_y})$. Here we denote by $\mathcal{E}_x^{(\dagger)}$ the trivial object in $\mathcal{C}_x$ and $(\ ,\ )_{\mathcal{C}_y}$ is the non-degenerate pairing on $M^\vee \otimes_{\mathcal{E}_y^{(\dagger)}} M$ into $\mathcal{E}_y^\dagger$. Since $\mathcal{E}_y^{(\dagger)}/\mathcal{E}_x^{(\dagger)}$ is a finite separable extension and by (5.1.2), the above pairing is a morphism in $\mathcal{C}_x$ and non-degenerate. □

For an object M of $\mathcal{C}_y$ (resp. $\mathcal{C}_x$) the natural map $f^*f_*M \to M$ (resp. $M \to f_*f^*M$), which is defined by $a \otimes m \mapsto am$ (resp. $m \mapsto 1 \otimes m$), is a morphism of $\mathcal{C}_y$ (resp. $\mathcal{C}_x$). By the standard arguments we have

(5.1.8) PROPOSITION. *The functors f^* and f_* are adjoint. In other words, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{C}_y}(f^*M_1, M_2) \cong \mathrm{Hom}_{\mathcal{C}_x}(M_1, f_*M_2).$$

By definition we have

(5.1.9) PROPOSITION. *The functors f^* and f_* commute with the functor $\iota_{\Lambda'/\Lambda}$* (3.1.6).

(5.2) We describe the relation between the functor of direct images and the functor D_σ (resp. $\mathrm{D}_\sigma^\dagger$).

(5.2.1) THEOREM. *Let V be an object in $\underline{\mathbf{Rep}}_\Lambda(G_{F_y})$* (resp. $\underline{\mathbf{Rep}}_\Lambda^{(\mathrm{fin})}(G_{F_y})$) *and put $M = \mathrm{D}_\sigma(V)$* (resp. $M = \mathrm{D}_\sigma^\dagger(V)$). *Then we have*

$$\mathrm{D}_\sigma(f_*V) \cong f_*(M) \qquad (\text{resp. } \mathrm{D}_\sigma^\dagger(f_*V) \cong f_*(M)).$$

Here f_*V is the induced representation $\Lambda[G_{F_x}] \otimes_{\Lambda[G_{F_y}]} V$ of V.

Proof. Since $\underline{\mathbf{M\Phi}}_{\mathcal{E}^\dagger,\sigma}^{\nabla\mathrm{et}}$ is a full subcategory of $\underline{\mathbf{M\Phi}}_{\mathcal{E},\sigma}^{\mathrm{et}}$ [TN1, (4.2.1)], it is sufficient to show the case of D_σ. Since $M = \mathrm{D}_\sigma(V)$, there is a canonical pairing

$$(\ ,\ )_y \colon M^\vee \bigotimes_\Lambda V \to \widehat{\mathcal{E}}^{un},$$

which is $\mathcal{E}_y$-linear, Frobenius- and G_{F_y}-equivariant and non-degenerate. Here Frobenius acts by $\varphi_{M^\vee} \otimes \mathrm{id}$ on the left-hand side and σ on the right-hand side, and $\tau \in G_{F_y}$ acts $\mathrm{id} \otimes \tau$ on the left-hand side. We define a pairing

$$(\ ,\ )_x \colon f_*(M^\vee) \bigotimes_\Lambda f_*V \to \widehat{\mathcal{E}}^{un},$$

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by $(m, g \otimes v)_x = g(m, v)_y$ for $m \in f_*(M^\vee) = M^\vee$ and for $g \otimes v \in f_*V = \Lambda[G_{F_x}] \otimes_{\Lambda[G_{F_y}]} V$. Since $(m, gh \otimes h^{-1}v)_x = gh(m, h^{-1}v)_y = g(m, v)_y$ for $h \in G_{F_y}$, $(\ ,\ )_x$ is well-defined. One can easily see that the pairing $(\ ,\ )_x$ is $\mathcal{E}_x$-linear, Frobenius- and G_{F_y}-equivariant, and non-degenerate. Therefore, we obtain

$$\mathrm{D}_\sigma(f_*V) \cong (f_*(M^\vee))^\vee \cong f_*(M),$$

by (5.1.7).

(5.3) Put $F = F_{x,k}$ and $\mathcal{E}^{(\dagger)} = \mathcal{E}^{(\dagger)}_{x,K}$. In the rest of this section we study direct images of φ-∇-modules for a morphism $f\colon \mathcal{E}^{(\dagger)} \to \mathcal{E}^{(\dagger)'}$ which corresponds to a finite separable extension $f\colon F \to F'$.

First we discuss the unramified case. In this case x is also a uniformizer of F' and, if we denote by k' the residue class field of F' and put $K' = K(\Lambda, k')$. Then $F' = F_{x,k'}$, $\mathcal{E}^{(\dagger)'}$ is isomorphic to $\mathcal{E}^{(\dagger)}_{x,K'}$ and f is the natural injection. In this situation, if we denote by $u_1, \ldots, u_{[k':k]}$ a basis of $W(k')$ over $W(k)$, the subfield $K'(x)$ (resp. $\mathcal{L}_{x,K'}$) of $\mathcal{E}^{(\dagger)}_{x,K'}$ is an extension of degree $[k':k]$ over the subfield $K(x)$ (resp. $\mathcal{L}_{x,K}$) of $\mathcal{E}^{(\dagger)}_{x,K}$ with a basis $\{u_j\}$.

(5.3.1) PROPOSITION. *Under the above situation, let M be a ∇-module over $\mathcal{E}^{(\dagger)}_{x,K'}$. If $\{e_i\}$ is a ∇-rational basis* (resp. *a ∇-formally rational basis*), *then $\{u_je_i\}$ is a ∇-rational basis* (resp. *a ∇-formally rational basis*) *of f_*M.*

(5.4) From now we see the case of a totally ramified extension F'/F. Hence the field of coefficients of differential structures is always K and we omit K from the notation. First we forget the extension F' over F and discuss in the following situation. Let $f\colon \mathcal{E}^{(\dagger)} \to \mathcal{E}^{(\dagger)'} = \mathcal{E}^{(\dagger)}_{y,K}$ be a morphism with $f(x) = a(y) = \sum_{n=d}^\infty a_ny^n \in S_y$ $(a_d \in O_K^\times)$ for some positive integer d. One can easily see that the field $\mathcal{L}_y$ includes the field $\mathcal{L}_x$ and there is a relation

$$y^d + b_1y^{d-1} + \cdots + b_d = 0 \quad (b_i \in S_x \ \text{ for all } \ i),$$

such that $|b_d|_G = 1$ and $b_d = -a_d^{-1}x +$ (higher terms on x). The above equation modulo π is an Eisenstein polynomial, so that the extension of residue class fields is totally ramified of degree d or an inseparable extension.

(5.4.1) LEMMA *Under the above situation, $\mathcal{L}_y$ is a finite extension of degree d over $\mathcal{L}_x$.*

Proof. Let $\mathcal{M} = \mathcal{L}_x + \mathcal{L}_xy + \cdots + \mathcal{L}_xy^{d-1}$ in $\mathcal{L}_y$. By the above relation for y $\mathcal{M}$ is a domain. Since $\mathcal{M}$ is finite over the field $\mathcal{L}_x$, $\mathcal{M}$ is a field. One can easily see that S_y is contained in $\mathcal{M}$. Hence, we have $\mathcal{L}_y = \mathcal{M}$. □

(5.4.2) REMARK. In general, $\mathcal{L}_y$ is not a Galois extension over $\mathcal{L}_x$ even if F_y is a Galois extension over F_x. For examples, if F_y/F_x is an extension with $(y^{-1})^p -$

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$y^{-1} = x^{-1}$, then $\mathcal{L}_y$ is not a Galois extension over $\mathcal{L}_x$. Put $u_\tau = y\tau(y^{-1})$ for $\tau \in \mathrm{Gal}(F_y/F_x)$. Then $u_\tau^p - y^{p-1}u_\tau = 1 - y^{p-1}$. If u_τ is contained in $\mathcal{L}_y$, u_τ is contained in $S_{y,K}$ by the normality. One can easily see that we find only 1 which is a solution of the equation $u_\tau^p - y^{p-1}u_\tau = 1 - y^{p-1}$ for u_τ in $S_{y,K}$.

(5.4.3) PROPOSITION. *Under the notation as above, let M be a ∇-module over $\mathcal{E}_y^{(\dagger)}$. If $\{e_i\}$ is a ∇-formally rational basis, then $\{y^j e_i\}$ is a ∇-formally rational basis of f_*M.*

Proof. Let L_y be a $\mathcal{L}_y$-subspace of M which is generated by e_i, then L_y is stable under the connection ∇. Denote by L_x the $\mathcal{L}_y$-space L_y as an $\mathcal{L}_x$-module . Then $\{y^j e_i\}$ generates L_x over $\mathcal{L}_x$ by (5.3.2) and $\mathcal{E}_x^{(\dagger)} \otimes_{\mathcal{L}_x} L_x = f_*M$. Since $x/\delta_y(x)$ is contained in $\mathcal{L}_y$ (2.2.3), $\nabla(L_x) \subset \omega_{\mathcal{L}_x} \otimes_{\mathcal{L}_x} L_x$.

(5.4.4) REMARK. In the case that the extension F'/F is an inseparable extension, we can define the direct image under the assumption on $a(y)$ and (5.4.1–3) also make sense.

(5.5) Assume that the extension F'/F is totally tamely ramified of degree N $(p \not| N)$. In this case we can choose a uniformizer of y in F' with $x = uy^N$ for some $u \in k^\times$. (If k is algebraically closed, then we can choose y such that $u = 1$.) If we denote by $\widehat{u}$ a lifting of u in O_K, there is an element $y \in O_{\mathcal{E}'}^{(\dagger)}$ which satisfies the equation $x = \widehat{u}y^N$ since $O_{\mathcal{E}'}^{(\dagger)}$ is henselian. Hence, $\mathcal{E}^{(\dagger)'}$ (resp. F') is canonically isomorphic to $\mathcal{E}_y^{(\dagger)}$ (resp. F_y) and f is given by $x \mapsto \widehat{u}y^N$ $(a(y) = \widehat{u}y^N$ in the notation of (5.4)). Moreover, the subfield $K(y)$ in $\mathcal{E}^{(\dagger)'}$ is an extension of degree N over the subfield $K(x)$ in $\mathcal{E}^{(\dagger)}$ with a basis $1, y, \ldots, y^{N-1}$.

(5.5.1) PROPOSITION. *Let M be a ∇-module over $\mathcal{E}_x^{(\dagger)}$. If $\{e_i\}$ is a ∇-rational basis* (resp. *a ∇-formally rational basis*), *then $\{y^j e_i\}$ is a ∇-rational basis* (resp. *a ∇-formally rational basis*) *of f_*M.*

(5.6) We consider an Artin–Schreier extension F'/F of degree p with a totally wild ramification. Then $F' = F(z)$ with

$$z^p - z = ux^{-N} + (\text{a polynomial of degree } N-1 \text{ in } k[x^{-1}]),$$

for some $u \in k^\times$ and a positive integer N $(p \not| N)$. Since $O_{\mathcal{E}'}^{(\dagger)}$ is henselian, there is a lifting $z \in O_{\mathcal{E}'}^{(\dagger)}$ which satisfies the relation

$$z^p - z = \widehat{u}x^{-N} + (\text{a polynomial of degree } N-1 \text{ in } O_K[x^{-1}]),$$

where the right-hand side of the equation is a lifting of the equation above into $O_K[x^{-1}]$. Denote by v (resp. $\widehat{v}$) the unique element in k with $v^p = u$ (resp. a lifting of v in O_K). Since the valuation of z is $-N/p$ in F' $(\mathrm{ord}\,(x) = 1)$, there is an element y in F' such that $y^N = v^{-1}z^{-1}$ and $x \equiv y^p \pmod{y^{p+1}}$ by an easy

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computation. So there is an element y in $O_{\mathcal{E}'}^{(\dagger)}$ which satisfies the conditions that $y^N = \widehat{v}^{-1} z^{-1}$ and that $x \equiv y^p \pmod{(y^{p+1}, \pi)}$. Hence, $F' = F_y$, $\mathcal{E}^{(\dagger)'} = \mathcal{E}_y^{(\dagger)}$ and f is given by $x \mapsto a(y) = y^p w(y)$ for $w(y) \equiv 1 + (Nv^{p-1})^{-1} y^{N(p-1)} + O_K[y^p] \pmod{(y^{N(p-1)+1}, \pi)}$.

(5.6.1) LEMMA. *Under the notation as above, let M be a φ-module over $\mathcal{E}_y^{(\dagger)}$ with a π-unipotent basis $\{e_i\}$. Then, $e_1, ze_1, \ldots, z^{p-1}e_1, e_2, ze_2, \ldots, z^{p-1}e_r$ is a π-unipotent basis of f_*M.*

Proof. Put $A_{M,\boldsymbol{e}} = (a_{ij})$ and $s(x) = \widehat{u}x^{-N} + \cdots$. Then,

$$\begin{aligned}
\varphi(z^k e_j) &\equiv z^{pk} e_j + \sum_{i=1}^{j-1} z^{pk} a_{ij} e_i \\
&\equiv (z + s(x))^k e_j + \sum_{i=1}^{j-1} z^{pk} a_{ij} e_i \\
&\equiv z^k e_j + (\text{lower order terms}) \pmod{\pi}.
\end{aligned}$$

(5.6.2) LEMMA. *Under the notation as in* (5.6)*, let M be a ∇-module over $\mathcal{E}_y^{(\dagger)}$ with a ∇-formally rational basis $\{e_i\}$. Then the basis $e_1, ze_1, \ldots, z^{p-1}e_r$ is a ∇-formally rational basis of f_*M.*

Proof. Since $1, z, \ldots, z^{p-1}$ is a basis of $\mathcal{L}_{y,K}$ over $\mathcal{L}_{x,K}$, the assertion follows (5.4.1) and (5.4.3). □

From (5.6.1), (5.6.2), (4.2.2) and (4.2.1) we have

(5.6.3) PROPOSITION. *Under the notation as in* (5.6)*, let M be a φ-∇-module of rank r over $\mathcal{E}_y^{(\dagger)}$ with a π-unipotent and ∇-formally rational basis $\{e_i\}$. Then there is a matrix $Q \in \mathrm{GL}_{pr}(S_x)$ such that $(z^*\boldsymbol{e})Q$ is a π-unipotent and ∇-rational basis of f_*M, where $z^*\boldsymbol{e}$ is the basis of f_*M as in* (5.6.1)*. In particular, assume furthermore that the Frobenius σ satisfies the condition that $\sigma(x)/x^q \in \mathcal{H}_x^{(\dagger)}$, then there is a matrix $Q \in \mathrm{GL}_{pr}(S_x)$ such that $(z^*\boldsymbol{e})Q$ is a strictly π-unipotent and ∇-rational basis of f_*M.*

6. Calculation of the difference of local indices for direct images

In this section we study the behavior of local indices for direct images. The theory of the local index was studied by Adolphson, Dwork, Robba and many people. Applying their theory to our situation, we calculate the difference of local indices for direct images of differential operators.

(6.1) Following the notation of [Ro1] [Ro2], for Ω-vector spaces E and F and for Ω-homomorphism $L\colon E \to F$, we say L has an index if and only if both the kernel

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and the cokernel of L are finite dimensional Ω-vector spaces. If L has an index, we define the index $\chi_\Omega(L, E, F)$ ($\chi_\Omega(L, E)$ if $E = F$) by $\dim_\Omega \ker L - \dim_\Omega \operatorname{coker} L$.

(6.2) Let Ω be a complete field of characteristic 0 under a non-Archimedean absolute value $|\ |$. Let Γ be a subgroup of the group of continuous automorphism of Ω and denote by Ω^Γ the Γ-invariant subfield of Ω. Then Ω^Γ is complete. Define an action of Γ on $\mathcal{R}_{x,\Omega}$ by $\tau(\sum a_n x^n) = \sum \tau(a_n)x^n$ for $\tau \in \Gamma$. The action of Γ commutes with the ring structures and the derivation δ_x and preserves the subrings $\mathcal{A}_{x,\Omega}$ and $\mathcal{H}^\dagger_{x,\Omega}$. The Γ-invariant subalgebra $\mathcal{R}^\Gamma_{x,\Omega}$ (resp. $\mathcal{A}^\Gamma_{x,\Omega}$, resp. $(\mathcal{H}^\dagger_{x,\Omega})^\Gamma$) is $\mathcal{R}_{x,\Omega^\Gamma}$ (resp. $\mathcal{A}_{x,\Omega^\Gamma}$, resp. $\mathcal{H}^\dagger_{x,\Omega^\Gamma}$). Define an action on $(\mathcal{R}_{x,\Omega})^r$ of Γ by $\tau({}^t(a_1, \ldots, a_r)) = {}^t(\tau(a_1), \ldots, \tau(a_r))$. Denote by R one of $(\mathcal{A}_{x,\Omega})^r$, $(\mathcal{H}^\dagger_{x,\Omega})^r$ and $(\mathcal{R}_{x,\Omega})^r$ and by R^Γ the corresponding $(\mathcal{A}_{x,\Omega^\Gamma})^r$, $(\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r$ or $(\mathcal{R}_{x,\Omega^\Gamma})^r$, respectively.

(6.2.1) LEMMA. *Let L be a first order differential operator in $M_r(R^\Gamma[\delta_x])$. If the dimension of* $\ker(L, R^r)$ *is finite over Ω, then the natural map* $\Omega \otimes_{\Omega^\Gamma} \ker(L, (R^\Gamma)^r) \to \ker(L, R^r)$ *of Ω-vector spaces is an isomorphism and the natural map* $\Omega \otimes_{\Omega^\Gamma} \operatorname{coker}(L, (R^\Gamma)^r) \to \operatorname{coker}(L, R^r)$ *of Ω-vector spaces is an injection. In particular, if L has an index in R^r, then L has an index in $(R^\Gamma)^r$ and we have*

$$\chi_{\Omega^\Gamma}(L, (R^\Gamma)^r) \geqslant \chi_\Omega(L, R^r).$$

Proof. Let $u_k = {}^t(u_{k,1}, \ldots, u_{k,r})$ $(1 \leqslant k \leqslant s)$ be a basis of $\ker(L, R^r)$ over Ω. After an elementary modification, we may assume that there is a sequence of pairs $(k_1, l_1), (k_2, l_2), \ldots, (k_s, l_s)$ of integers such that $(k_i, l_i) \neq (k_j, l_j)$ for $i \neq j$ and that the l_i-th coefficient of u_{i,k_i} is 1 for all i and that the l_j-th coefficient of u_{i,k_j} is 0 for all $j \neq i$. Since $\tau(u_i)$ is also a solution of the differential operator L, $\tau(u_i)$ is expressed by a linear sum of u_j over Ω. Comparing coefficients, we have $\tau(u_i) = u_i$ for all $\tau \in \Gamma$ by the assumption, hence $u_i \in \ker(L, (R^\Gamma)^r)$ for all i. We have shown the surjectivity of the map between kernels. The injectivity is easy.

Assume that for an element $a \in (R^\Gamma)^r$ there is an element $b = {}^t(b_1, \ldots, b_r) \in R^r$ such that $a = L(b)$. We may assume that the l_i-th coefficient of b_{k_i} is 0 for all $1 \leqslant i \leqslant s$. We know that $\tau(b) - b$ is a linear sum of $u_1, \ldots, u_s$ over Ω for each $\tau \in \Gamma$. Comparing coefficients, we have $\tau(b) - b = 0$ for all $\tau \in \Gamma$. Hence, b is contained in $(R^\Gamma)^r$ and the map $\operatorname{coker}(L, (R^\Gamma)^r) \to \operatorname{coker}(L, R^r)$ is an injection. Let $v_1, \ldots, v_t$ be a system of representation of $\operatorname{coker}(L, (R^\Gamma)^r)$. Assume that, renumbering if we need, $\alpha_1 v_1 + \cdots + \alpha_{t'} v_{t'} = 0$ $(\alpha \in \Omega)$ is a minimal relation of v_i in $\operatorname{coker}(L, R^r)$. Then, one of α_i/α_1 is not contained in Ω^Γ. Since Ω^Γ is the Γ-invariant field of Ω, there is a relation of length less than t'. Hence, we have proven the assertion. □

(6.2.2) LEMMA. *Under the assumption of* (6.2.1) *and assume furthermore that Γ is a finite group. If a differential operator $L \in M_r(R^\Gamma[\delta_x])$ has an index in R^r, then L has an index in $(R^\Gamma)^r$ and we have*

$$\chi_{\Omega^\Gamma}(L, (R^\Gamma)^r) = \chi_\Omega(L, R^r).$$

Proof. Since $H^1(\Gamma, \mathrm{GL}_t(\Omega^\times)) = \{1\}$ for any positive integer t [Sel, X.1.Prop. 3.], we can choose a system $v_1, v_2, \ldots, v_t$ of elements in R^r such that its image is a basis of coker$(L, (R^\Gamma)^r)$ and that $\tau(v_i) - v_i$ is contained in the image of L for all $\tau \in \Gamma$ and for all i. So the image of $\{\sum_{\tau\in\Gamma} \tau(v_1), \ldots, \sum_{\tau\in\Gamma} \tau(v_t)\}$ is a basis of coker(L, R^r). On the other hand, $\sum_{\tau\in\Gamma} \tau(v_i)$ is contained in $(R^\Gamma)^r$ for each i. The assertion follows from (6.2.1). □

(6.2.3) REMARK. If Ω_0 is a complete subfield of Ω such that $[\Omega\colon \Omega_0] < \infty$, then the same assertion of (6.2.2) also holds. Pick a finite Galois extension Ω_1 of Ω_0 such that Ω is included in Ω_1. Then, apply (6.2.2) to both extensions Ω_1/Ω and Ω_1/Ω_0.

(6.3) Keep the notation in (6.2) and assume furthermore Ω is algebraically closed and complete under a valuation which is an extension of that on $\mathbf{Q}_p$. We recall the fundamental results of indices in [Ro1] [Ro2] and restate them in our context. For an element $a = \sum a_n x^n$ in $\mathcal{R}_{x,\Omega}$, put

$$\mathrm{ord}_x^-(a) = \min\{n \mid |a_n| = |a|_G\},$$

when the minimum exists. ($|\ |_G$ is defined in (2.2).)

(6.3.1) LEMMA. *Let $s \neq 0$ be an element of $\mathcal{A}_{x,\Omega^\Gamma}$. Then s has an index in $\mathcal{A}_{x,\Omega^\Gamma}$ if and only if there is a unit u of $\mathcal{A}_{x,\Omega^\Gamma}$ such that $su \in \Omega^\Gamma[x]$. If s has an index in $\mathcal{A}_{x,\Omega^\Gamma}$,* ord$_x^-(s)$ *can be defined and we have*

$$\chi_{\Omega^\Gamma}(s, \mathcal{A}_{x,\Omega^\Gamma}) = -\mathrm{ord}_x^-(s).$$

In particular, if Ω^Γ is a complete discrete valuation field under the absolute value $|\ |$, then, for $s \in \mathcal{A}_{x,\Omega^\Gamma}$, s has an index in $\mathcal{A}_{x,\Omega^\Gamma}$ if and only if s is contained in $S_{x,\Omega^\Gamma}[p^{-1}]$.

Proof. For $s \in \mathcal{A}_{x,\Omega^\Gamma}$, if there is a unit u of $\mathcal{A}_{x,\Omega^\Gamma}$ such that $su \in \Omega^\Gamma[x]$, then there exists only finitely many zero points of s in $\mathbf{B}(0, 1^-) = \{z \in \Omega \mid |z| < 1\}$. By the theory of Newton polygon of formal power series [DGS, II.2.], ord$_x^-(s)$ is defined and ord$_x^-(su) = $ ord$_x^-(s)$. We may assume that s is a monic polynomial such that deg$(s) = $ ord$_x^-(s)$ by the definition of Ω^Γ. In the case that $\Gamma = \{1\}$, the assertion holds by [Ro2, 3.4.]. Since $\{1, x, \ldots, x^{\mathrm{ord}_x^-(s)-1}\}$ is a representation system of coker$(s, \mathcal{A}_{x,\Omega})$, s has an index in $\mathcal{A}_{x,\Omega^\Gamma}$ and

$$\chi_{\Omega^\Gamma}(s, \mathcal{A}_{x,\Omega^\Gamma}) = \chi_\Omega(s, \mathcal{A}_{x,\Omega}) = -\mathrm{ord}_x^-(s),$$

by (6.2.1). Assume that s has an index in $\mathcal{A}_{x,\Omega^\Gamma}$. Then the intersection of $\Omega^\Gamma[x]$ and $s\mathcal{A}_{x,\Omega^\Gamma}$ contains nonzero elements. Let su ($u \in O^{(\dagger)}_{\mathcal{A}_{x,\Omega^\Gamma}}$) be a nonzero polynomial of minimal degree. By the minimality of degree, u has no zero in $\mathbf{B}(0, 1^-)$ and u has no minus slopes by [DGS, II.Thm.2.1]. Hence, u is a unit. □

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(6.3.2) LEMMA. *For an element $s \neq 0$ of $\mathcal{H}^\dagger_{x,\Omega^\Gamma}$, s has an index in $\mathcal{H}^\dagger_{x,\Omega^\Gamma}$ and we have*

$$\chi_{\Omega^\Gamma}(s, \mathcal{H}^\dagger_{x,\Omega^\Gamma}) = \mathrm{ord}^-_x(s).$$

Proof. We may assume that $|s|_G = 1$. Put $\alpha = \mathrm{ord}^-_x(s)$. Define $pr_\alpha \colon \mathcal{H}^\dagger_{x,\Omega^\Gamma} \to \mathcal{H}^\dagger_{x,\Omega^\Gamma}$ by $pr_\alpha(\sum_{n=0}^{-\infty} a_n x^n) = \sum_{n=\alpha}^{-\infty} a_n x^n$ and put $\Omega^\Gamma[x^{-1}]^{>\alpha} = \Omega^\Gamma \oplus \cdots \oplus \Omega^\Gamma x^{\alpha+1}$. Consider the following diagram

$$\begin{tikzcd}
 & & \mathcal{H}^\dagger_{x,\Omega^\Gamma} \arrow[r, equal] \arrow[d, "s"'] & \mathcal{H}^\dagger_{x,\Omega^\Gamma} \arrow[d, "pr_\alpha \circ s"] & \\
0 \arrow[r] & \Omega^\Gamma[x^{-1}]^{>\alpha} \arrow[r] & \mathcal{H}^\dagger_{x,\Omega^\Gamma} \arrow[r, "pr_\alpha"'] & x^\alpha \mathcal{H}^\dagger_{x,\Omega^\Gamma} \arrow[r] & 0\,.
\end{tikzcd}$$

To show the assertion, we have only to prove the right vertical arrow is bijective. The injectivity can be easily shown. Now we will show the surjectivity. Put $s = \sum s_i x^i$ and choose $0 < \gamma < 1$ such that $|s_i| \leqslant \gamma^{\alpha-i}$. By definition of α, there is an element u_n of degree $n-\alpha$ in $O_{\Omega^\Gamma}[x^{-1}]$ such that $pr_\alpha(su_n) = x^n +$ (lower term) and that the absolute value of the ith coefficient of $pr_\alpha(su_n)$ is less than or equal to γ^{n-i} for all $n \leqslant \alpha$. For any element $\sum b_i x^i \in x^\alpha \mathcal{H}^\dagger_{x,\Omega^\Gamma}$ such that there is a $0 < \delta < 1$ with $|b_i| \leqslant \delta^{-i}$ for all i, define c_n $(n \leqslant \alpha)$ by b_α if $n = \alpha$ and by the nth coefficient of $\sum b_i x^i - pr_\alpha(s \sum_{i=\alpha}^{n+1} c_i u_i)$ for $n < \alpha$. One can easily see that $|c_n| \leqslant \max\{\gamma, \delta\}^{\alpha-n}$ for all n. Therefore, $\sum c_n u_n$ is convergent in $\mathcal{H}^\dagger_{x,\Omega^\Gamma}$ and $pr_\alpha(s \sum c_n u_n) = \sum b_n x^n$. Therefore, $pr_\alpha \circ s$ is surjective. □

(6.3.3) REMARK. The number of $\mathrm{ord}^-_x(s)$ in (6.3.1) (resp. (6.3.2)) is the number of zeros of s in $\mathbf{B}(0, 1^-)$ (resp. $\mathbf{B}(\infty, 1^+) = \{z \in \Omega |\ |t| \geqslant 1\} \cup \{\infty\}$).

Let L be a differential operator in $M_r(\mathcal{R}_{x,\Omega^\Gamma}[\delta_x])$. We say L has an index in $(\mathcal{A}_{x,\Omega^\Gamma})^r$ (resp. $(\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r$) if and only if there is an element $s \in \Omega^\Gamma[x]$ (resp. $\Omega^\Gamma[x^{-1}]$) such that $sL \in M_r(\mathcal{A}_{x,\Omega^\Gamma}[\delta_x])$ (resp. $M_r(\mathcal{H}^\dagger_{x,\Omega^\Gamma}[\delta_x])$) and that sL has an index. We define an index by

$$\chi_{\Omega^\Gamma}(L, (\mathcal{A}_{x,\Omega^\Gamma})^r) = \chi_{\Omega^\Gamma}(sL, (\mathcal{A}_{x,\Omega^\Gamma})^r) - \chi_{\Omega^\Gamma}(s, (\mathcal{A}_{x,\Omega^\Gamma})^r),$$

$$(\text{resp. } \chi_{\Omega^\Gamma}(L, (\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r) = \chi_{\Omega^\Gamma}(sL, (\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r) - \chi_{\Omega^\Gamma}(s, (\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r)).$$

The notion and the definition of index are independent of the choice of s by (6.3.1) (resp. (6.3.2)).

From [Ro2, 3.11.] we have

(6.3.4) LEMMA. *If a differential operator $L \in M_r(\Omega^\Gamma(x)[\delta_x])$ has indices in two of $(\mathcal{A}_{x,\Omega^\Gamma})^r$, $(\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r$ and $(\mathcal{R}_{x,\Omega^\Gamma})^r$, then L has an index in the third and we have*

$$\chi_{\Omega^\Gamma}(L, (\mathcal{R}_{x,\Omega^\Gamma})^r) = \chi_{\Omega^\Gamma}(L, (\mathcal{A}_{x,\Omega^\Gamma})^r) + \chi_{\Omega^\Gamma}(L, (\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r).$$

(6.3.5) DEFINITION. We say a differential operator $L \in M_r(\mathcal{R}_{x,\Omega^\Gamma}[\delta_x])$ has indices on Ω^Γ if and only if L has indices in all three of $(\mathcal{A}_{x,\Omega^\Gamma})^r$, $(\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r$ and $(\mathcal{R}_{x,\Omega^\Gamma})^r$.

From [Ro2, 3.14.] we have the following two lemmas.

(6.3.6) LEMMA. *Let L and L' be differential operators in $M_r(\mathcal{R}_{x,\Omega^\Gamma}[\delta_x])$ with a matrix Q in $\mathrm{GL}_r(\mathcal{R}_{x,\Omega^\Gamma})$ such that $L' = Q^{-1}LQ$.*

(1) *L has an index in $(\mathcal{R}_{x,\Omega^\Gamma})^r$ if and only if L' has an index in $(\mathcal{R}_{x,\Omega^\Gamma})^r$. If so, we have*

$$\chi_{\Omega^\Gamma}(L', (\mathcal{R}_{x,\Omega^\Gamma})^r) = \chi_{\Omega^\Gamma}(L, (\mathcal{R}_{x,\Omega^\Gamma})^r).$$

(2) *Assume that Q is contained in $\mathrm{GL}_r(\Omega^\Gamma(x) \otimes_{\Omega^\Gamma[x]} \mathcal{A}_{x,\Omega^\Gamma})$. Then, L has an index in $(\mathcal{A}_{x,\Omega^\Gamma})^r$ if and only if L' has an index in $(\mathcal{A}_{x,\Omega^\Gamma})^r$. If so, we have*

$$\chi_{\Omega^\Gamma}(L', (\mathcal{A}_{x,\Omega^\Gamma})^r) = \chi_{\Omega^\Gamma}(L, (\mathcal{A}_{x,\Omega^\Gamma})^r).$$

(3) *Assume that Q is contained in $\mathrm{GL}_r(\Omega^\Gamma(x) \otimes_{\Omega^\Gamma[x^{-1}]} \mathcal{H}^\dagger_{x,\Omega^\Gamma})$. Then, L has an index in $(\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r$ if and only if L' has an index in $(\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r$. If so, we have*

$$\chi_{\Omega^\Gamma}(L', (\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r) = \chi_{\Omega^\Gamma}(L, (\mathcal{H}^\dagger_{x,\Omega^\Gamma})^r).$$

(6.3.7) LEMMA. *Let L be a differential operator in $M_r(\Omega^\Gamma(x)[\delta_x])$ and let L' be a differential operator in $M_r(\Omega^\Gamma(x) \otimes_{\Omega^\Gamma[x]} \mathcal{A}_{x,\Omega^\Gamma}[\delta_x])$. Assume that there exists a matrix Q in $\mathrm{GL}_r(\mathcal{R}_{x,\Omega^\Gamma})$ such that $L' = Q^{-1}LQ$. L has indices on Ω^Γ if and only if L' has indices both in $\mathcal{R}_{x,\Omega^\Gamma}$ and in $\mathcal{A}_{x,\Omega^\Gamma}$. If so, the formulas*

$$\chi_{\Omega^\Gamma}(L', (\mathcal{R}_{x,\Omega^\Gamma})^r) = \chi_{\Omega^\Gamma}(L, (\mathcal{R}_{x,\Omega^\Gamma})^r),$$
$$\chi_{\Omega^\Gamma}(L', (\mathcal{A}_{x,\Omega^\Gamma})^r) = \chi_{\Omega^\Gamma}(L, (\mathcal{A}_{x,\Omega^\Gamma})^r),$$

hold. The assertion is also valid if we replace $\mathcal{A}$ into $\mathcal{H}^\dagger$.

(6.3.8) REMARK. (1) By [Ch, Cor.3.3] Q is decomposed into $x^{Q_{\mathrm{diag}}}Q_{\mathcal{H}}Q_{\mathcal{A}}$ for a diagonal matrix $Q_{\mathrm{diag}} \in M_r(\mathbf{Z})$, $Q_{\mathcal{H}} \in \mathrm{GL}_r(\mathcal{H}^\dagger_{x,\Omega^\Gamma})$ and $Q_{\mathcal{A}} \in \mathrm{GL}_r(\mathcal{A}_{x,\Omega^\Gamma})$. (Consider the Galois action defined in (6.2).)

(2) In Section 8 we use (6.3.7) when Q belongs to $\mathrm{GL}_r(\mathcal{E}^\dagger_{x,K})$ for a complete discrete valuation field K. In this case Q is decomposed into $Q_{\mathcal{H}}Q_{\mathcal{L}}$ such that

$Q_{\mathcal{H}} \in 1 + \pi x^{-1} M_r(O^{\dagger}_{\mathcal{H}_{x,K}})$ and $Q_{\mathcal{L}} \in \mathrm{GL}_r(\mathcal{L}_{x,K})$. We can prove (6.3.7) by the same method of [Ro2, 3.11] using this type of decomposition.

(6.4) Let K be a discrete valuation field of mixed characteristic $(0, p)$ with a perfect residue class field and denote by $\widehat{K}^{\mathrm{alg}}$ the completion of an algebraically closure of K under the valuation $|\ |$. Let Γ be a subgroup of the group of continuous automorphisms of $\widehat{K}^{\mathrm{alg}}$ over K and put $K(\Gamma) = (\widehat{K}^{\mathrm{alg}})^{\Gamma}$ the Γ-invariant subfield of $\widehat{K}^{\mathrm{alg}}$. Assume that $a(y) = \sum_{n=d}^{\infty} a_n y^n$ is an element in $S_{y,K}$ for a positive integer d and that $a_N \in O_K^{\times}$. Then the K-algebra homomorphism $f\colon \mathcal{R}_{x,K(\Gamma)} \to \mathcal{R}_{y,K(\Gamma)}$ (resp. $f\colon \mathcal{A}_{x,K(\Gamma)} \to \mathcal{A}_{y,K(\Gamma)}$) which is defined by $x \mapsto a(y)$ is injective and preserves the Gauss norm $|\ |_G$.

(6.4.1) LEMMA. *Via the injection f as above, we have*

(1) $\mathcal{A}_{y,K(\Gamma)} = \mathcal{A}_{x,K(\Gamma)} \oplus y\mathcal{A}_{x,K(\Gamma)} \oplus \cdots \oplus y^{d-1}\mathcal{A}_{x,K(\Gamma)}$;
(2) $\mathcal{R}_{y,K(\Gamma)} = \mathcal{R}_{x,K(\Gamma)} \oplus y\mathcal{R}_{x,K(\Gamma)} \oplus \cdots \oplus y^{d-1}\mathcal{R}_{x,K(\Gamma)}$.

Proof. (1) Consider the extension $K(\Gamma)[[x]] \to K(\Gamma)[[y]]$ $(x \mapsto a(y))$. If $s_0 + s_1 y + \cdots + s_{d-1}y^{d-1} = 0$ for some $s_i \in \mathcal{A}_{x,K(\Gamma)}$, then we have $s_0 = s_1 = \cdots = s_{d-1} = 0$. So the map from the right-hand side of (1) into $\mathcal{A}_{y,K(\Gamma)}$ is injective. Every element $t = \sum t_n y^n \in \mathcal{A}_{y,K(\Gamma)}$ is expressed by $s_0 + s_1 y + \cdots + s_{d-1}y^{d-1}$ for some $s_i = \sum s_{i,n}x^n \in K(\Gamma)[[x]]$. By the calculation of valuation, we have

$$|s_{i,n}| \leqslant \max\{|t_m|\ | m \leqslant nd + i\},$$

for all $n \geqslant 0$ inductively. Since $t \in \mathcal{A}_{y,K(\Gamma)}$, s_i is contained in $\mathcal{A}_{x,K(\Gamma)}$ for all i.

(2) If $s_0 + s_1 y + \cdots + s_{d-1}y^{d-1} = 0$ for some $s_i \in \mathcal{R}_{x,K(\Gamma)}$, then we have $s_i = 0$ for all i. Indeed, if one of s_i is not contained in $\mathcal{A}_{y,K(\Gamma)}$, then, considering the lowest nagative term of all s_i for x whose absolute value is largest among all absulute values of coefficients of negative power terms of x, we have a contradiction. So s_i is contained in $\mathcal{A}_{y,K(\Gamma)}$ and we have $s_i = 0$ for all i by (1). Therefore, the map from the right-hand side of (2) into $\mathcal{R}_{y,K(\Gamma)}$ is injective. For any element $t = \sum t_n y^n \in \mathcal{H}^{\dagger}_{y,K(\Gamma)}$, there are $s_i = \sum s_{i,n}x^n \in \mathcal{H}^{\dagger}_{x,K(\Gamma)}$ such that

$$t = s_0 + s_1 y + \cdots + s_{d-1}y^{d-1} + (\text{an element of } S_{x,K(\Gamma)}[p^{-1}]).$$

In fact, we construct s_i inductively on the power of y. Since t is contained in $\mathcal{H}^{\dagger}_{y,K(\Gamma)}$, each s_i can be defined in $\mathcal{H}_{x,K(\Gamma)}$ and the rest term is convergent in $S_{x,K(\Gamma)}[p^{-1}]$. Of course, we use the property that $K(\Gamma)$ is complete. By our construction of s_i, we have

$$|s_{i,n}| \leqslant \max\{|t_m|\ | m \leqslant nd + i\},$$

for all $n \leqslant 0$. $(s_{i,0} = 0$ for $i \neq 0)$. Hence, s_i is contained in $\mathcal{H}^{\dagger}_{x,K(\Gamma)}$ so that the assertion (2) holds by (1). □

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Let L_y be a differential operator in $M_r(\mathcal{R}_{x,K}[\delta_y])$. Define the direct image $L_x = f_*L_y \in M_r(\mathcal{R}_{x,K}[\delta_x])$ by the induced endomorphism through the isomorphism $(\mathcal{R}_{y,K})^r \cong (\mathcal{R}_{x,K})^{dr}$ as $\mathcal{R}_{x,K}$-modules in (6.4.1).

(6.4.2) THEOREM. *Under the notation as above, we have*

(1) *there is an element* $s_y \in K[y]$ *such that* $s_yL_y \in M_r(\mathcal{A}_{y,K}[\delta_y])$ *if and only if there is an element* $s_x \in K[x]$ *such that* $s_xL_x \in M_{dr}(\mathcal{A}_{x,K}[\delta_x])$.

(2) L_y *has an index in* $(\mathcal{A}_{y,K(\Gamma)})^r$ (resp. $(\mathcal{R}_{y,K(\Gamma)})^r$) *if and only if* L_x *has an index in* $(\mathcal{A}_{x,K(\Gamma)})^{dr}$ (resp. $(\mathcal{R}_{x,K(\Gamma)})^{dr}$). *If so, we have*

$$\chi_{K(\Gamma)}(L_x, (\mathcal{A}_{x,K(\Gamma)})^{dr}) = \chi_{K(\Gamma)}(L_y, (\mathcal{A}_{y,K(\Gamma)})^{r}),$$
$$\chi_{K(\Gamma)}(L_x, (\mathcal{R}_{x,K(\Gamma)})^{dr}) = \chi_{K(\Gamma)}(L_y, (\mathcal{R}_{y,K(\Gamma)})^{r}).$$

Proof. Let s_y be an element in $K[y]$ such that s_yL_y is contained in $M_r(\mathcal{A}_{y,K}[\delta_y])$. By (5.4.1) there is an element $s_x \in K[x]$ such that $s_x(a(y))L_y \in M_r(\mathcal{A}_{y,K}[\delta_y])$. (For example, we can put $s_x = \det_{\mathcal{L}_{y,K}/\mathcal{L}_{x,K}}(s_y) \times$ (some element in $S_{x,K}$)). By (6.4.1), $s_xL_x \in M_r(\mathcal{A}_{x,K}[\delta_x])$. The converse of (1) is easy. To prove (2), consider the following diagram

$$\begin{tikzcd}
(\mathcal{A}_{x,K(\Gamma)})^{dr} \arrow[r, "s_xL_x"] \arrow[d, equal, "(6.4.1)"'] & (\mathcal{A}_{x,K(\Gamma)})^{dr} \arrow[d, equal, "(6.4.1)"] \\
(\mathcal{A}_{y,K(\Gamma)})^{r} \arrow[r, "s_x(a(y))L_y"'] & (\mathcal{A}_{y,K(\Gamma)})^{r}
\end{tikzcd},$$

is commutative, where s_x is an element in $O_K[x]$ such that $s_xL_x \in M_r(\mathcal{A}_{x,K}[\delta_x])$. Since $s_x(a(y))$ is contained in $S_{x,K}$, L_y has an index in $(\mathcal{A}_{y,K(\Gamma)})^r$ if and only if L_x has an index in $(\mathcal{A}_{x,K(\Gamma)})^{dr}$. By the choice of a, $\mathrm{ord}_y^-(s_x(a(y))) = d\,\mathrm{ord}_x^-(s(x))$. We have

$$\begin{aligned}
&\chi_{K(\Gamma)}(L_x, (\mathcal{A}_{x,K(\Gamma)})^{dr}) \\
&\quad = \chi_{K(\Gamma)}(s_xL_x, (\mathcal{A}_{x,K(\Gamma)})^{dr}) + dr\,\mathrm{ord}_x^-(s(x)) \\
&\quad = \chi_{K(\Gamma)}(s_x(a(y))L_y, (\mathcal{A}_{y,K(\Gamma)})^{r}) + r\,\mathrm{ord}_y^-(s_x(a(y))) \\
&\quad = \chi_{K(\Gamma)}(L_y, (\mathcal{A}_{y,K(\Gamma)})^{r}),
\end{aligned}$$

by (6.3.1). The case of $\mathcal{R}$ is same as above. □

We denote by $[a(y)/\delta_y(a(y))]$ the induced automorphism on $\mathcal{R}_{x,K(\Gamma)}$ from the map $a(y)/\delta_y(a(y))$ on $\mathcal{R}_{y,K(\Gamma)}$ via the isomorphism in (6.4.1).

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(6.4.3) LEMMA. $[a(y)/\delta_y(a(y))]$ *has an index in* $\mathcal{A}_{x,K(\Gamma)}$ *and we have*

$$\chi_{K(\Gamma)}\left(\left[\frac{a(y)}{\delta_y(a(y))}\right], \mathcal{A}_{x,K(\Gamma)}\right) = -\mathrm{ord}_y^-(a(y)) + \mathrm{ord}_y^-(\delta_y(a(y))).$$

Proof. Since $a(y)/\delta_y(a(y))$ is contained in $\mathcal{L}_{y,K}$, $a(y)/\delta_y(a(y))$ have an index in $\mathcal{A}_{y,K(\Gamma)}$ by (6.3.1). Hence, $[a(y)/\delta_y(a(y))]$ has an index in $\mathcal{A}_{x,K(\Gamma)}$ by definition and the formula follows (6.3.1). □

(6.4.4) COROLLARY. *Let* $L_y = C_1\delta_y - C_0$ *be a differential operator of first order in* $M_r(\mathcal{R}_{y,K}[\delta_y])$. L_y *has an index in* $(\mathcal{A}_{y,K(\Gamma)})^r$ (resp. $(\mathcal{R}_{y,K(\Gamma)})^r$) *if and only if* $[a(y)/\delta_y(a(y))]L_x$ *has an index in* $(\mathcal{A}_{x,K(\Gamma)})^{dr}$ (resp. $(\mathcal{R}_{x,K(\Gamma)})^{dr}$). *If so, then we have*

$$\chi_{K(\Gamma)}\left(\left[\frac{a(y)}{\delta_y(a(y))}\right]L_x, (\mathcal{A}_{x,K(\Gamma)})^{dr}\right)$$
$$= -r(\mathrm{ord}_y^-(a(y)) - \mathrm{ord}_y^-(\delta_y(a(y)))) + \chi_{K(\Gamma)}(L_y, (\mathcal{A}_{y,K(\Gamma)})^r),$$
$$\chi_{K(\Gamma)}\left(\left[\frac{a(y)}{\delta_y(a(y))}\right]L_x, (\mathcal{R}_{x,K(\Gamma)})^{dr}\right) = \chi_{K(\Gamma)}(L_y, (\mathcal{R}_{y,K(\Gamma)})^r).$$

Proof. Since $f_*((a(y)/\delta_y(a(y)))L_y) = [a(y)/\delta_y(a(y))]L_x$, $(a(y)/\delta_y(a(y)))L_y$ has an index in $(\mathcal{A}_{y,K(\Gamma)})^r)$ if and only if $[a(y)/\delta_y(a(y))]L_x$ has an index in $(\mathcal{A}_{x,K(\Gamma)})^{dr})$ by (6.4.2). The formula follows (6.4.2) and (6.4.3). □

(6.5) We apply our calculation of the difference of indices for direct images of a morphism which corresponds to a finite separable extension F' of $F = F_{x,k}$. The notation follows (5.3)–(5.5).

First we study the totally ramified case. Let F' be a finite separable extension over F. Choose a coordinate of $\mathcal{E}^{\dagger'} = \mathcal{E}_y^\dagger$ as in (5.4).

(6.5.1) PROPOSITION. *Under the notation as in* (5.4.3)*, let* M *be a* ∇*-module over* $\mathcal{E}_y^\dagger$ *with a* ∇*-formally rational basis* $\{e_i\}$. $L_{M,\boldsymbol{e}}$ *has an index in* $(\mathcal{A}_{y,K(\Gamma)})^r$ (resp. $(\mathcal{R}_{y,K(\Gamma)})^r$) *if and only if* $L_{f_*M,y^*\boldsymbol{e}}$ *has an index in* $(\mathcal{A}_{x,K(\Gamma)})^{dr}$ (resp. $(\mathcal{R}_{x,K(\Gamma)})^{dr}$), *where* $y^*\boldsymbol{e}$ *is a basis as in* (5.4.3). *If so, then we have*

$$\chi_{K(\Gamma)}(L_{f_*M,y^*\boldsymbol{e}}, (\mathcal{A}_{x,K(\Gamma)})^{dr})$$
$$= r\,\mathrm{length}_{O_{F'}}\omega_{O_{F'}/O_F} + \chi_{K(\Gamma)}(L_{M,\boldsymbol{e}}, (\mathcal{A}_{y,K(\Gamma)})^r),$$
$$\chi_{K(\Gamma)}(L_{f_*M,y^*\boldsymbol{e}}, (\mathcal{R}_{x,K(\Gamma)})^{dr}) = \chi_{K(\Gamma)}(L_{M,\boldsymbol{e}}, (\mathcal{R}_{y,K(\Gamma)})^r).$$

Here $r = \mathrm{rank}\, M$ *and* $\omega_{O_{F'}/O_F}$ *is the logarithmic differential module of* $O_{F'}$ *over* O_F [KK, (1.7)].

Proof. By (5.4.3) $y^*\boldsymbol{e}$ is a ∇-formally rational basis of f_*M and we have $L_{f_*M,y^*\boldsymbol{e}} = [a(y)/\delta_y(a(y))]L_x$. Since the extension F'/F is separable, we have $|\delta_y(a(y))|_G = 1$. The assertion follows from (6.4.4) and the equalities

$$\begin{aligned}\mathrm{length}_{O_{F'}}\omega_{O_{F'}/O_F} &= \mathrm{length}_{O_{F'}}O_{F'}\Big/\left(\frac{\overline{\delta_y(a(y))}}{\overline{a(y)}}\right)\\ &= \mathrm{ord}_y^-(\delta_y(a(y))) - \mathrm{ord}_y^-(a(y)).\end{aligned}$$

Here, for $a \in O_{\mathcal{E}'}^{\dagger}$, we denote by $\overline{a}$ the natural projection of a in F'. □

(6.5.2) REMARK. The formulas in (6.4.4) also hold if the extension F'/F is inseparable and if we choose the coordinate of $\mathcal{E}^{\dagger'}$ as in (5.4) (See (5.4.4).). But the author does not know what these identities mean.

In the case of a totally tamely ramified extension F'/F of degree N $(p \nmid N)$, choose a coordinate of $\mathcal{E}^{\dagger'} = \mathcal{E}_y^{\dagger}$ as in (5.5). Then we also have

$$\mathcal{H}_{y,K(\Gamma)}^{\dagger} = \mathcal{H}_{x,K(\Gamma)}^{\dagger} \oplus y^{-1}\mathcal{H}_{x,K(\Gamma)}^{\dagger} \oplus \cdots \oplus y^{-N+1}\mathcal{H}_{x,K(\Gamma)}^{\dagger}.$$

By (6.3.2) and by the same method of (6.4) we have

(6.5.3) PROPOSITION. *Under the notation as above, let M be a ∇-module over $\mathcal{E}_y^{(\dagger)}$ with a ∇-rational basis $\{e_i\}$ and denote by $\{y^j e_i\}$ a basis of f_*M as in* (5.5.1). *$L_{M,\boldsymbol{e}}$ has an index in $(\mathcal{H}_{y,K(\Gamma)}^{\dagger})^r$ if and only if $L_{f_*M,y^*\boldsymbol{e}}$ has an index in $(\mathcal{H}_{x,K(\Gamma)}^{\dagger})^{Nr}$. If so, we have*

$$\chi_{K(\Gamma)}(L_{f_*M,y^*\boldsymbol{e}}, (\mathcal{H}_{x,K(\Gamma)}^{\dagger})^{Nr}) = \chi_{K(\Gamma)}(L_{M,\boldsymbol{e}}, (\mathcal{H}_{y,K(\Gamma)}^{\dagger})^{r}).$$

In particular, $L_{M,\boldsymbol{e}}$ has indices on $K(\Gamma)$ if and only if $L_{f_*M,y^*\boldsymbol{e}}$ has so.

We now discuss the case of unramified extension. Fix the situation as in (5.3). Put $\Gamma' = \mathrm{Aut}_{\mathrm{cont}}(\widehat{K^{\mathrm{alg}}}/K') \cap \Gamma$ and $K(\Gamma') = (\widehat{K^{\mathrm{alg}}})^{\Gamma'}$. Then one can easily see that $K(\Gamma')$ is finite over $K(\Gamma)$.

(6.5.4) PROPOSITION. *Assume that M is a ∇-module with a ∇-formally rational basis $\{e_i\}$. Then, $L_{M,\boldsymbol{e}}$ has an index in $(\mathcal{A}_{x,K(\Gamma')})^r$* (resp. $(\mathcal{R}_{y,K(\Gamma')})^r$) *if and only if $L_{f_*M,u_*\boldsymbol{e}}$ has an index in $(\mathcal{A}_{x,K(\Gamma)})^{[k':k]r}$* (resp. $(\mathcal{R}_{x,K(\Gamma)})^{[k':k]r}$). *If so, then we have*

$$\begin{aligned}\chi_{K(\Gamma)}(L_{f_*M,u_*\boldsymbol{e}}, (\mathcal{A}_{x,K(\Gamma)})^{[k':k]r}) &= [k':k]\chi_{K(\Gamma')}(L_{M,\boldsymbol{e}}, (\mathcal{A}_{x,K(\Gamma')})^{r}),\\ \chi_{K(\Gamma)}(L_{f_*M,u_*\boldsymbol{e}}, (\mathcal{R}_{x,K(\Gamma)})^{[k':k]r}) &= [k':k]\chi_{K(\Gamma')}(L_{M,\boldsymbol{e}}, (\mathcal{R}_{y,K(\Gamma')})^{r}).\end{aligned}$$

Assume furthermore that $L_{M,\boldsymbol{e}}$ has rational coefficients, then $L_{M,\boldsymbol{e}}$ has indices on $K(\Gamma')$ if and only if $L_{f_*M,u_*\boldsymbol{e}}$ has indices on $K(\Gamma)$.

Proof. By (6.2.3) we may assume that $K(\Gamma') = K(\Gamma)$. Since $\{u_j\}$ is a basis of $\mathcal{E}^\dagger_{x,K'}$ over $\mathcal{E}^\dagger_{x,K}$, we have

$$L_{f_*M,u_*\boldsymbol{e}} = \begin{pmatrix} L_{M,\boldsymbol{e}} & & 0 \\ & \ddots & \\ 0 & & L_{M,\boldsymbol{e}} \end{pmatrix}.$$

The assertion easily follows the formula above. □

(6.5.5) REMARK. If F' is at worst tamely ramified over F, then the logarithmic differential module $\omega_{O_{F'}/O_F} = 0$.

(6.6) We study the difference of local indices for a direct image which corresponds to an Artin–Schreier extension of degree p. Follow the situation and notations as in (5.6). Let M be a ∇-module of rank r over $\mathcal{E}^\dagger_y$ with a ∇-formally rational basis $\{e_i\}$. Denote by $L_{M,\boldsymbol{e}}$ (resp. $L_{f_*M,z^*\boldsymbol{e}}$) the differential operator which corresponds to the pair $(M, \boldsymbol{e})$ (resp. $(f_*M, z^*\boldsymbol{e})$), where $z^*\boldsymbol{e}$ is the basis $e_1, ze_1, \ldots, z^{p-1}e_r$. (See (4.1).) Then $L_{f_*M,z^*\boldsymbol{e}}$ is contained in $M_{pr}(\mathcal{L}_{x,K}[\delta_x])$ by (5.4.2).

(6.6.1) PROPOSITION. *Under the situation above, $L_{M,\boldsymbol{e}}$ has an index in $(\mathcal{A}_{y,K(\Gamma)})^r$ (resp. $(\mathcal{R}_{y,K(\Gamma)})^r$) if and only if $L_{f_*M,z^*\boldsymbol{e}}$ has an index in $(\mathcal{A}_{x,K(\Gamma)})^{pr}$ (resp. $(\mathcal{R}_{x,K(\Gamma)})^{pr}$). If so, we have*

$$\begin{aligned} &\chi_{K(\Gamma)}(L_{f_*M,z^*\boldsymbol{e}}, (\mathcal{A}_{x,K(\Gamma)})^{pr}) \\ &\quad = r\,\mathrm{length}_{O_{F_y}}\omega_{O_{F_y}/O_{F_x}} + \chi_{K(\Gamma)}(L_{M,\boldsymbol{e}}, (\mathcal{A}_{y,K(\Gamma)})^r), \end{aligned}$$

$$\chi_{K(\Gamma)}(L_{f_*M,z^*\boldsymbol{e}}, (\mathcal{R}_{x,K(\Gamma)})^{pr}) = \chi_{K(\Gamma)}(L_{M,\boldsymbol{e}}, (\mathcal{R}_{y,K(\Gamma)})^r).$$

Proof. The assertion follows (5.6.2), (6.3.6) and (6.5.1). □

(6.6.2) REMARK. In (6.6.1) the length of the logarithmic differential module $\omega_{O_{F_y}/O_{F_x}}$ is $N(p-1)$.

7. Irregularity for overconvergent etale φ-∇-modules

In this section we define an irregularity for overconvergent etale φ-modules and state our main theorems. Fix a pair (Λ, k) as in (2.1.1). Put $F = F_{x,k}$, $K = K(\Lambda, k)$ and $\widehat{K}^{un} = K(\Lambda, k^{\mathrm{alg}})$. Let Γ be a subgroup of the continuous automorphism group $\mathrm{Aut}_{\mathrm{cont}}(\widehat{K}^{\mathrm{alg}}/K)$. Put $K(\Gamma) = (\widehat{K}^{\mathrm{alg}})^\Gamma$ and put $\widehat{K}^{un}(\Gamma)$ to be the subfield of $\widehat{K}^{\mathrm{alg}}$ invariant under $\mathrm{Aut}_{\mathrm{cont}}(\widehat{K}^{\mathrm{alg}}/\widehat{K}^{un}) \cap \Gamma$.

(7.1) Fix a Frobenius σ on $\mathcal{E}^\dagger_{x,K}$ such that $\sigma(x)/x^q \in \mathcal{H}^\dagger_{x,K}$.

Let M be an object of $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger_{x,K},\sigma}$ and denote by M^a the pull back of M by the natural map $\mathcal{E}^\dagger_{x,K} \to \mathcal{E}^\dagger_{x,\widehat{K}^{un}}$. Then M^a is an object in $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger_{x,\widehat{K}^{un}},\sigma}$. By (3.2.4) and (3.5.3) there is a positive integer N $(p \nmid N)$ such that the pull back $[N]^*M^a$ is a strictly π-unipotent φ-∇-module in $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger_{y,\widehat{K}^{un}},\sigma}$. Here $[N]\colon \mathcal{E}^\dagger_{x,\widehat{K}^{un}} \to \mathcal{E}^\dagger_{y,\widehat{K}^{un}}$ is the natural map which is defined by $x \mapsto y^N$. One can easily see that the induced Frobenius σ on $\mathcal{E}^\dagger_{y,\widehat{K}^{un}}$ satisfies the condition $(\sigma(y)/y^q) \in \mathcal{H}^\dagger_{y,\widehat{K}^{un}}$ for $p \nmid N$. Therefore, there is a basis $\{e_i\}$ of $[N]^*M^a$ such that it is strictly π-unipotent and $C_{[N]^*M^a,\boldsymbol{e}}$ is contained in $x^{-1}M_r(\mathcal{H}^\dagger_{x,\widehat{K}^{un}})$ by (3.2.14).

We define the irregularity of M as follows. If $K(\Gamma) = \widehat{K}^{\mathrm{alg}}$ and if M has a ∇-rational basis, our irregularity coincides with the irregularity $i_0(L,1)$ which is defined by Robba [Ro2, 10.1.] from (6.3.7).

(7.1.1) DEFINITION. Under the above situation, we define the irregularity of M on $\widehat{K}^{un}(\Gamma)$ with respect to N by

$$i_{\widehat{K}^{un}(\Gamma)}(M,N) = -\frac{1}{N}\chi_{\widehat{K}^{un}(\Gamma)}(L_{[N]^*(M^a),\boldsymbol{e}}, \mathcal{H}^\dagger_{y,\widehat{K}^{un}(\Gamma)}).$$

Our definition of irregularity is independent of the choice of the strictly π-unipotent basis $\{e_i\}$ of $[N]^*M^a$ by (3.3.2) and (6.3.6). From (3.3.5) we have

(7.1.2) LEMMA. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence in $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger_{x,K},\sigma}$. *Assume that* $[N]^*M_2^a$ *is* π*-unipotent for a positive integer* N $(p \nmid N)$. *If two of* $i_{\widehat{K}^{un}(\Gamma)}(M_1,N)$, $i_{\widehat{K}^{un}(\Gamma)}(M_2,N)$ *and* $i_{\widehat{K}^{un}(\Gamma)}(M_3,N)$ *are finite, then the rest is also finite and the identity*

$$i_{\widehat{K}^{un}(\Gamma)}(M_2,N) = i_{\widehat{K}^{un}(\Gamma)}(M_1,N) + i_{\widehat{K}^{un}(\Gamma)}(M_3,N)$$

holds.

Our definition of irregularity depends a priori on the choice of the positive integer N and the field $\widehat{K}^{un}(\Gamma)$ of coefficients of connection in a glance. However, one can easily see that it is independent of the extension of the coefficient field Λ of representations.

Let Λ' be a finite extension of Λ such that the pair (Λ', k) satisfies the condition (2.1.1). Denote by $\sigma' = \sigma^f$ the induced Frobenius on $\mathcal{E}^\dagger_{x,K'}$, then σ' satisfies the condition $\sigma'(x)/x^{q'} \in \mathcal{H}^\dagger_{x,K'}$. Here $q' = q^f$ is the cardinal of the residue class field of Λ'. Put $K' = K(\Lambda', k)$, $\Gamma' = \mathrm{Aut}_{\mathrm{cont}}(\widehat{K}^{\mathrm{alg}}/K\Lambda') \cap \Gamma$ and $\widehat{K}^{un}(\Gamma') =$ the $(\mathrm{Aut}_{\mathrm{cont}}(\widehat{K}^{\mathrm{alg}}/\widehat{K}^{un}\Lambda') \cap \Gamma')$-invariant subfield of $\widehat{K}^{\mathrm{alg}}$.

(7.1.3) LEMMA. *Keep the notation as above and as in* (3.1.6). *For an object* M *in* $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger_{x,K},\sigma}$ *and for an integer* N $(p \nmid N)$ *such that* $[N]^*M^a$ *is* π*-unipotent,*

$i_{\widehat{K}^{un}(\Gamma)}(M,N)$ is finite if and only if $i_{\widehat{K}^{un}(\Gamma')}(\iota_{\Lambda'/\Lambda}(M),N)$ is finite. If they are finite, then

$$i_{\widehat{K}^{un}(\Gamma')}(\iota_{\Lambda'/\Lambda}(M),N) = i_{\widehat{K}^{un}(\Gamma)}(M,N).$$

Proof. The assertion follows from the definition of the functor $\iota_{\Lambda'/\Lambda}$ and (6.2.3) ($\widehat{K}^{un}(\Gamma')$ is finite over $\widehat{K}^{un}(\Gamma)$).

(7.1.4) REMARK. In our definition of irregularities we omit the tame part. The author does not know how to define irregularities without pull backs to the cases of π-unipotent (i.e. the case of a totally wild ramification on the Galois representation side). Of course, if one can define irregularities over the field $\mathcal{E}^\dagger_{x,K}$ of definition of φ-∇-modules, then it is expected that they coincide with those of (7.1.1). (See (8.3.2).) In the case of rank one, we can define irregularities over the field $\mathcal{E}^\dagger_{x,K}$ of definition and they coincide with those of our irregularity. (See (8.2).)

(7.1.5) REMARK. Our irregularity must be related to the local terms of the Euler characteristic of overconvergent unit-root F-isocrystals on a curve via the theory of canonical extension as same as l-adic theory in [KN]. In the case of rank one it is studied in [Be] [Ga].

(7.2) Keep the notations as in (7.1) and assume that the Frobenius σ on $\mathcal{E}^\dagger_{x,K}$ satisfies the condition that $\sigma(x)/x^q \in \mathcal{H}^\dagger_{x,K}$ in (7.2). Now we state our main theorems.

(7.2.1) THEOREM. *Let M be an object in $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger_{x,K},\sigma}$. Then the irregularity $i_{\widehat{K}^{un}(\Gamma)}(M,N)$ is finite and independent of the choice of the positive integer N $(p \nmid N)$ such that $[N]^*M^a$ is π-unipotent.*

For any object M in $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger_{x,K},\sigma}$, we denote by $i_{\widehat{K}^{un}(\Gamma)}(M)$ the irregularity on $\widehat{K}^{un}(\Gamma)$ which is determined independently of the choice of N.

(7.2.2) THEOREM. *Let V be an object in $\underline{\mathbf{Rep}}^{\mathrm{fin}}_{\Lambda}(G_F)$ and put $M = \mathrm{D}^\dagger_\sigma(V)$ the corresponding etale φ-∇-module with respect to $(\mathcal{E}^\dagger_{x,K},\sigma)$. Then, we have an identity*

$$i_{\widehat{K}^{un}(\Gamma)}(M) = \mathrm{Swan}(V).$$

Here $\mathrm{Swan}(V)$ *is the Swan conductor of the representation* V.

We remark that, if we denote by $a_{[N]}$ the extension $F = F_{x,k} \to F_{y,k^a}$ which corresponds to the extension $\mathcal{E}^\dagger_{x,K} \to \mathcal{E}^\dagger_{y,\widehat{K}^{un}}$, then $\mathrm{D}^\dagger_\sigma(a^*_{[N]}V) \cong [N]^*(M^a)$.

The irregularity is independent of the choice of the coefficient field $\widehat{K}^{un}(\Gamma)$ of differential structures by (7.2.2). Especially, we have the identity

$$i_{\widehat{K}^{\mathrm{alg}}}(M) = i_{\widehat{K}^{un}}(M) = \mathrm{Swan}(V).$$

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In the case of rank one, Matsuda showed this comparison theorem if p is an odd prime [Ma, 5.4, 5.5]. Using the Kummer–Artin–Schreier–Witt complex, he constructed a differential equation with polynomial coefficients explicitly for a p-adic representation of rank one, which is minimal in some sense. Then he applied Robba's calculation of local indices to the explicit formula.

We can easily reduce (7.2.1) and (7.2.2) to the following case.

(7.2.3) LEMMA. *Assume that, for any π-unipotent φ-∇-module M with* $\mathrm{D}^{\dagger}_{\sigma}(V) = M$, $i_{K(\Gamma)}(M,1)$ *is finite and the identity*

$$i_{\widehat{K}^{un}(\Gamma)}(M,1) = \mathrm{Swan}(V)$$

holds in the case that k is algebraically closed. Then the assertions of (7.2.1) *and* (7.2.2) *are true.*

Proof. One can easily reduce (7.2.1) and (7.2.2) to the case that k is algebraically closed and that M is π-unipotent (in other word, the image $\mathrm{im}(G_F \to \mathrm{GL}(V))$ is a finite p-group by (3.5.1)) since the functor $\mathrm{D}^{\dagger}_{\sigma}$ commutes with pull backs. If k is algebraically closed and if M is π-unipotent, then $i_{K(\Gamma)}(M,N) = N^{-1} i_{K(\Gamma)}([N]^{*}M, 1)$ by definition. On the other hand, we know that $\mathrm{Swan}(V) = N^{-1}\mathrm{Swan}([N]^{*}V)$ by [Fo1, II. Prop. 6.1.]. □

We will prove (7.2.1) and (7.2.2) in Section 8. First we will show the theorem in the case of rank one. Then we will prove that $i_{\widehat{K}^{un}(\Gamma)}(M,N)$ is finite and coincides with the Swan conductor using the method of Brauer induction. It is important that, for any irreducible π-unipotent φ-∇-module, there is a basis which is ∇-rational in the proof.

(7.3) Fix a Frobenius σ on $\mathcal{E}^{\dagger}_{x,K}$ arbitrarily. For any ∇-module M over $\mathcal{E}^{\dagger}_{x,K}$, we define a $K(\Gamma)$-connection

$$\nabla_{M,\mathcal{R}_{K(\Gamma)}}\colon \mathcal{R}_{x,K(\Gamma)} \bigotimes_{\mathcal{E}^{\dagger}_{x,K}} M \to \omega_{\mathcal{R}_{x,K(\Gamma)}} \bigotimes_{\mathcal{E}^{\dagger}_{x,K}} M$$

by the extension $\mathcal{E}^{\dagger}_{x,K} \to \mathcal{R}_{x,K(\Gamma)}$. It can be calculated using any basis of M over $\mathcal{E}^{\dagger}$ by (7.3.6).

The theorem below is related to the Robba's conjecture [Ro2, 3.12.].

(7.3.1) THEOREM. *For any object M in* $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^{\dagger}_{x,K},\sigma}$, $\nabla_{\mathcal{R}_{K(\Gamma)}}$ *has an index and we have*

$$\chi_{K(\Gamma)}(\nabla_{M,\mathcal{R}_{K(\Gamma)}}) = 0.$$

We will prove (7.3.1) in the Section 8. We prove the following assertions which are useful to prove our main theorems in the next section.

(7.3.2) PROPOSITION. *Let M be an object of* rank r *in* $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^{\dagger}_{x,K},\sigma}$*, If* $\ker\nabla_{M,\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}}$ *is of finite dimension over* $\widehat{K}^{\mathrm{alg}}$*, then the natural map*

$$\ker\nabla_{M,\mathcal{E}^{\dagger}_{K(\Gamma)}} \to \ker\nabla_{M,\mathcal{R}_{K(\Gamma)}}$$

is bijective. Here $\nabla_{M,\mathcal{E}^{\dagger}_{K(\Gamma)}}$ *is the induced* $K(\Gamma)$*-connection by the extension* $\mathcal{E}^{\dagger}_{x,K} \to \mathcal{E}^{\dagger}_{x,K(\Gamma)}$.

Proof. The injectivity is trivial. By the same method of the proof (6.2.1) one can show that the natural map $\widehat{K}^{\mathrm{alg}} \otimes_{K(\Gamma)} \ker\nabla_{M,\mathcal{E}^{\dagger}_{K(\Gamma)}} \to \ker\nabla_{M,\mathcal{E}^{\dagger}_{\widehat{K}^{\mathrm{alg}}}}$ is bijective. By (6.2.1) and this fact we may assume that $\Gamma = \{1\}$, that is, $K(\Gamma) = \widehat{K}^{\mathrm{alg}}$. So one can easily see that we may assume that k is algebraically closed. Denote by V the representation of G_{F_x} such that $\mathrm{D}^{\dagger}(V) = M$. By [Sel, IV.2.] there is a sequence $F_{x,k} = F_0 \subset F_1 \subset \cdots \subset F_n$ of finite separable extensions of $F_{x,k}$ such that

$G_{F_n} = \mathrm{Gal}(F^{\mathrm{sep}}_{x,k}/F_n)$ acts trivially on V;
F_1/F_0 is totally tamely ramified;
F_{i+1}/F_i is an Artin–Schreier extension of degree p with wild ramification $(1 \leqslant i < n)$.

Then we can inductively determine a sequence of unramified extensions $\mathcal{E}^{\dagger}_{x_0,K} \subset \mathcal{E}^{\dagger}_{x_1,K} \subset \cdots \subset \mathcal{E}^{\dagger}_{x_n,K}$ $(x_0 = x)$ which corresponds to the sequence $F_0 \subset F_1 \subset \cdots \subset F_n$ as in (5.5) and (5.6). In this situation we know that an element $a \in \mathcal{R}_{x_i,\widehat{K}^{\mathrm{alg}}}$ is contained in $\mathcal{E}^{\dagger}_{x_i,\widehat{K}^{\mathrm{alg}}}$ if and only if a is contained in $\mathcal{E}^{\dagger}_{x_{i+1},\widehat{K}^{\mathrm{alg}}}$ by (6.4.1). So, for an element $m \in \ker\nabla_{M,\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}}$, we have only to check m is contained in $\mathcal{E}^{\dagger}_{x_n,\widehat{K}^{\mathrm{alg}}} \otimes_{\mathcal{E}^{\dagger}_{x,K}} M$. Since G_{F_n} acts trivially on V, there is an isomorphism $\mathcal{E}^{\dagger}_{x_n,K} \otimes_{\mathcal{E}^{\dagger}_{x,K}} M \to (\mathcal{E}^{\dagger}_{x_n,K})^r$ of ∇-modules over $\mathcal{E}^{\dagger}_{x_n,K}$ and, therefore, we have

$$\ker\nabla_{M,\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}} = (\widehat{K}^{\mathrm{alg}})^r = \ker\nabla_{M,\mathcal{E}^{\dagger}_{\widehat{K}^{\mathrm{alg}}}}. \qquad \square$$

(7.3.3) COROLLARY. *Let M be an object in* $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^{\dagger}_{x,K},\sigma}$*. If* $\nabla_{M,\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}}$ *has an index, then* $\nabla_{M,\mathcal{R}_{K(\Gamma)}}$ *has an index and we have*

$$\chi_{K(\Gamma)}(M_{\mathcal{R}_{K(\Gamma)}}) \leqslant 0.$$

Proof. By (6.2.1) $\nabla_{M,\mathcal{R}_{K(\Gamma)}}$ has an index. If $\ker\nabla_{M,\mathcal{R}_{K(\Gamma)}} = 0$, then we obtain the formula by definition. Assume that $\ker\nabla_{M,\mathcal{R}_{K(\Gamma)}} \neq 0$. We show the assertion by the induction on the rank of M. Since the $\mathrm{Aut}_{\mathrm{cont}}(\widehat{K}^{\mathrm{alg}}/K)$-invariant field of $\widehat{K}^{\mathrm{alg}}$ is K [Ta, 3.3 Thm.1], the natural map $K(\Gamma)\otimes_K \ker\nabla_{\mathcal{E}^{\dagger}_K} \to \ker\nabla_{M,\mathcal{R}_{K(\Gamma)}}$ is bijective by (6.2.1) and (7.3.2). (Compare both vector spaces in $\mathcal{R}_{x,\widehat{K}^{\mathrm{alg}}}\otimes_K M$). By definition of φ-∇-modules and the theory of slopes, we know that $M' = \mathcal{E}^\dagger_{x,K} \otimes_K \ker \nabla_{\mathcal{E}^\dagger_K}$ is a subobject of M in $\underline{\mathbf{M\Phi}}^{\nabla \text{et}}_{\mathcal{E}^\dagger_{x,K},\sigma}$. Since the connection M' is $d \otimes \mathrm{id}$, we have $\chi_{K(\Gamma)}(M'_{\mathcal{R}_{K(\Gamma)}}) = 0$ by easy computations. So we reduce to the case of the quotient M/M'. $\square$

8. Brauer induction

In this section we prove our main theorems in (7.2). In (8.1) we give the formula of indices of direct images of overconvergent φ-∇-modules. In (8.2) we prove our main theorems for the object of rank one. In (8.3) we show (7.2.1) and (7.2.2). In (8.4) we prove (7.3.1). Keep the notations as in the Section 7.

(8.1) Consider the following situation. Let F' be a finite separable extension of $F = F_{x,k}$ such that there is a sequence $F = F_0 \subset F_1 \subset \cdots \subset F_n = F'$ of finite separable extensions of F which satisfies the conditions that F_{i+1}/F_i is either (1) unramified, (2) totally tamely ramified or (3) an Artin–Schreier extension of degree p with a totally wild ramification for all i. Then we can inductively determine a sequence of finite unramified extensions $\mathcal{E}^\dagger_{x_0,K_0} \subset \mathcal{E}^\dagger_{x_1,K_1} \subset \cdots \subset \mathcal{E}^\dagger_{x_n,K_n}$ ($x_0 = x$, $K_0 = K$) which corresponds to the sequence $F_0 \subset F_1 \subset \cdots \subset F_n$ as in (5.3), (5.5) and (5.6). Here k_i is the residue class field of F_i and we put $K_i = K(\Lambda, k_i)$. We also use the notation x' (resp. k', resp. K') for x_n (resp. k_n, resp. K_n). We denote by f (resp. f_i) both extensions $F \to F'$ and $\mathcal{E}^\dagger_{x,K} \to \mathcal{E}^\dagger_{x',K'}$ (resp. $F_i \to F_{i+1}$ and $\mathcal{E}^\dagger_{x_i,K_l} \to \mathcal{E}^\dagger_{x_i,K_l}$). Denote by d (resp. d_i) the degree of the extension F'/F (resp. F_{i+1}/F_i).

Fix a Frobenius σ on $\mathcal{E}^\dagger_{x,K}$ arbitrarily and we also use the notation σ for the unique extension of the Frobenius σ on $\mathcal{E}^\dagger_{x',K'}$ (resp. $\mathcal{E}^\dagger_{x_i,K_i}$). Put $\Gamma' = \mathrm{Aut}_{\mathrm{cont}}(\widehat{K}^{\mathrm{alg}}/K') \cap \Gamma$ and $K(\Gamma') = (\widehat{K}^{\mathrm{alg}})^{\Gamma'}$.

(8.1.1) PROPOSITION. *Under the situation as above, let M be a ∇-module of* rank r *over $\mathcal{E}^\dagger_{x',K'}$ with a ∇-formally rational basis $\{e_i\}$. If $\{\tilde{e}_j\}$ is a basis f_*M which is induced from the basis $\{e_i\}$ of M by the method as in* (5.3.1), (5.5.1) *and* (5.6.2) *inductively* (*here we use* (5.1.6)) *and if $L_{M,\boldsymbol{e}}$ has an index in* $(\mathcal{A}_{x',K(\Gamma')})^r$ (resp. *in* $(\mathcal{R}_{x',K(\Gamma')})^r$), *$L_{f_*M,\tilde{\boldsymbol{e}}}$ also has an index in* $(\mathcal{A}_{x,K(\Gamma)})^{dr}$ (resp. *in* $(\mathcal{R}_{x,K(\Gamma)})^{dr}$) *and*

$$\begin{aligned}&\chi_{K(\Gamma)}(L_{f_*M,\tilde{\boldsymbol{e}}}, (\mathcal{A}_{x,K(\Gamma)})^{dr})\\ &\quad = [k' : k](r\,\mathrm{length}_{O_{F'}}\omega_{O_{F'}/O_F} + \chi_{K(\Gamma')}(L_{M,\boldsymbol{e}}, (\mathcal{A}_{x',K(\Gamma')})^r)),\end{aligned}$$

$$\chi_{K(\Gamma)}(L_{f_*M,\tilde{\boldsymbol{e}}}, (\mathcal{R}_{x,K(\Gamma)})^{dr}) = [k' : k]\chi_{K(\Gamma')}(L_{M,\boldsymbol{e}}, (\mathcal{R}_{x',K(\Gamma')})^r).$$

Proof. The formulas follow from the fact that the sequence

$$0 \to O_{F'} \bigotimes_{O_{F_{i+1}}} \omega_{O_{F_{i+1}}/O_{F_i}} \to \omega_{O_{F'}/O_{F_i}} \to \omega_{O_{F'}/O_{F_{i+1}}} \to 0$$

of $O_{F'}$-modules is exact for all i and also (5.1.6), (6.5.3), (6.5.4), (6.5.5) and (6.6.1). □

(8.1.2) COROLLARY. *Under the situation as above, assume furthermore that the extensions of type* (1) *and type* (2) *appear only in* F_1/F_0 *or in both the extensions* F_1/F_0 *and* F_2/F_1. *Let* M *be an object of rank* r *of* $\underline{\mathbf{M\Phi}}^{\nabla \mathrm{et}}_{\mathcal{E}^\dagger_{x,K'},\sigma}$, *with a* π-*unipotent and* ∇-*formally rational basis* $\{e_i\}$. *Then* f_*M *has a* ∇-*rational basis. Moreover, if* $L_{M,\boldsymbol{e}}$ *has an index in* $(\mathcal{A}_{x',K(\Gamma')})^r$ (resp. *in* $(\mathcal{R}_{x',K(\Gamma')})^r$) *and if* $\tilde{\boldsymbol{e}}$ *is a* ∇-*rational basis of* f_*M, *then* $L_{f_*M,\tilde{\boldsymbol{e}}}$ *also has an index in* $(\mathcal{A}_{x,K(\Gamma)})^{dr}$ (resp. *in* $(\mathcal{R}_{x,K(\Gamma)})^{dr}$) *and*

$$\begin{aligned}
&\chi_{K(\Gamma)}(L_{f_*M,\tilde{\boldsymbol{e}}}, (\mathcal{A}_{x,K(\Gamma)})^{dr}) \\
&\quad = [k':k](r\, \mathrm{length}_{O_{F'}} \omega_{O_{F'}/O_F} + \chi_{K(\Gamma')}(L_{M,\boldsymbol{e}}, (\mathcal{A}_{x',K(\Gamma')})^r)), \\
&\chi_{K(\Gamma)}(L_{f_*M,\tilde{\boldsymbol{e}}}, (\mathcal{R}_{x,K(\Gamma)})^{dr}) = [k':k]\chi_{K(\Gamma')}(L_{M,\boldsymbol{e}}, (\mathcal{R}_{x',K(\Gamma')})^r).
\end{aligned}$$

Proof. By (5.4.3) and (5.1.6) there is a π-unipotent and ∇-rational basis in the direct image of M for successive extensions of type (3) if M has a π-unipotent and ∇-rational basis. (If an extension either of type (1) or of type (2) appears in some extension F_{i+1}/F_i ($i \geqslant 2$), then $\{y^j e_i\}$ as in (5.5.1) (resp. $\{u_j e_i\}$ as in (5.3.1) is not π-unipotent)). Hence f_*M has a ∇-rational basis by (5.3.1) and (5.5.1). The formulas follow from (8.1.1) and (6.3.7). □

If F'/F is a finite Galois extension, then there is a sequence of finite separable extensions of F as in the above situation. If F'' is a middle field of a finite Galois extension F'/F such that F'/F'' is totally wild ramified, then there is a sequence of finite extensions of F''/F as above, because a finite p-group is nilpotent [Se2, II.9.3. Thm. 18.].

Let us introduce the corresponding formula for Swan conductor. Let $f\colon F \to F'$ be a finite separable extension as above and let V be a representation of the absolute Galois group $G_{F'}$ of F' over the field of characteristic 0. Assume that the image $\mathrm{im}(G_{F'} \to \mathrm{GL}(V))$ is finite. Then we know the following formula for the induced representation f_*V [Sel, VI.2. Prop. 4., Cor.] [Fol, II.6. Prop. 6.1.].

(8.1.3) PROPOSITION. *Under the above assumption, we have*

$$\mathrm{Swan}(f_*V) = [k':k](\mathrm{rank}\, V\, \mathrm{length}_{O_{F'}} \omega_{O_{F'}/O_F} + \mathrm{Swan}(V)).$$

(8.2) We prove the theorem in the case of rank one. In this case Matsuda's results imply (8.2.1) and (8.2.2) for odd prime p. Here we give a new proof and show it for general p. Fix a Frobenius σ on $\mathcal{E}^{\dagger}_{x,K}$ arbitrarily in (8.2.1) and (8.2.2). Denote by $\mathbf{Z}_{(p)}$ the localization of the ring $\mathbf{Z}$ of rational integer at the prime ideal (p).

(8.2.1) PROPOSITION. *Let M be a φ-∇-module of rank one over $\mathcal{E}^{\dagger}_{x,K}$. Then M has a base e such that $C_{M,e} \in \mathbf{Z}_{(p)} + \pi x^{-1} O_K[x^{-1}]$. Moreover, if M is π-unipotent, then M has a π-unipotent base e such that $C_{M,e} \in \pi x^{-1} O_K[x^{-1}]$.*

Proof. We may assume that M is etale and that the Frobenius σ satisfies the condition $\sigma(x)/x^q \in \mathcal{H}^{\dagger}$ by (3.4.4). Let e' be a basis of M and put $a = A_{M,e'}$ and $c = C_{M,e'}$. Then a is a unit in $O^{\dagger}_{\mathcal{E}}$. Choose an integer N and an element $u \in k^{\times}$ such that $(ux^N)^{-1} a \pmod{\pi} \in 1 + xk[[x]]$. Choose a lifting $v \in S_{x,K}$ of the element $\prod_{n=0}^{\infty}(\pi^{-s}(ux^N)^{-1}a \pmod{\pi})^{q^n}$. Then $v^{-1}(ux^N)^{-1}a\sigma(v) \equiv 1 \pmod{\pi}$. By (3.2.5) there is an element $v' \in 1 + \pi x S_{x,K}$ and $u' \in 1 + \pi O_K$ such that $(vv')^{-1}(uu'x^N)^{-1}a\sigma(vv') \in 1 + \pi x^{-1} O^{\dagger}_{\mathcal{H}}$. So we may assume $(ux^N)^{-1}a \in 1 + \pi x^{-1} O^{\dagger}_{\mathcal{H}}$ and $c = N/(q-1) + \pi x^{-1} O^{\dagger}_{\mathcal{H}}$ by (3.2.14). Now we put $c = c_0 + c_1$ such that $c_0 \in O_K[x^{-1}], c_1 \in \pi x^{-1} O^{\dagger}_{\mathcal{H}}$ and that $|c_1|_G$ is sufficiently small, so that $w = \exp(-\int c_1(\mathrm{d}x/x))$ can be defined in $1 + \pi x^{-1} O^{\dagger}_{\mathcal{H}}$. Then we have $w^{-1}\delta(w) = -c_1$ and $e = we'$ is a desired base of M. In the π-unipotent case we begin with $a \equiv 1 \pmod{\pi}$. $\square$

In the case of rank one we have the following stronger result than (7.2.2). The assertions of (7.2.1) and (7.2.2) for objects of rank one easily follow (6.3.7), (7.2.3) and (8.2.2).

(8.2.2) THEOREM. *Let V be an object of rank one in $\underline{\mathbf{Rep}}^{\mathrm{fin}}_{\Lambda}(G_F)$ and put $M = \mathrm{D}^{\dagger}(V)$ the corresponding etale φ-∇-module over $\mathcal{E}^{\dagger}_{x,K}$ with respect to σ. For a ∇-rational base e of M, $L_{M,e}$ has indices on $K(\Gamma)$ and we have*

$$\chi_{K(\Gamma)}(L_{M,e}, \mathcal{A}_{x,K(\Gamma)}) = -\chi_{K(\Gamma)}(L_{M,e}, \mathcal{H}^{\dagger}_{x,K(\Gamma)}) = \mathrm{Swan}(V),$$
$$\chi_{K(\Gamma)}(L_{M,e}, \mathcal{R}_{x,K(\Gamma)}) = 0.$$

First we reduce (8.2.2) to the case that $\Gamma = \{1\}$, that is, $K(\Gamma) = \widehat{K}^{\mathrm{alg}}$, and k is algebraically closed. Indeed, if $L_{M,e}$ has indices on $\widehat{K}^{\mathrm{alg}}$, then $L_{M,e}$ has indices on $K(\Gamma)$ by (6.2.1). If $\chi_{\widehat{K}^{\mathrm{alg}}}(L_{M,e}, \mathcal{R}_{x,\widehat{K}^{\mathrm{alg}}}) = 0$, then we obtain $\chi_{K(\Gamma)}(L_{M,e}, \mathcal{R}_{x,K(\Gamma)}) = 0$ by (6.2.1) and (7.3.3). Moreover, we have identities $\chi_{\widehat{K}^{\mathrm{alg}}}(L_{M,e}, \mathcal{A}_{x,\widehat{K}^{\mathrm{alg}}}) = \chi_{K(\Gamma)}(L_{M,e}, \mathcal{A}_{x,K(\Gamma)})$ and $\chi_{\widehat{K}^{\mathrm{alg}}}(L_{M,e}, \mathcal{H}^{\dagger}_{x,\widehat{K}^{\mathrm{alg}}}) = \chi_{K(\Gamma)}(L_{M,e}, \mathcal{H}^{\dagger}_{x,K(\Gamma)})$ by (6.2.1) and (6.3.4). Hence, we reduce the assertion to the case that $\Gamma = \{1\}$. Now we can easily reduce it to the case that k is algebraically closed.

To finish the proof of (8.2.2), we use Robba's formula. (See [Ro2].) Let us explain it as needed. Let t_r be a $(0, r)$-generic point. For a differential operator of

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first order $L = \delta_x + \eta \in \widehat{K}^{\mathrm{alg}}(x)[\delta_x]$, define

$$\begin{aligned}&\rho_0(L, r)\\&\quad = \min\{\text{radius of convergence of a solution } (\neq 0) \text{ of } L = 0 \text{ at } t_r, r\}.\end{aligned}$$

We say L is solvable on the $(0,1)$-generic disk if and only if $\rho_0(L, 1) = 1$. By [Ro2, 5.6.] we have

(8.2.3) THEOREM. *Under the notations as above, unless simultaneously L is solvable on the $(0,1)$-generic disk, $(d\log\rho_0(L,r)/d\log r)^-|_{r=1} = 1$, and* $\mathrm{Res}_0(\eta(\mathrm{d}x/x), 1)$ *is of Liouville* [Ro2, 5.5.], *then L has indices on $\widehat{K}^{\mathrm{alg}}$ and*

$$\chi_{\widehat{K}^{\mathrm{alg}}}(L, \mathcal{A}_{x,\widehat{K}^{\mathrm{alg}}}) = -\chi_{\widehat{K}^{\mathrm{alg}}}(L, \mathcal{H}^{\dagger}_{x,\widehat{K}^{\mathrm{alg}}}) = \left(\frac{d\log\rho_0(L,r)}{d\log r}\right)^{-}|_{r=1} - 1,$$

$$\chi_{\widehat{K}^{\mathrm{alg}}}(L, \mathcal{R}_{x,\widehat{K}^{\mathrm{alg}}}) = 0.$$

Here $(\mathrm{d}v/\mathrm{d}s)^-$ denotes the left-hand side derivative of v with respect to s.

By [Ro2, 10.7] we have

(8.2.4) COROLLARY. *Under the assumption as above, unless simultaneously L is solvable on the $(0,1)$-generic disk, $(d\log\rho_0(L,r)/d\log r)^-|_{r=1} = 1$, and* $\mathrm{Res}_0(\eta(\mathrm{d}x/x), 1)$ *is of Liouville, $L+\alpha$ has indices on $\widehat{K}^{\mathrm{alg}}$ and the formulas*

$$\chi_{\widehat{K}^{\mathrm{alg}}}(L+\alpha, \mathcal{A}_{x,\widehat{K}^{\mathrm{alg}}}) = \chi_{\widehat{K}^{\mathrm{alg}}}(L, \mathcal{A}_{x,\widehat{K}^{\mathrm{alg}}}),$$

$$\chi_{\widehat{K}^{\mathrm{alg}}}(L+\alpha, \mathcal{H}^{\dagger}_{x,\widehat{K}^{\mathrm{alg}}}) = \chi_{\widehat{K}^{\mathrm{alg}}}(L, \mathcal{H}^{\dagger}_{x,\widehat{K}^{\mathrm{alg}}}),$$

$$\chi_{\widehat{K}^{\mathrm{alg}}}(L+\alpha, \mathcal{R}_{x,\widehat{K}^{\mathrm{alg}}}) = 0,$$

hold for any non-Liouville element $\alpha \in \widehat{K}^{\mathrm{alg}}$.

Now we return to the proof of (8.2.2). We point out that we have only to show the assertion for a ∇-rational base e of M by (6.3.7). Choose a π-unipotent base e such that $C_{M,e} \in \mathbf{Z}_{(p)} + \pi x^{-1}O_K[x^{-1}]$. (We can always do by (8.2.1)). Since rational numbers are of non-Liouville, L has indices on $\widehat{K}^{\mathrm{alg}}$ and $\chi_{\widehat{K}^{\mathrm{alg}}}(L_{M,e}, \mathcal{R}_{x,\widehat{K}^{\mathrm{alg}}}) = 0$ by (8.2.3). So we have only to show

$$\chi_{\widehat{K}^{\mathrm{alg}}}(L_{M,e}, \mathcal{A}_{x,\widehat{K}^{\mathrm{alg}}}) = \mathrm{Swan}(V),$$

for a base e as in (8.2.1).

We reduce to the case M is π-unipotent, in other words, V is totally wild ramified. Let N $(p \nmid N)$ be a positive integer such that $[N]^*M$ is π-unipotent, where we use the notation $[N]$ for the both maps $\mathcal{E}^{\dagger}_{x,K} \to \mathcal{E}^{\dagger}_{y,K}$ and $F_{x,k} \to F_{y,k}$

defined by $(x \mapsto y^N)$. We also denote by e the base $1 \otimes e$ of $[N]^*M$. Then e is a base of $[N]^*M$ as in (7.2.1) and we have

$$L_{[N]_*[N]^*M,y^*e} = \begin{pmatrix} L_{M,e} & & 0 \\ & \ddots & \\ 0 & & L_{M,e} + \dfrac{N-1}{N} \end{pmatrix}.$$

So the formula

$$\chi_{\widehat{K}^{\mathrm{alg}}}(L_{[N]^*M,e}, \mathcal{A}_{y,\widehat{K}^{\mathrm{alg}}}) = N\chi_{\widehat{K}^{\mathrm{alg}}}(L_{M,e}, \mathcal{A}_{x,\widehat{K}^{\mathrm{alg}}}),$$

holds by (6.5.2) and (8.2.3). On the other hand, we know that $\mathrm{D}^\dagger_\sigma([N]^*V) \cong [N]^*M$ by [TN2, 4.2.6] and $\mathrm{Swan}([N]^*V) = N\,\mathrm{Swan}(V)$ by [Fol, II. Prop. 6.1.]. Therefore, we reduce to the case that M is π-unipotent.

Assume that M is π-unipotent. Denote by F' the invariant field of $\ker(G_F \to \mathrm{GL}(V))$ in F^{sep} and put $p^s = [F' \colon F]$. By the assumption there is a sequence of extension $F = F_0 \subset \cdots \subset F_s = F'$ which satisfies the condition in (8.1) and let $\mathcal{E}^\dagger_{x_0,K} \subset \mathcal{E}^\dagger_{x_1,K} \subset \cdots \subset \mathcal{E}^\dagger_{x_s,K}$ $(x_0 = x)$ be the sequence of corresponding finite unramified extensions. Denote by f (resp. f_s) both extensions $F \to F'$ and $\mathcal{E}^\dagger_{x,K} \to \mathcal{E}^\dagger_{x_s,K}$ (resp. $F \to F_{s-1}$ and $\mathcal{E}^\dagger_{x,K} \to \mathcal{E}^\dagger_{x_{s-1},K}$). Since the extension F'/F is cyclic of order p^s, we have the decomposition

$$f_*\Lambda \cong \bigoplus_{0 \leqslant i < p^s} V^{\otimes i} \cong (f_s)_*\Lambda \bigoplus \bigoplus_{p \nmid i} V^{\otimes i}, \tag{8.2.5}$$

as $\Lambda[G_F]$-modules. Here Λ is the trivial representation and $V^{\otimes i} = V \otimes \cdots \otimes V$ (i-times). On the side of φ-∇-modules we also have the decomposition

$$f_*\mathcal{E}^\dagger_{x_s,K} \cong \bigoplus_{0 \leqslant i < p^s} M^{\otimes i} \cong (f_s)_*\mathcal{E}^\dagger_{x_{s-1},K} \bigoplus \bigoplus_{p \nmid i} M^{\otimes i}, \tag{8.2.6}$$

where $\mathcal{E}^\dagger$ is the trivial object in $\underline{\mathbf{M\Phi}}^{\nabla \mathrm{et}}_{\mathcal{E}^\dagger,\sigma}$ and $M^{\otimes i} = V \otimes \cdots \otimes V$ (i-times), and

$$\mathrm{D}^\dagger(V^{\otimes i}) \cong M^{\otimes i},$$

by (5.2.1). Since $V^{\otimes i}$ is totally wild ramified, $M^{\otimes i}$ is π-unipotent. Let $e^{(i)}$ be a base of $M^{\otimes i}$ as in (8.2.1). Put $e = e^{(1)}$ to be a base of M.

We prove the identity ‘index = Swan’ by induction on s. In the case when $s = 0$ both sides are 0. By (8.1.2), (8.1.3), (8.2.5), (8.2.6) and the assumption of induction we have

$$\sum_{0<i<p^s, p \nmid i} \chi_{\widehat{K}^{\mathrm{alg}}}(L_{M^{\otimes i}, e^{(i)}}, \mathcal{A}_{x,\widehat{K}^{\mathrm{alg}}}) = \sum_{0<i<p^s, p \nmid i} \mathrm{Swan}(V^{\otimes i}).$$

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Since $\mathrm{Swan}(V^{\otimes i}) = \mathrm{Swan}(V)$ for all i $(p \nmid i)$ [Fol, II. Prop. 6.1.], we have only to show

$$\chi_{\widehat{K}^{\mathrm{alg}}}(L_{M^{\otimes i}, e^{(i)}}, \mathcal{A}_{x, \widehat{K}^{\mathrm{alg}}}) = \chi_{\widehat{K}^{\mathrm{alg}}}(L_{M,e}, \mathcal{A}_{x, \widehat{K}^{\mathrm{alg}}})$$

for all i $(p \nmid i)$. These identities follow from (8.2.3) and the lemma below.

(8.2.7) LEMMA. *If* $r < 1$ *is close to* 1, *then we have*

$$\rho_0(L_{M^{\otimes i}, e^{(i)}}, r) = \rho_0(L_{M,e}, r),$$

for all i $(p \nmid i)$.

Proof. For any element α of $\mathcal{E}^{\dagger}_{x,K}$ and for any real number $0 < r < 1$ close to 1, the radius of convergence of α at the $(0, r)$-generic point t_r is r. Therefore, $\rho_0(L_{M,e}, r)$ is independent of the choice of the basis e if r is close to 1. If $u \neq 0$ is a solution of $L_{M,e}$ at t_r, then u^i is a solution of $L_{M^{\otimes i}, e^{\otimes i}}$. Here $e^{\otimes i} = e \otimes \cdots \otimes e$ (i-times). So we have

$$\rho_0(L_{M^{\otimes i}, e^{(i)}}, r) \geqslant \rho_0(L_{M,e}, r).$$

If i is prime to p, then there is a positive integer j such that $ij \equiv 1 \pmod{p^s}$, so that $(M^{\otimes i})^{\otimes j} \cong M$. By the same argument as above we have

$$\rho_0(L_{M^{\otimes i}, e^{(i)}}, r) \leqslant \rho_0(L_{M,e}, r).$$

Hence, we obtain the assertion. □

(8.3) Now we prove (7.2.1) and (7.2.2). We have already reduced to the case that k is algebraically closed and M is π-unipotent by (7.2.3). Let F' be the fixed field of the kernel $\ker(G_F \to \mathrm{GL}(V))$ and put $G = \mathrm{Gal}(F'/F)$. Since G is finite, V is a successive extension of absolutely irreducible representation of G after a finite extension of Λ. By (7.1.3) and (7.1.2) we can reduce to the case that V is an absolutely irreducible Λ-representation of G. Since G is a finite p-group, V is an induced representation of rank one of a subgroup H of G [Se2, II.10.5. Thm. 20.]. Denote by W the Λ-adic representation of rank one of H and put $F'' = (F')^H$ the H-invariant subfield of F', and $g\colon F \to F''$. Then $g_*W = V$ and $g_*M_W = g_*\mathrm{D}^{\dagger}(W) \cong M$ by (5.2.1). Since H is a p-group, M_W has a π-unipotent and ∇-rational basis e such that

$$\chi_{K(\Gamma)}(L_{M,e}, \mathcal{A}) = \mathrm{Swan}(W),$$
$$\chi_{K(\Gamma)}(L_{M,e}, \mathcal{R}) = 0,$$

by (8.2.1) and (8.2.2). Since G is a finite p-group, there is a sequence of extension $F = F_0 \subset F_1 \subset \cdots \subset F_n = F''$ which satisfies the condition in (8.1). So there is a π-unipotent and ∇-rational basis $\tilde{e}_j$ such that $L_{M,\tilde{e}_j}$ has an index on $K(\Gamma)$ and we have

$$\begin{aligned}&\chi_{K(\Gamma)}(L_{M,\tilde{e}_j},(\mathcal{A}_{x,K(\Gamma)})^{[F'':F]})\\&\quad=\mathrm{rank}(M)\ \mathrm{length}_{O_{F''}}\omega_{O_{F''}/O_F}+\chi_{K(\Gamma)}(L_{M_W,e},\mathcal{A}_{x_n,K(\Gamma)}),\\&\chi_{K(\Gamma)}(L_{M,\tilde{e}_j},(\mathcal{R}_{x,K(\Gamma)})^{[F'':F]})=0,\end{aligned}$$

by (8.1.2). Therefore, we obtain the formula

$$i_{K(\Gamma)}(M,1)=\mathrm{Swan}(V),$$

by (6.3.4) and (8.1.3). □

As another consequence of the proof we have

(8.3.1) PROPOSITION. *Assume k is an algebraically closed field. Let M be a π-unipotent φ-∇-module. Then $\nabla_{M,\mathcal{R}_{K(\Gamma)}}$ has an index and we have*

$$\chi_{K(\Gamma)}(M_{\mathcal{R}_{K(\Gamma)}})=0.$$

(8.3.2) REMARK. It is expected that, for any φ-∇-module M with a ∇-rational basis $\{e_i\}$, the identity $\chi(L_{M,e_i},\mathcal{A})=i(M)=\mathrm{Swan}(V)$ holds like the case of rank one. To show this there is a difficulty which is related to Robba's conjecture [Ro2, 8.3.].

(8.4) We prove (7.3.1). We may assume that $\Gamma=\{1\}$, that is, $K(\Gamma)=\widehat{K}^{\mathrm{alg}}$, and that k is algebraically closed by the same argument of the proof of (8.2.2). Let M be an object in $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger_{x,K},\sigma}$. By (3.5.3) there is a positive integer N ($p\nmid N$) such that $[N]^*M$ is π-unipotent ($[N]\colon\mathcal{E}^\dagger_x\to\mathcal{E}^\dagger_y$ $(x\mapsto y^N)$). We can regard M as a subobject of $[N]_*[N]^*M$ in the category $\underline{\mathbf{M\Phi}}^{\nabla\mathrm{et}}_{\mathcal{E}^\dagger_{x,K},\sigma}$ by the natural map, and denote by M' the quotient $[N]_*[N]^*M/M$. (See (5.1)). Since

$$[N]_*[N]^*M\cong M\oplus yM\oplus\cdots\oplus y^{N-1}M$$

as ∇-modules over $\mathcal{E}^\dagger_{x,K}$, $\nabla_{M,\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}}$ has an index and we have

$$\chi_{\widehat{K}^{\mathrm{alg}}}(\nabla_{M,\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}})+\chi_{\widehat{K}^{\mathrm{alg}}}(\nabla_{M',\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}})=\chi_{\widehat{K}^{\mathrm{alg}}}(\nabla_{[N]_*[N]^*M,\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}})=0,$$

by (6.5.2), (8.3.1). We also know that both $\chi_{\widehat{K}^{\mathrm{alg}}}(\nabla_{M,\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}})$ and $\chi_{\widehat{K}^{\mathrm{alg}}}(\nabla_{M',\mathcal{R}_{\widehat{K}^{\mathrm{alg}}}})$ are less than or equal to 0 by (7.3.3). Therefore, we have the assertion. □

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