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DUAL SYMMETRIC INVERSE MONOIDS AND REPRESENTATION THEORY

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Abstract

There is a substantial theory (modelled on permutation representations of groups) of representations of an inverse semigroup S in a symmetric inverse monoid $\mathscr{I}_X$, that is, a monoid of partial one-to-one selfmaps of a set X. The present paper describes the structure of a categorical dual $\mathscr{I}_X^*$ to the symmetric inverse monoid and discusses representations of an inverse semigroup in this dual symmetric inverse monoid. It is shown how a representation of S by (full) selfmaps of a set X leads to dual pairs of representations in $\mathscr{I}_X$ and $\mathscr{I}_X^*$, and how a number of known representations arise as one or the other of these pairs. Conditions on S are described which ensure that representations of S preserve such infima or suprema as exist in the natural order of S. The categorical treatment allows the construction, from standard functors, of representations of S in certain other inverse algebras (that is, inverse monoids in which all finite infima exist). The paper concludes by distinguishing two subclasses of inverse algebras on the basis of their embedding properties.

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1. Background information

In this paper we consider (i) the dual symmetric inverse monoid $\mathscr{I}_X^*$ of all bijections between the quotient sets of a given set X, and more generally, the dual symmetric inverse monoid $\mathscr{I}_A^*$ of all isomorphisms between the quotient objects of an object A in any sufficiently well-endowed category; and (ii) representations of arbitrary inverse monoids in dual symmetric inverse monoids. To do so with sufficient generality requires that we first recall the category-theoretic framework of symmetric inverse monoids. This approach directs consideration to both duality, and to the existence of extra structure (that of complete inverse algebras) in both the symmetric and dual symmetric cases.

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We adopt the concepts and notation of category theory given in [9] and the theory of inverse semigroups given in [10], with the following exceptions. Firstly, rather than use the standard definition of a subobject as an equivalence class of monics into Y, we wish 'X is a subobject of Y' to be read in the naive sense that there is a monic from the object X to the object Y. Secondly, morphisms (arrows) in a category, such as set functions and group homomorphisms, will be written to the right of their arguments, and so composition will occur from left to right, in diagram order. Functors, however, will be written to the left of their arguments and their composition read from right to left, as usual in [9].

A *monosetting* is a pair $(\mathbf{M}, X)$, where $\mathbf{M}$ is a category and X is a distinguished object of $\mathbf{M}$, such that every object of $\mathbf{M}$ is a subobject of X, every morphism of $\mathbf{M}$ is monic, and $\mathbf{M}$ has finite intersections. We define in dual fashion an *episetting*. An *equivalence* [*duality*] of settings is an equivalence [duality] between the categories preserving the distinguished objects. Settings are encountered in ordinary categories as follows:

If X is an object in a category $\mathbf{K}$ which has finite intersections, denote by $\mathbf{M}(X)$ the subcategory of all monomorphisms between subobjects of X in $\mathbf{K}$. Then the pair $(\mathbf{M}(X), X)$ is a monosetting. If $\mathbf{K}$ is well powered (locally small), then every monosetting $(\mathbf{M}(X), X)$ is equivalent to a *small approximation* $(\mathbf{M}, X)$, where $\mathbf{M}$ is a small subcategory of $\mathbf{M}(X)$ which also contains X and for which the inclusion $\mathbf{M} \subseteq \mathbf{M}(X)$ is an equivalence. Any pair of small approximations of $(\mathbf{M}(X), X)$ are equivalent. Our interest in small approximations rests on the following

CONSTRUCTION 1.1. Let $(\mathbf{M}, X)$ be a small monosetting. Consider parallel pairs of morphisms $(g, g') : A \to X$ in $\mathbf{M}$ (where the common domain A varies with the pair). Two such pairs (g, g') and (h, h') are equivalent if there is an isomorphism $u \in \mathbf{M}$ such that both $h = ug$ and $h' = ug'$. A *fractional morphism* is an equivalence class of pairs, with the class of (g, g') denoted by $[g, g']$. Let $\mathscr{I}(\mathbf{M}, X)$ denote the set of all such fractional morphisms. A multiplication is defined on $\mathscr{I}(\mathbf{M}, X)$ by setting $[g, g'][h, h'] = [mg, nh']$ where m and n arise from the intersection in $\mathbf{M}$ of the middle pair: $mg' = nh$.

It is easy to check, using the universal property of intersections, that the multiplication is well defined. In fact it is straightforward to verify the following theorem, a dual (and slightly more abstract) version of which may also be found in [6] or in [4, Section VII.8]

THEOREM 1.2. (i) *Under the multiplication described above, $\mathscr{I}(\mathbf{M}, X)$ forms an inverse monoid. The idempotents are classes of the form $[g, g]$, with the identity given by $[1_X, 1_X]$ and inversion by $[g, g']^{-1} = [g', g]$.*

(ii) *An equivalence $F : (\mathbf{M}, X) \simeq (\mathbf{N}, Y)$ of small monosettings induces an* *isomorphism* $F^* : \mathscr{I}(\mathbf{M}, X) \cong \mathscr{I}(\mathbf{N}, Y)$ *given by the rule* $[g, g'] \to [Fg, Fg']$. *Conversely, if derived inverse monoids* $\mathscr{I}(\mathbf{M}, X)$ *and* $\mathscr{I}(\mathbf{N}, Y)$ *are isomorphic, then their settings are equivalent.*

The inverse monoid $\mathscr{I}(\mathbf{M}, X)$ is thus called the *classifying monoid* of the monosetting $(\mathbf{M}, X)$. Clearly any small episetting $(\mathbf{E}, X)$ also has a *classifying monoid* $\mathscr{I}(\mathbf{E}, X)$ defined on the set of equivalence classes $[g, g']$ of pairs (g, g') of epimorphisms from X to some common codomain. Upon defining multiplication by $[g, g'][h, h'] = [gm, h'n]$, where $g'm = hn$ is the co-intersection of g' with h, the dual assertions of the above theorem also hold.

Let $\mathbf{K}$ be a well-powered category with object X. If $\mathbf{M}(X)$ is a monosetting, then the *symmetric inverse monoid* of X in $\mathbf{K}$ is the classifying monoid of any small approximation $(\mathbf{M}, X)$ of $(\mathbf{M}(X), X)$, and is denoted $\mathscr{I}_X$ (or $\mathscr{I}(\mathbf{K}, X)$ when the ambient category needs reference). Since any two choices for $(\mathbf{M}, X)$ are equivalent, $\mathscr{I}_X$ is well defined to within isomorphism. For each object X in a mathematically interesting concrete category $\mathbf{K}$, there is usually a standard choice for $(\mathbf{M}, X)$ and thus a standard form for $\mathscr{I}_X$. Given the dual conditions at X, the *dual symmetric inverse monoid* $\mathscr{I}_X^*$ of X in $\mathbf{K}$ is the classifying monoid of any small approximation $(\mathbf{E}, X)$ of the episetting $(\mathbf{E}(X), X)$. (Equivalently, $\mathscr{I}_X^*$ in $\mathbf{K}$ is the classifying monoid of the monosetting at X in the *dual* category $\mathbf{K}^{opp}$.) Again $\mathscr{I}_X^*$ is unique to within isomorphism. A category $\mathbf{K}$ has *[dual] symmetric inverse monoids* if such monoids exist at each of its objects. In many categories both types of monoids exist at each object, with the two monoids usually being nonisomorphic. Clearly we have:

COROLLARY 1.3. *A well-powered category has symmetric inverse monoids if and only if it has finite intersections. Dually, a co-well-powered category has dual symmetric inverse monoids if and only if it has finite co-intersections.*

Most important categories of mathematical objects are usually complete and cocomplete. It follows that the [dual] symmetric inverse monoids of their objects must be correspondingly well endowed. To see what occurs, recall first that any inverse monoid S comes equipped with a natural partial ordering defined in S by $x \geq y$ if and only if $y = yy^{-1}x$, or equivalently $y = xy^{-1}y$, so that y is in some sense a restriction of x. This ordering is compatible with both multiplication and inversion: $x \geq y$ and $u \geq v$ imply $xu \geq yv$ and $x^{-1} \geq y^{-1}$. This leads us to the following definitions. An *inverse algebra* is an inverse monoid which forms a meet semilattice under the natural partial ordering: for all $x, y \in S$, $x \wedge y$ exists in S. Equivalently, an inverse algebra may be defined as an inverse monoid S such that for each $x \in S$, there exists a greatest idempotent $f[x] \in E(S)$ which annihilates x. Thus $x \geq f[x]$, and for any idempotent $e \in S$ such that $x \geq e$ it follows that $f[x] \geq e$. We refer to $f[x]$ as the *fixed point idempotent* of x, and to the induced unary operator $f : S \to E(S)$ as the *fixed point operator* of S. The binary operation $x \wedge y$ and the unary operator $f[x]$ are connected by the identities: $x \wedge y = f[xy^{-1}]y = xf[x^{-1}y]$ and $f[x] = 1 \wedge x$.

An inverse algebra is *complete* if all nonempty subsets have infima. For an inverse algebra S to be complete, it is necessary and sufficient that its set of idempotents $E(S)$ form a complete lattice under the natural partial ordering. In the complete case, infima of nonempty subsets are induced from infima in $E(S)$ by $\inf\{x_i \mid i \in I\} = x_k \inf\{f[x_i^{-1}x_j] \mid i, j \in I\}$, where k is any element of the index set I. Inverse algebras are introduced and described in [7].

A category $\mathbf{K}$ has [*dual*] *symmetric inverse algebras* if it has [dual] symmetric inverse monoids existing at each of its objects, with all such monoids being inverse algebras. If the algebras are also complete, we say that $\mathbf{K}$ has *complete* [*dual*] *symmetric inverse algebras*. When does a category have possibly complete [dual] symmetric inverse algebras? The answer is given by the following extension of Corollary 1.3 together with its epic dual.

THEOREM 1.4. *A well-powered category has* [*complete*] *symmetric inverse algebras if and only if it has finite* [*arbitrary*] *intersections and parallel pairs of monomorphisms have equalizers. In particular, a finitely complete* [*small-complete*], *well-powered category has* [*complete*] *symmetric inverse algebras.*

The details of the proof, as well as a more thorough discussion of foundational issues, are given in [8]. We remark, however, that in the fractional construction given above, the fixed point idempotent of the class $[g, g']$, if it exists, is the class $[mg, mg']$ where m is the equalizer of g and g' in the monosetting.

The final assertion of Theorem 1.4 explains why [dual] symmetric inverse monoids of objects in categories of interest to mathematicians typically form complete inverse algebras: such categories tend to be both complete and cocomplete as categories. This is indeed the case with the following categories of interest in this paper: the category **Set** of sets and functions between them; the category of all algebras in a given variety $\mathscr{V}$ of algebras of a given type, together with all homomorphisms between them; in particular, for a given ring R, the category R-**Mod** of left R-modules and module homomorphisms; and the category **Ab** of abelian groups and homomorphisms.

In what follows we shall be particularly interested in three questions: What can be said about dual symmetric inverse monoids? How do they differ from symmetric inverse monoids? What can be said about representations of inverse monoids and algebras in dual symmetric inverse algebras of objects in the above categories?

We begin in the next two sections by examining the dual symmetric inverse monoid $\mathscr{I}_X^*$ of a set X and comparing it, both as monoid and inverse algebra, with the more familiar symmetric inverse monoid $\mathscr{I}_X$. The fourth section treats representations of inverse monoids in dual symmetric inverse monoids of sets, and the fifth, representations of inverse monoids and algebras in [dual] symmetric inverse algebras of members of a variety, particularly abelian groups. In these final sections, particular attention will be given to the question of when an inverse algebra has a faithful representation as an algebra in the dual symmetric inverse algebra of a set or of some abelian group. Our study will involve Cayley-like representations such as the [dual] Wagner-Preston Theorem, as well as induced representations obtained as instances of the following useful result and its dual.

THEOREM 1.5. *Suppose* $\mathbf{K}$ *has symmetric inverse monoids, and let* $F : \mathbf{K} \to \mathbf{K}'$ *be a functor. If* F *preserves monics and intersections, then at each object* X *of* $\mathbf{K}$, F *induces a homomorphism of symmetric inverse monoids* $f_X : \mathscr{I}_X \to \mathscr{I}_{FX}$ *given by:* $[g, g'] \mapsto [Fg, Fg']$. *If* F *also preserves equalizers of parallel pairs of monics, then* f_X *is a homomorphism of inverse algebras. If, further,* F *reflects isomorphisms, then* f_X *is injective.*

2. The dual symmetric inverse monoid of a set

When X is a set (an object in the category **Set**) one may identify $\mathscr{I}_X = \mathscr{I}(\mathbf{M}(X), X)$ with the symmetric inverse monoid (also denoted $\mathscr{I}_X$) of all partial 1-1 selfmaps of X, under the isomorphism $[g, g'] \mapsto g^{-1}g'$ for pairs of monics (g, g') with codomain X. We now describe $\mathscr{I}^*_X = \mathscr{I}(\mathbf{E}(X), X)$ using the notions of block bijections and biequivalences.

Recall that, on partitioning X into disjoint, nonempty subsets or *blocks* $A, B, \dots$, there results the *quotient set* $\{A, B, \dots\}$. Then a *block bijection* of X is a bijection μ between two quotient sets of X. Such a μ may be depicted using a variant form of the usual permutation notation where, for example, $\mu = \left(\begin{array}{c|c|c} A_1 & A_2 & \cdots \\ B_1 & B_2 & \cdots \end{array}\right)$ indicates that $(A_1 \mid A_2 \mid \cdots)$ and $(B_1 \mid B_2 \mid \cdots)$ are respectively the domain and codomain partitions of X, and that, for each member i of a common index set I, block A_i maps under μ to block B_i. Since $\bigcup_{i \in I} (A_i \times B_i)$ is then a special kind of binary relation, it is convenient to make the following definition. A binary relation α on X is a *biequivalence* if it is both *full*, that is $X\alpha = \alpha X = X$, and *bifunctional*, as we translate *difonctionelle* [12]; that is, $\alpha \circ \alpha^{-1} \circ \alpha \subseteq \alpha$ where $\circ$ denotes the usual composition of binary relations on a set. Since the inclusion $\alpha \circ \alpha^{-1} \circ \alpha \supseteq \alpha$ always holds, the condition $\alpha \circ \alpha^{-1} \circ \alpha \subseteq \alpha$ is equivalent to asserting that $\alpha \circ \alpha^{-1} \circ \alpha = \alpha$. Note that the relational inverse α^{-1} of a biequivalence α is also a biequivalence. Now biequivalences are essentially block bijections:

LEMMA 2.1. *If* α *is a biequivalence on X, then both* $\alpha \circ \alpha^{-1}$ *and* $\alpha^{-1} \circ \alpha$ *are equivalence relations on X. Moreover the map* $\widetilde{\alpha}$ *defined by* $\widetilde{\alpha} : x(\alpha \circ \alpha^{-1}) \mapsto x\alpha$ *for* $x \in X$ *is a block bijection of* $X/\alpha \circ \alpha^{-1}$ *to* $X/\alpha^{-1} \circ \alpha$. *Conversely, given equivalence* *relations β and γ on X together with a block bijection $\mu : X/\beta \to X/\gamma$, a unique biequivalence $\hat{\mu}$ on X inducing μ is given by: $x\hat{\mu}y$ if and only if $x\beta \mapsto y\gamma$ under the block bijection μ (in which case $\beta = \hat{\mu} \circ \hat{\mu}^{-1}$ and $\gamma = \hat{\mu}^{-1} \circ \hat{\mu}$). Finally, the two processes are reciprocal: $\hat{\tilde{\alpha}} = \alpha$ and $\tilde{\hat{\mu}} = \mu$.*

We now define the *dual symmetric inverse monoid on a set X*, also denoted by $\mathscr{I}_X^*$, to consist of all biequivalences on X with multiplication

$$\alpha\beta = \alpha \circ (\alpha^{-1} \circ \alpha \vee \beta \circ \beta^{-1}) \circ \beta,$$

where $\vee$ denotes the familiar join in the lattice of equivalence relations. It is straightforward to check that $\mathscr{I}_X^*$ is an inverse monoid. Equivalent formulations of the product are

$$\alpha\beta = \bigcap\{\gamma \in \mathscr{I}_X^* \mid \alpha \circ \beta \subseteq \gamma\} = \bigcup\{\alpha \circ \beta \circ (\beta^{-1} \circ \alpha^{-1} \circ \alpha \circ \beta)^n \mid n \in \mathbb{N}\}.$$

It is also useful to record here that $\alpha\alpha^{-1} = \alpha \circ \alpha^{-1}$ and $\alpha\alpha^{-1}\alpha = \alpha \circ \alpha^{-1} \circ \alpha$ for any α, and that $\alpha\beta = \alpha \vee \beta$ when α and β are equivalence relations. Further, a biequivalence is an idempotent if, and only if, it is an equivalence, and the zero and identity elements of $\mathscr{I}_X^*$ are $\nabla = X \times X$ and $\Delta = \{(x, x) \mid x \in X\}$ respectively. Lastly, it is easy to check that $\mathscr{I}_X^*$ forms a complete inverse algebra, with the infimum of any $\mathscr{A} \subseteq \mathscr{I}_X^*$ given by $\inf \mathscr{A} = \bigcap\{\beta \in \mathscr{I}_X^* \mid \alpha \subseteq \beta$ for all $\alpha \in \mathscr{A}\}$, and the fixed point idempotent of $\alpha \in \mathscr{I}_X^*$ by the equivalence relation generated by α.

Regarding biequivalences as block bijections, the connection with the categorical definition of $\mathscr{I}_X^*$ becomes apparent: an equivalence class $[f, g]$ (where f and g are mappings from X onto some common codomain Y) induces a block bijection μ from the partition $\{yf^{-1} \mid y \in Y\}$ to the partition $\{yg^{-1} \mid y \in Y\}$ defined by the rule $\mu : yf^{-1} \longmapsto yg^{-1}$. It is again easy to check that this produces a well-defined 1-1 correspondence, and that the multiplication in the categorical description of Section 1 matches the multiplication of biequivalences as described above.

Biequivalences are studied by Schein in [14], where they are termed bifunctional multipermutations. However, for a multiplication operation, Schein considers only composition of binary relations, and in contrast to the multiplication in $\mathscr{I}_X^*$ defined above, the set of *all* biequivalences on a set is not closed under composition.

Note that the empty function $e : \emptyset \to \emptyset$ in **Set** is an epi, so that $\mathscr{I}_\emptyset^*$ consists of the single element $[e, e]$ (corresponding to the empty biequivalence on $\emptyset$). Henceforth we consider only those $\mathscr{I}_X^*$ with X nonempty. Let $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ be termed respectively the *left* and *right equivalences* (or *partitions*) of α. By the *rank* of α is meant the cardinality of $X/\alpha\alpha^{-1}$. Combining basic inverse semigroup theory with elementary combinatorics allows aspects of the local structure of $\mathscr{I}_X^*$ to be described as follows.

THEOREM 2.2. *Let* $\alpha, \beta \in \mathscr{I}_X^*$.

(i) $\alpha \mathscr{L} \beta$ [*respectively*, $\alpha \mathscr{R} \beta$] *if and only if* α *and* β *have the same right* [*left*] *equivalences.*

(ii) $\mathscr{D} = \mathscr{J}$; *moreover,* $\alpha \mathscr{D} \beta$ *if and only if* α *and* β *have the same rank.*

(iii) *For each cardinal* λ *such that* $1 \le \lambda \le |X|$ *there is a* $\mathscr{J}$*-class* D_λ *of that rank. Moreover, in the usual ordering of* $\mathscr{J}$*-classes,* $D_\lambda \le D_\mu$ *if and only if* $\lambda \le \mu$, *so that the* $\mathscr{J}$*-classes of* $\mathscr{I}_X^*$ *are totally ordered.*

(iv) *The maximal subgroup associated with an idempotent of rank* r *is isomorphic with* $\mathfrak{S}_r$, *the symmetric group of all permutations on a set of cardinality* r.

(v) *When* X *is finite with* $|X| = n$, *then for all* $1 \le r \le n$ *the number of idempotents in the* $\mathscr{J}$*-class* D_r *of rank* r *is the Stirling number of the second kind* $S_{n,r}$, *and* D_r *has cardinality* $r!S_{n,r}^2$.

(vi) *When* X *is finite with* $|X| = n$, *then the total number of idempotents of* $\mathscr{I}_X^*$ *is the Bell number* $B_n = \sum_{r=1}^n S_{n,r}$ *and* $\left|\mathscr{I}_X^*\right| = \sum_{r=1}^n r!S_{n,r}^2$.

From this theorem follow two immediate observations.

First, upon comparing the size of $\mathscr{I}_X^*$ as given by Theorem 2.2(vi) above with $|\mathscr{I}_X| = \sum_{r=0}^n \binom{n}{r} P_{n,r} = \sum_{r=0}^n r! \binom{n}{r}^2$ (where $n = |X|$), it follows that, for $n \ge 4$, $\left|\mathscr{I}_X^*\right| > |\mathscr{I}_X|$ and in fact $\left|\mathscr{I}_X^*\right| / |\mathscr{I}_X| \to \infty$ as $n \to \infty$. Comparative values for small n are given in the following table:

$\lvert X\rvert$	0	1	2	3	4	5	6
$\lvert I_X\rvert$	1	2	7	34	209	1546	13 327
$\lvert I_X^*\rvert$	1	1	3	25	339	6721	179 643

Second, for all finite nonempty X, $\mathscr{I}_X$ cannot be embedded into $\mathscr{I}_X^*$. Indeed if $|X| = n$, then a maximal chain of idempotents in $\mathscr{I}_X$ has $n+1$ elements, while by Theorem 2.2(iii) a maximal such chain in $\mathscr{I}_X^*$ has only n elements. Setting $X_0 = X \cup \{0\}$ where $0 \notin X$, an embedding $\theta_X : \mathscr{I}_X \to \mathscr{I}_{X_0}^*$ is defined as follows. Given $\alpha \in \mathscr{I}_X$, first identify α with the subset of $X \times X$ given by its graph $\{(x, y) \mid x\alpha = y\}$ and then set

$$\theta_X(\alpha) = \alpha \cup (A_\alpha \times B_\alpha) \subseteq X_0 \times X_0,$$

where $A_\alpha = X_0 \backslash \text{domain}(\alpha)$ and $B_\alpha = X_0 \backslash \text{codomain}(\alpha)$. The map θ_X is clearly well-defined and one-to-one. That it is a homomorphism is easy to check directly (and is also a consequence of Theorem 4.1 below). Composing θ_X with the Wagner-Preston embedding of S in $\mathscr{I}_X$, it follows that any given inverse monoid S can be embedded in a dual symmetric inverse monoid on a set X of cardinality at most $|S| + 1$. In Section 4, where we examine embeddings and more generally representations, it will be seen that this cardinality may be reduced to $|S|$, by virtue of a dual of the Wagner-Preston Theorem. It will also be seen that $\mathscr{I}_X$ and $\mathscr{I}_X^*$ differ in their properties as the codomains of embeddings; for this, we need first to distinguish certain aspects of their structures.

3. Comparing the structures of $\mathscr{I}_X$ and $\mathscr{I}_X^*$

Recall that an inverse monoid S with group of units G and semilattice of idempotents E is *factorizable* if $S = GE$; or equivalently, if for each $x \in S$ there is some $g \in G$ such that $x \leq g$. In general, $F(S) = GE = EG$ is the greatest factorizable inverse submonoid of any S. If X is finite, then $\mathscr{I}_X$ is factorizable; otherwise, $F(\mathscr{I}_X)$ is a proper submonoid of $\mathscr{I}_X$. For $|X| \geq 2$, every element in $F(\mathscr{I}_X)$, except for those partial bijections α for which both the domain and range of α have singleton complements in X, may be expressed as the infimum of some nonempty subset of the group of units. By contrast, in the dual symmetric inverse monoid we have

PROPOSITION 3.1. *If $|X| \geq 3$, then $\mathscr{F}_X^* = F(\mathscr{I}_X^*)$ is a proper submonoid of $\mathscr{I}_X^*$. For all X, each element of $\mathscr{F}_X^*$ arises as the infimum of some nonempty subset of the group of units $\mathfrak{S}_X$ of $\mathscr{I}_X^*$.*

PROOF. Given $\alpha \in \mathscr{I}_X^*$, $\alpha \in \mathscr{F}_X^*$ if and only if $\sigma \subseteq \alpha$ for some permutation σ of X. But this is equivalent to asserting that each pair of corresponding blocks A and A' under the induced block bijection $\widetilde{\alpha}$ have common cardinality. In this case $\widetilde{\alpha}$ and α are called *uniform*. Suppose that $|X| \geq 3$ and let $x \in X$. Then the block transposition interchanging $\{x\}$ with $X \backslash \{x\}$ is not uniform. The assertion about $\mathscr{F}_X^*$ being generated from $\mathfrak{S}_X$ via infima is a consequence of the following observation: let A and B be subsets of common cardinality, and let $\Gamma(A, B)$ denote the set of all bijections $\gamma : A \to B$ with each bijection viewed as a subset of $A \times B$. Then $A \times B = \bigcup \Gamma(A, B)$. Hence if the uniform biequivalence α decomposes as $\alpha = \bigcup_{i \in I}(A_i \times B_i)$, then α is the union of all permutations ρ of X of the form $\rho = \bigcup_{i \in I} \gamma_i$ where $\gamma_i \in \Gamma(A_i, B_i)$ for all $i \in I$. That is, in the algebra $\mathscr{F}_X^*$, α is the infimum of all such ρ. □

The $\mathscr{D}$-classes of $\mathscr{F}_X^*$, unlike those of $\mathscr{I}_X$ and $\mathscr{I}_X^*$, are not linearly ordered by rank. Let $|X| = n$. A partition η of X is *of type* $1^{r_1}2^{r_2} \dots n^{r_n}$ if there are r_i blocks of size i for $i = 1, 2, \dots, n$. If η has rank r, then the restrictions on the integers r_i are that $r_i \geq 0$, $\sum_{i=1}^n r_i = r$, and $\sum_{i=1}^n ir_i = n$. Due to uniformity, $\mathscr{D}$-classes in $\mathscr{F}_X^*$ are classified by type, with a $\mathscr{D}$-class D consisting of all biequivalences α whose left and right equivalences $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ share the common *type* of D. If this type is $1^{r_1}2^{r_2} \dots n^{r_n}$, then any maximal subgroup of D is isomorphic to the product of the permutation groups $\mathfrak{S}_{r_i}$ and so contains $g_D = r_1!r_2! \cdots r_n!$ elements. The number of idempotents in D equals the number of partitions of type $1^{r_1}2^{r_2}\dots n^{r_n}$, which is

$$\pi_D = \frac{n!}{(1!)^{r_1}(2!)^{r_2}\cdots(n!)^{r_n}r_1!r_2!\cdots r_n!}.$$

Since the number of elements in D is $g_D\pi_D^2$, the cardinality of $\mathscr{F}_X^*$ is

$$\sum \frac{(n!)^2}{r_1!r_2!\cdots r_n!(1!)^{2r_1}(2!)^{2r_2}\cdots(n!)^{2r_n}},$$

the sum being taken over all non-negative integer n-tuples $(r_1, r_2, \dots r_n)$ such that $\sum_{i=1}^n ir_i = n$. A brief table of computed values for $|\mathscr{F}_X^*|$ is given below.

$\lvert X\rvert$	0	1	2	3	4	5	6
$\lvert\mathscr{F}_X^*\rvert$	1	1	3	16	131	1496	22 482

Our present interest in $\mathscr{F}_X^*$ lies in the way the fixed point structure of elements in $\mathscr{F}_X^*$ distinguishes $\mathscr{I}_X^*$ from $\mathscr{I}_X$ and other inverse algebras. First we summarise some arithmetic information which follows from the description in the proof of Proposition 3.1.

PROPOSITION 3.2. *Let σ be a permutation of finite order $n > 1$ having orbits $O_1, O_2, \dots$ with $|O_k| = n_k$. Then* (i) *n is the least common multiple of the n_k, and* (ii) *the equivalence classes of $f[\sigma]$ in $\mathscr{F}_X^*$ (and in $\mathscr{I}_X^*$) are precisely the orbits O_k. It follows that if τ is a permutation of differing order, then $f[\sigma] \neq f[\tau]$. In particular,* (iii) *if m and p are distinct positive divisors of n, then $f[\sigma^m] \neq f[\sigma^p]$; and* (iv) *if m is relatively prime to n, then $f[\sigma^m] = f[\sigma]$ since σ^m and σ have the same orbits. Thus,* (v) *σ creates distinct idempotents $f[\sigma^m]$ in $\mathscr{F}_X^*$, for distinct positive divisors m of n.*

EXAMPLE 3.3. In $\mathscr{I}_{\{1,2,3,4\}}$, by contrast, $f[(1234)] = f[(13)(24)]$ where $(13)(24) = (1234)^2$ has order 2. Hence, while the monoid $\mathscr{I}_{\{1,2,3,4\}}$ can be embedded into $\mathscr{I}^*_{\{0,1,2,3,4\}}$, it cannot be embedded *as an inverse algebra* (that is, with preservation of fixed point idempotents or equivalently of natural meets) into the dual inverse algebra $\mathscr{I}_X^*$ of *any* set X.

Recall (see [7], Section 1) that inverse algebras form a variety of (universal) algebras of type $\langle 2, 2, 1, 0\rangle$. An inverse algebra S is *generated by* $x \in S$ if it is the only subalgebra containing x. If, in addition, x satisfies $x^n = 1$ for some $n \geq 1$ with n the least such exponent, then S is said to be *n-cyclic* on x. It is *freely n-cyclic* on x if $x^n = 1$ determines the structure of S, in which case the group of units is cyclic on x of order n. S is just *freely cyclic* on x if S is generated by x and is determined solely by the relations $xx^{-1} = 1 = x^{-1}x$, in which case the group of units is infinite cyclic on x.

Either type of freely cyclic inverse algebra is unique to within isomorphism and may be constructed as follows. If G is a cyclic group on x, let $\mathbf{K}(G)$ denote the coset algebra consisting of cosets $\langle x^m \rangle x^k$ of subgroups $\langle x^m \rangle$ of G, where m divides n if $n < \infty$. Multiplication is given by $\langle x^m \rangle x^k \cdot \langle x^p \rangle x^q = (\langle x^m \rangle \vee \langle x^p \rangle) x^{k+q}$ with the fixed point operator given by $f[\langle x^m \rangle x^k] = \langle x^m \rangle \vee \langle x^k \rangle$. If we identify x with the coset $\{1\}x$, then results in [7, Section 3] imply that $\mathbf{K}(G)$ is the freely (n-)cyclic algebra on x.

THEOREM 3.4. *Let σ be a permutation on X, and let S be the subalgebra generated by σ in $\mathscr{I}_X^*$. If σ has order n, then S is freely n-cyclic on σ. If X is infinite and σ has an infinite orbit, then S is freely cyclic on σ.*

PROOF. If G is the subgroup of $\mathscr{I}_X^*$ generated from σ, then the inclusion $\{\sigma\} \subseteq \mathscr{I}_X^*$ induces an algebra homomorphism of $\mathbf{K}(G)$ upon S defined by $\langle \sigma^m \rangle \sigma^k \mapsto f[\sigma^m]\sigma^k$. By [7, Proposition 1.18], in order to show that this map is one-to-one, we need only show that it is one-to-one on idempotents. So suppose that subgroups $\langle \sigma^m \rangle$ and $\langle \sigma^p \rangle$ map to the same idempotent $f[\sigma^m] = f[\sigma^p]$ in $\mathscr{I}_X^*$. If σ has finite order n, then m and p are positive divisors of n, so that by Proposition 3.2(v), $m = p$. If $(\dots x_{-2}\, x_{-1}\, x_0\, x_1\, x_2 \dots)$ is an infinite cycle of σ corresponding to an infinite orbit, then $f[\sigma^m]$ has the equivalence class $\{x_{km} \mid k \in \mathbb{Z}\}$, $f[\sigma^p]$ has the class $\{x_{kp} \mid k \in \mathbb{Z}\}$, and again $m = p$. □

The infinite orbit is required in Theorem 3.4. For consider the permutation $\tau = (12)(3456)(78\dots)\dots$ of infinite order, with orbits of increasing orders $2, 4, 8, 16, \dots$, but having no infinite orbit. In $\mathscr{I}_X^*$, $f[\tau] = f[\tau^3]$, so that the subalgebra generated by τ is not freely cyclic.

The $\mathscr{I}_X$ variant of the above theorem fails. A finite counterexample is provided by the 4-cycle (1234) in $\mathscr{I}_{\{1,2,3,4\}}$. For an infinite counterexample, take X as the set $\mathbb{Z}$ of integers, and σ as the infinite cycle $(\dots, -2, -1, 0, 1, 2, \dots)$. Then for all $n \neq 0$, σ^n has no fixed points and so $f[\sigma^n] = f[\sigma] = \emptyset$. Thus the subalgebra of $\mathscr{I}_{\mathbb{Z}}$ generated from σ is G^0, where G is the cyclic group on σ, while $\mathbf{K}(G)$ is a lattice of infinitely many cyclic groups.

We have seen how the behavior of cyclic subalgebras distinguishes dual symmetric inverse algebras $\mathscr{I}_X^*$ from symmetric inverse algebras $\mathscr{I}_X$. We now consider another distinguishing feature involving permutations of order 2, and more generally, biequivalences α such that $\alpha^3 = \alpha$.

PROPOSITION 3.5. *If $\alpha, \beta \in \mathscr{I}_X^*$ with $\alpha^3 = \alpha$ and $\alpha \geq \beta \geq f[\alpha]$, then $\beta^3 = \beta$.*

PROOF. Suppose first that α is a permutation, and so factors as a product of disjoint transpositions (a_i, b_i). Thus each equivalence class of $f[\alpha]$ is a binary subset $\{a_i, b_i\}$

or a singleton $\{c\}$, with $c \notin \bigcup_i \{a_i, b_i\}$. All biequivalences β such that $\alpha > \beta > f[\alpha]$ arise by replacing, in $f[\alpha]$, some of its subrelations $\{a_i, b_i\} \times \{a_i, b_i\}$ by the original transposition (a_i, b_i), and all such β satisfy $\beta^3 = \beta$. In the general case where $\alpha^3 = \alpha$, α is its own inverse α^{-1} and lies in the maximal subgroup of the idempotent $\varepsilon = \alpha^2$. Now if β be such that $\alpha \geq \beta \geq f[\alpha]$, then α, $\beta$and $f[\alpha]$ all lie in the local monoid $\varepsilon \mathscr{I}_X^* \varepsilon$, which is isomorphic with the dual symmetric algebra $\mathscr{I}_Y^*$, where Y denotes the quotient set X/ε. Then $\beta^3 = \beta$ follows by the first part of the proof. $\square$

This proposition allows the strengthening of an observation in Example 3.3.

COROLLARY 3.6. *If $|X| \geq 2$, then no semigroup embedding of $\mathscr{I}_X$ into some $\mathscr{I}_Y^*$ exists which preserves all instances of fixed point idempotents and natural meets.*

PROOF. Let a and b be distinct elements of X and let σ be the transposition (a, b) interchanging a and b, but fixing all other elements of X. In $\mathscr{I}_X$, $f[\sigma]$ is the identity map on the subset $X \setminus \{a, b\}$. Next let τ be the partial bijection which sends a to b and fixes all $x \in X \setminus \{a, b\}$. Note that $\sigma > \tau > f[\sigma]$ and $\tau^2 = f[\sigma]$, so that $\tau^3 \neq \tau$. Let $\theta : \mathscr{I}_X \to \mathscr{I}_Y^*$ be a monoid embedding with $\sigma\theta = \alpha$ and $\tau\theta = \beta$. Then in $\mathscr{I}_Y^*$, $\alpha^3 = \alpha$, $\beta^3 \neq \beta$, but $\alpha \geq \beta \geq (f[\sigma])\theta$. By Proposition 3.5, $(f[\sigma])\theta$ cannot be $f[\sigma\theta]$ and so θ does not always preserve fixed point idempotents. By the identity $f[x] = 1 \wedge x$, neither are meets always preserved. $\square$

We conclude this section with some further comparisons and contrasts between $\mathscr{I}_X$ and $\mathscr{I}_X^*$. For $|X| \geq 2$, both $\mathscr{I}_X$ and $\mathscr{I}_X^*$ have a unique 0-minimal $\mathscr{D}$-class M, lying directly above the 0-class: in $\mathscr{I}_X$ this is the class of singleton relations, in $\mathscr{I}_X^*$ the class D_2 of biequivalences of rank 2. In both $\mathscr{I}_X$ and $\mathscr{I}_X^*$ one has $\alpha = \sup\{\beta \in M \mid \beta \leq \alpha\}$ for any α, with the zero element arising as the empty supremum. However the fact that the class D_2 has subgroups of order 2 leads to yet another distinction between $\mathscr{I}_X$ and $\mathscr{I}_X^*$. Recall that the coarsest idempotent-separating congruence μ on $\mathscr{I}_X$ is the equality relation. By way of contrast we have

PROPOSITION 3.7. *The coarsest idempotent-separating congruence μ on $\mathscr{I}_X^*$ is the relation $\{(\alpha, \beta) | \alpha = \beta$ or else $\alpha \mathscr{H} \beta$ with $\alpha, \beta \in D_2\}$.*

PROOF. Recall that for any inverse semigroup S, μ is given by $\mu = \{(a, b) \in S \times S \mid a^{-1}fa = b^{-1}fb$ for all $f \in E(S)\}$ and that $\mu \subseteq \mathscr{H}$.

Suppose that S has a zero, that $e > 0$ is a primitive idempotent, and that $a \in H_e$. Then for any idempotent f one has both $a^{-1}fa = a^{-1}aa^{-1}fa = a^{-1}efa$ and $e^{-1}fe = ef$. But ef is either e or 0, as e is primitive, and since $a \in H_e$, one has in either case that $a^{-1}fa = e^{-1}fe = ef$ and so $(a, e) \in \mu$. It follows that—letting ν temporarily denote the relation described in the proposition—at least $\nu \subseteq \mu$.

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For the reverse inclusion, suppose that $\alpha \mathscr{H} \beta$. Then α and β share both the same left equivalence, with blocks say $\{A_i \mid i \in I\}$, and the same right equivalence, with blocks $\{B_i \mid i \in I\}$, where the common index set I has cardinality equal to the common rank r of α and β. If $r = 1$, then $\alpha = \nabla = \beta$, and if $r = 2$, then $(\alpha, \beta) \in \nu$ by definition. If $r \geq 3$ but $\alpha \neq \beta$, then it is the case that some block A_i corresponds under α to B_i, but under β to B_j , where $B_i \neq B_j$. Let η be the idempotent determined by the partition, $\{A_i, X \setminus A_i\}$. Then the idempotent $\alpha^{-1}\eta\alpha$ is determined by the partition $\{B_i, X \setminus B_i\}$ while $\beta^{-1}\eta\beta$ is determined by $\{B_j, X \setminus B_j\}$, and hence is distinct. Hence $(\alpha, \beta) \notin \mu$. Thus for rank $r \geq 3$, $(\alpha, \beta) \in \mu$ implies $\alpha = \beta$, so that $\mu = \nu$ follows. □

As is easily checked, the Munn representation of $\mathscr{I}_X^*$ is surjective, and so (analogously to $\mathscr{I}_X$) $\mathscr{I}_X^*/\mu$ is isomorphic with its own T_E, namely the monoid of all isomorphisms between principal ideals of the lattice of equivalence relations on X.

A monoid congruence θ on an inverse algebra S is a [*complete*] *algebra congruence* if the homomorphism it induces is a homomorphism of [complete] inverse algebras, that is, preserves meets [respectively, arbitrary infima]. Recall from [7, Section 2] that θ is an algebra congruence if and only if it is the minimum congruence in its trace class, that is, θ is the monoid congruence generated from its restriction $\theta|_{E(S)}$ to the semilattice of idempotents $E(S)$. Thus if θ_1, θ_2 are algebra congruences such that $\theta_1|_{E(S)} = \theta_2|_{E(S)}$, then $\theta_1 = \theta_2$. This observation is used in the next theorem, which determines all complete inverse algebra congruences on $\mathscr{I}_X$ and $\mathscr{I}_X^*$. Here we denote idempotents in $\mathscr{I}_X$ by ι_A (the identity on $A \subseteq X$), and idempotents in $\mathscr{I}_X^*$ by ε (an equivalence on X).

THEOREM 3.8. *The complete algebra congruences on $\mathscr{I}_X$, and those on $\mathscr{I}_X^*$, are precisely the Rees ideal congruences.*

PROOF. It is easy to verify directly that the Rees ideal congruences on a complete inverse algebra are indeed complete algebra congruences. For the converse statements, first let θ be a complete algebra congruence on $\mathscr{I}_X$, and suppose $(\iota_A, \iota_B) \in \theta$ with $A \neq B$. Without loss of generality we may suppose that $B \subsetneq A$, since $(\iota_A, \iota_{A\cap B}) \in \theta$. Let Σ be the set of permutations of A, so that $\Sigma \subseteq \mathscr{I}_X$. Then for all $\sigma \in \Sigma$, $\left(\sigma^{-1}\iota_A\sigma, \sigma^{-1}\iota_B\sigma\right) = (\iota_A, \iota_{B\sigma}) \in \theta$, whence $\left(\iota_A, \bigcap_{\sigma\in\Sigma} \iota_{B\sigma}\right) \in \theta$ since θ is complete. But $\bigcap_{\sigma\in\Sigma} \iota_{B\sigma} = \emptyset$, so $(\iota_A, \emptyset) \in \theta$. Thus $\theta|_E$ is the restriction to $E = E(\mathscr{I}_X)$ of a Rees ideal congruence, which latter is then identified with θ.

Secondly, let θ be a complete algebra congruence on $\mathscr{I}_X^*$, and suppose $(\varepsilon, \varepsilon') \in \theta$ with $\varepsilon \neq \varepsilon'$. Again without loss of generality we may suppose that $\varepsilon' < \varepsilon$. Let Σ' be the set of permutations of X/ε. For $\sigma' \in \Sigma'$, set $\sigma = \{(x, y) \in X \times X : (x\varepsilon, y\varepsilon) \in \sigma'\}$. Let $\Sigma = \{\sigma : \sigma' \in \Sigma'\}$, so $\Sigma \subseteq \mathscr{I}_X^*$. Then for all $\sigma \in \Sigma$, $(\sigma^{-1}\varepsilon\sigma, \sigma^{-1}\varepsilon'\sigma) = (\varepsilon, \sigma^{-1}\varepsilon'\sigma) \in \theta$, and so $(\varepsilon, \inf_{\sigma\in\Sigma}(\sigma^{-1}\varepsilon'\sigma)) \in \theta$ since θ is complete. But

$\inf_{\sigma\in\Sigma}(\sigma^{-1}\varepsilon'\sigma) = X \times X = \nabla$ and so $(\varepsilon, \nabla) \in \theta$. As above, θ is a Rees ideal congruence on $\mathscr{I}_X^*$. □

COROLLARY 3.9. *If X is finite, the algebra congruences on $\mathscr{I}_X$, and those on $\mathscr{I}_X^*$, are precisely the Rees ideal congruences.*

There are algebra congruences on $\mathscr{I}_X$ and $\mathscr{I}_X^*$ (with X infinite) which are not complete (and so not Rees ideal congruences). To show this, it is enough to produce a normal congruence on $E = E(\mathscr{I}_X)$ [respectively $E^* = E(\mathscr{I}_X^*)$] which is not complete; for the congruence which it generates on $\mathscr{I}_X$ [respectively $\mathscr{I}_X^*$] is an algebra congruence which is not complete. The congruence on E which relates ι_A and ι_B if A and B differ by a finite set is such a congruence, as is, analogously, the congruence on E^* which relates two equivalences if their partitions differ (as subsets of 2^X) by a finite set.

4. Representations

Let S be an inverse semigroup and T a semigroup. By a *representation* of S in T is meant a (semigroup) homomorphism of S into T. A representation is *faithful* (or, as previously, an embedding) if it is one-to-one. In the case that S and T are inverse algebras, a representation $\varphi : S \to T$ will be called *algebraic* if it preserves natural meets (equivalently, preserves fixed point idempotents) and preserves the identity ($1_S\varphi = 1_T$).

Our interest here is in the case where T is one of the following monoids: $\mathscr{T}_X$, the monoid of all transformations on a given set X; $\mathscr{P}\mathscr{T}_X$, the monoid of all partial transformations on a set X; $\mathscr{I}_X$, the inverse monoid of all one-to-one partial transformations on X; $\mathscr{B}_X$, the monoid of all binary relations on X; and $\mathscr{I}_X^*$, here understood as the inverse monoid of all biequivalences on X. The monoids $\mathscr{T}_X$, $\mathscr{P}\mathscr{T}_X$ and $\mathscr{I}_X$ are all submonoids of $\mathscr{B}_X$. While $\mathscr{I}_X^*$ is only a subset of $\mathscr{B}_X$, observe that for every full relation α on X, there is a unique smallest biequivalence $\eta \in \mathscr{I}_X^*$ such that $\alpha \subseteq \eta$, called the *closure* of α in $\mathscr{I}_X^*$. It is denoted by α^+ and may be described equivalently as $\alpha^+ = \bigcap\{\gamma \in \mathscr{I}_X^* \mid \alpha \subseteq \gamma\}$ or as $\alpha^+ = \bigcup\{\alpha \circ (\alpha^{-1} \circ \alpha)^n \mid n \in \mathbb{N}\}$; by previous remarks, $\alpha\beta = (\alpha \circ \beta)^+$ for all biequivalences α and β.

THEOREM 4.1. *Let φ be a representation of S in $\mathscr{T}_X$. Define mappings χ and ψ from S to $\mathscr{B}_X$ as follows:*

$$\chi : s \mapsto s\varphi \cap \left(s^{-1}\varphi\right)^{-1},$$
$$\psi : s \mapsto \left(s\varphi \cup \left(s^{-1}\varphi\right)^{-1}\right)^{+},$$

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where $(s^{-1}\varphi)^{-1}$ means the relation inverse to the function $(s^{-1}\varphi)$. Then χ is a representation of S in $\mathscr{I}_X$, ψ is a representation of S in $\mathscr{I}_X^*$, and $S\chi \cong S\varphi \cong S\psi$. In particular, all these representations are faithful if any one is.

PROOF. The statement and proof concerning χ were first given by Wagner [15]. We turn to ψ. Clearly $(st)\varphi = s\varphi \circ t\varphi \subseteq s\psi \circ t\psi$ and $((st)^{-1}\varphi)^{-1} = (s^{-1}\varphi)^{-1} \circ (t^{-1}\varphi)^{-1} \subseteq s\psi \circ t\psi$, whence $(st)\psi = (((st)\varphi) \cup ((st)^{-1}\varphi)^{-1})^+ \subseteq (s\psi \circ t\psi)^+ = (s\psi)(t\psi)$. To establish the reverse inclusion we first prove the following lemma.

LEMMA 4.2. *Let $\rho = \{(x, y) \in X \times X \mid x(s\varphi) = y(s^{-1}s)\varphi\}$; then $\rho = s\psi$.*

PROOF. For each $x \in X$, both $(x, xs\varphi), (xs^{-1}\varphi, x) \in \rho$ so that $s\varphi, (s^{-1}\varphi)^{-1} \subseteq \rho$ and moreover ρ is full. If $(x, y), (z, y), (z, w) \in \rho$, so that $xs\varphi = y(s^{-1}s)\varphi = zs\varphi = w(s^{-1}s)\varphi$, then $(x, w) \in \rho$ and it follows that $\rho \circ \rho^{-1} \circ \rho \subseteq \rho$ and hence that $\rho \in \mathscr{I}_X^*$. If, however, $\sigma \in \mathscr{I}_X^*$ is such that both $s\varphi, (s^{-1}\varphi)^{-1} \subseteq \sigma$, then

$$\begin{aligned}(x, y) \in \rho \Rightarrow xs\varphi &= y(s^{-1}s)\varphi \\ &\Rightarrow (x, xs\varphi), (y(s^{-1}\varphi), xs\varphi), (y(s^{-1}\varphi), y) \in \sigma \\ &\Rightarrow (x, y) \in \sigma \circ \sigma^{-1} \circ \sigma = \sigma,\end{aligned}$$

so that $\rho \subseteq \sigma$. Hence $\rho = (s\varphi \cup (s^{-1}\varphi)^{-1})^+$, and the lemma is proved. □

PROOF. Returning to the proof of the theorem, suppose $(x, y) \in s\psi$ and $(y, z) \in t\psi$. By the lemma, $xs\varphi = y(s^{-1}s)\varphi$ and $yt\varphi = z(t^{-1}t)\varphi$, so that

$$\begin{aligned}x(s\varphi t\varphi) &= y(s^{-1}s)\varphi t\varphi = y(s^{-1}stt^{-1}t)\varphi = y(tt^{-1}s^{-1}st)\varphi \\ &= y(t\varphi)(t^{-1}s^{-1}st)\varphi = z(t^{-1}t)\varphi(t^{-1}s^{-1}st)\varphi = z(t^{-1}s^{-1}st)\varphi,\end{aligned}$$

that is, $x(st)\varphi = z[(st)^{-1}st]\varphi$ and so (again by the lemma) $(x, z) \in (st)\psi$. Therefore $s\psi \circ t\psi \subseteq (st)\psi$, and $(s\psi)(t\psi) \subseteq (st)\psi$ follows. So ψ is a homomorphism.

To complete the proof, observe that the association $s\varphi \mapsto s\psi$ yields a well-defined map of $S\varphi$ upon $S\psi$; it is a homomorphism as both φ and ψ are thus. To show that it is one-to-one, let $s\psi = t\psi$ and take $x \in X$. Let $y \in X$ be such that $(x, y) \in s\varphi$ and so $(x, y) \in s\psi = t\psi$, whence $xt\varphi = y(t^{-1}t)\varphi = xs\varphi(t^{-1}t)\varphi$. Thus $t\varphi = s\varphi(t^{-1}t)\varphi$ and so $t\varphi \leq s\varphi$ in the natural order in $S\varphi$. By symmetry $s\varphi \leq t\varphi$ and hence $s\varphi = t\varphi$. □

A number of known representations arise as one or another of χ or ψ. As a first instance, consider the standard monoid embedding $\varphi_0 : \mathscr{P}\mathscr{T}_X \to \mathscr{T}_{X\cup\{0\}}$, where $0 \notin X$, defined by declaring $(x)\alpha\varphi_0 = x\alpha$ if $x\alpha$ is defined, and $(x)\alpha\varphi_0 = 0$ otherwise.

Since φ_0 is a monoid embedding, so is the restriction $\varphi = \varphi_0|_{\mathscr{I}_X} : \mathscr{I}_X \to \mathscr{T}_{X\cup\{0\}}$. The induced embedding $\psi : \mathscr{I}_X \to \mathscr{I}^*_{X\cup\{0\}}$ is just the map θ_X encountered at the end of Section 2. We remark that θ_X preserves suprema and the identity.

As another instance of Theorem 4.1, we take φ to be the right regular representation of S in $\mathscr{T}_S$ (so that $s\varphi : x \mapsto xs$). Then χ is the classic Wagner-Preston faithful representation $s \mapsto s\chi = \{(x, xs) \mid x \in Ss^{-1}\}$ whose image is a subsemigroup of the inverse semigroup of bijections between principal left ideals of S. The corresponding ψ is a faithful representation

$$s \mapsto s\psi = \{(x, y) \in S \times S \mid xs = ys^{-1}s\}$$

of S in $\mathscr{I}^*_S$, which deserves to be called the *dual Wagner-Preston representation*, and whose image is contained in the inverse semigroup of block bijections between left quotient sets, that is, quotients under left stable equivalences. Henceforth we reserve the symbols χ and ψ for the Wagner-Preston and the dual Wagner-Preston embeddings of S into $\mathscr{I}_S$ and $\mathscr{I}^*_S$ respectively.

It is immediate that if S has either an identity 1 or a zero 0, then $0\chi = \{(0, 0)\}$, $1\chi = \Delta = 1\psi$, and $0\psi = \nabla$. It was shown in [7, Theorem 1.20] that the Wagner-Preston representation χ is algebraic; analogous remarks apply when S is a complete inverse algebra. However ψ does not always preserve meets, as may easily be seen in the case of the algebra S formed by adjoining a zero to a group of order two. A characterization of those inverse algebras for which ψ is an embedding of algebras requires the following concept.

An element s of an inverse algebra S is *aperiodic* if there is a positive integer n such that $s^n = s^{n+1}$, in which case $f[s] = s^n$. All semigroup homomorphisms preserve this property and thus also preserve fixed point idempotents of aperiodic elements. An inverse algebra S is *aperiodic* if all its elements are aperiodic, or equivalently if S is periodic and combinatorial. This property, applied to inverse monoids, was called *E-nil* in [7]; such an inverse monoid is necessarily an inverse algebra, and every monoid homomorphism from S to any inverse algebra is algebraic.

THEOREM 4.3. *If S is an inverse algebra and $s \in S$, then $f[s\psi] = f[s]\psi$ if, and only if, s is aperiodic.*

PROOF. Let a relation $s\omega$ be defined on S as follows: $(x, y) \in s\omega$ if and only if there exist elements $u, v \in \langle s\rangle$, the inverse submonoid generated from s, such that $xu = yv$. Consider first the claim that $s\omega$ is the equivalence relation on S generated from $s\psi$. It is indeed immediate that $s\psi = \{(x, y) \mid xs = ys^{-1}s\} \subseteq s\omega$ and that $s\omega$ is reflexive and symmetric, so suppose that $(x, y), (y, z) \in s\omega$, that is, $xu = yv$ and $yw = zt$ for some $u, v, w, t \in \langle s\rangle$. Then $xuv^{-1}ww^{-1} = ztw^{-1}vv^{-1}$, so that $(x, z) \in s\omega$ and $s\omega$ is transitive. Thus the relation $s\omega$ is an equivalence containing

$s\psi$. On the other hand, let ρ be any equivalence containing $s\psi$. Since $(x, xs) \in s\psi$ and $(x, xs^{-1}) \in s^{-1}\psi = (s\psi)^{-1}$, we have $(x, xu) \in \rho$ for each $u \in \langle s \rangle$ and $x \in S$. Similarly, $(yv, y) \in \rho$ for each $v \in \langle s \rangle$ and $y \in S$. Thus $(x, y) \in s\omega$ implies $(x, y) \in \rho \circ \rho = \rho$. Hence $s\omega$ is indeed the smallest equivalence on S containing $s\psi$; stated otherwise, $s\omega = f[s\psi]$ in $\mathscr{I}_S^*$.

Now suppose that $f[s\psi] = f[s]\psi$, and put $e = f[s]$. Then $(s, e) \in e\psi = s\omega$, so that $su = ev = e$ for some pair $u, v \in \langle s \rangle$. Thus $e \in \langle s \rangle$, and in fact e is the zero of $\langle s \rangle$ and the sole element of its $\mathscr{D}$-class in $\langle s \rangle$. Since every $\mathscr{D}$-class of a monogenic inverse monoid, except possibly $\{1\}$, contains a positive power of the generator, it follows that s is aperiodic. Since the converse is immediate, the proof is complete. □

COROLLARY 4.4. ([7], Theorem 3.22.) *An inverse algebra S is aperiodic if, and only if, every monoid embedding of S into an arbitrary inverse algebra is algebraic.*

Let us now turn to the preservation of joins and suprema, which requires the following definitions. An inverse semigroup or monoid S has *conditional joins* [or has *conditional suprema*] if each pair s, t [each nonempty subset M] which is bounded above in the natural partial ordering has a supremum, denoted by $s \vee t$ [by $\sup M$]. S is *join-distributive* [*sup-distributive*] if, in addition, $u(s \vee t) = us \vee ut$ [$u(\sup M) = \sup(uM)$] holds for all $u \in S$ whenever $s \vee t$ [$\sup M$] exists in S. Join-distributivity [sup-distributivity] for monoids is equivalent to asserting that $E(S)$ is distributive [sup-distributive in itself] under the natural partial ordering (proof of this equivalence may be found in references cited in [7, sections 1.29 and 1.30]). We say that a homomorphism $\phi : S \to T$ of inverse semigroups or monoids *preserves joins* [*preserves suprema*] if $(s \vee t)\phi = s\phi \vee t\phi$ [$(\sup M)\phi = \sup(M\phi)$] whenever $s \vee t$ [$\sup M$] exists in S.

Now if S has a faithful representation in $\mathscr{I}_X$ which preserves joins [suprema], then S must be join-distributive [sup-distributive] since $\mathscr{I}_X$ is (thanks to D. A. Bredikhin for this observation). More is true of the (primal) Wagner-Preston representation: χ preserves any existing natural joins $x \vee y$ (for $x, y \in S$) and suprema $\sup X$ (for $X \subseteq S$) only in the trivial case in which the join or supremum is actually a maximum element. That is, $(\sup X)\chi = \sup(X\chi)$ holds if and only if X possesses a maximum element a, in which case both expressions reduce to $a\chi$. Thus by remarks above, if the semigroup S has conditional joins [suprema], then χ preserves joins [suprema] if and only if every pair [nonempty subset] of elements bounded above has a maximum element, equivalently if and only if $E(S)$ is a tree [in which every subchain is dually well-ordered]. If S is additionally a monoid, then χ preserves joins [suprema] if and only if $E(S)$ is a [dually well-ordered] chain. In the converse direction, if $E(S)$ is a tree [in which every subchain is dually well-ordered] then every representation of S preserves existing joins [suprema].

The situation is somewhat different for ψ. In $\mathscr{I}_X^*$, conditional suprema of biequivalences are intersections: for $\mathscr{A} \subseteq \mathscr{I}_X^*$, $\sup \mathscr{A} = \bigcap \mathscr{A}$ when the latter is in $\mathscr{I}_X^*$. Thus, to assert that ψ preserves joins is to assert that $(s \vee t)\psi = s\psi \cap t\psi$ holds whenever $s \vee t$ exists; likewise, ψ preserves suprema if $(\sup M)\psi = \bigcap_{s\in M} s\psi$ holds whenever $\sup M$ exists in S. Inverse monoids for which ψ preserves joins or suprema may then be characterized as follows:

PROPOSITION 4.5. *Let S be an inverse monoid having conditional joins. Then the dual Wagner-Preston representation ψ preserves joins if and only if S is join-distributive. Likewise, if S has conditional suprema, then ψ preserves suprema if and only if S is sup-distributive.*

PROOF. First let S be sup-distributive, with $\sup M$ existing in S, and let $(x, y) \in \bigcap_{s\in M} s\psi$. Then

$$x(\sup M) = \sup_{s\in M} xs = \sup_{s\in M} ys^{-1}s = y \sup_{s\in M} s^{-1}s = y \sup_{s\in M} s^{-1} \sup_{s\in M} s$$

so that $(x, y) \in (\sup M)\psi$. Hence $\bigcap_{s\in M} s\psi \subseteq (\sup M)\psi$. Since ψ preserves the natural order, the reverse inclusion always holds, and $\bigcap_{s\in M} s\psi = (\sup M)\psi$ whenever $\sup M$ exists. Conversely, assume that $\bigcap_{s\in M} s\psi = (\sup M)\psi$ whenever $\sup M$ exists. In particular, this holds for all $M \subseteq E(S)$. So take $r \in E(S)$; upon setting $e = \sup_{s\in M} rs$ and $f = r \sup M$, one obtains $es \leq fs = r(\sup M)s = rs \leq es$ so that $es = fs$. That is, $(e, f) \in s\psi$ for all $s \in M$, and so by hypothesis of sup-distributivity, $e(\sup M) = f(\sup M)$, which reduces to $\sup_{s\in M} rs = r \sup M$. Thus $E(S)$ is sup-distributive, and so is S. The statement regarding join-distributivity and ψ is verified in a similar manner. □

The above concepts can be generalized. A *directed supremum* is the supremum $\sup M$ of an upward directed subset M of S. An inverse monoid S has *directed suprema* if every upward directed subset has a supremum, in which case S is *upper continuous* if multiplication distributes over directed suprema. A homomorphism $\phi : S \to T$ of inverse monoids *preserves directed suprema* if $(\sup M)\phi = \sup(M\phi)$ for all occurrences of directed suprema in S. Arguments similar to those above yield: *If an inverse monoid S has directed suprema, then the dual Wagner-Preston representation ψ preserves directed suprema if and only if S is upper continuous.* (Regarding directed suprema, see [7, Section 6].) In another direction, Bredikhin [1] has shown that each inverse monoid possesses a monoid embedding in some $\mathscr{I}_X^*$ which preserves joins.

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5. Functorial representations

Let $\mathbf{A} = (A, \Omega)$ be a universal algebra with a set Ω of operations defined on the carrier set A, and consider the category $\mathbf{K}$ of (homomorphisms between) universal algebras of the same type as $\mathbf{A}$. Elements of $\mathscr{I}_{\mathbf{A}} = \mathscr{I}(\mathbf{K}, \mathbf{A})$ are termed *partial automorphisms*; elements of $\mathscr{I}^*_{\mathbf{A}} = \mathscr{I}^*(\mathbf{K}, \mathbf{A})$ are termed *dual partial automorphisms* or *block multiautomorphisms*. Since all monics in any variety of algebras are injective as functions, $\mathscr{I}_{\mathbf{A}}$ is a subset of $\mathscr{I}(\mathbf{Set}, A)$. Epis, however, need not be surmorphisms (surjective morphisms), although they are so in the cases of inverse semigroups, groups and R-modules. Thus, in what follows, $\mathscr{I}^*_{\mathbf{A}}$ *is redefined as the classifying inverse algebra of the episetting of all surmorphisms of* $\mathbf{A}$ *to its quotient algebras.* Thus $\mathscr{I}^*_{\mathbf{A}}$ is also a subset of $\mathscr{I}^*(\mathbf{Set}, A)$. Members of $\mathscr{I}^*_{\mathbf{A}}$ are characterised as biequivalences which are also subalgebras of $\mathbf{A} \times \mathbf{A}$ and are thus termed *bicongruences*. By Theorem 1.4 and its dual, both $\mathscr{I}_{\mathbf{A}}$ and $\mathscr{I}^*_{\mathbf{A}}$ are complete inverse algebras. We begin by comparing the ways in which $\mathscr{I}_{\mathbf{A}}$ and $\mathscr{I}^*_{\mathbf{A}}$ are included in $\mathscr{I}(\mathbf{Set}, A)$ and $\mathscr{I}^*(\mathbf{Set}, A)$ respectively.

THEOREM 5.1. *The inclusion* $\mathscr{I}_{\mathbf{A}} \hookrightarrow \mathscr{I}(\mathbf{Set}, A)$ *is an embedding of complete inverse algebras which need not preserve joins. The inclusion* $\mathscr{I}^*_{\mathbf{A}} \hookrightarrow \mathscr{I}^*(\mathbf{Set}, A)$ *is an embedding of inverse monoids which preserves suprema and infima of idempotents, but need not preserve arbitrary natural meets.*

PROOF. Let $\mathscr{V}$ be the variety of algebras generated from $\mathbf{A}$. Let $U : \mathscr{V} \to \mathbf{Set}$ be the forgetful functor taking algebras and homomorphisms to their underlying sets and functions. Then U creates (and preserves) arbitrary intersections and equalizers, but not joins. Thus all calculations of multiplication and infima in $\mathscr{I}_{\mathbf{A}}$ are particular calculations for $\mathscr{I}(\mathbf{Set}, A)$, so that the inclusion $\mathscr{I}_{\mathbf{A}} \hookrightarrow \mathscr{I}(\mathbf{Set}, A)$ is an embedding of complete inverse algebras.

On the other hand, while U creates co-intersections of surmorphisms (as the congruence lattice of any algebra is a complete sublattice of the lattice of all equivalences on its carrier set), U need not create co-equalizers. Hence the inclusion $\mathscr{I}^*_{\mathbf{A}} \hookrightarrow \mathscr{I}^*(\mathbf{Set}, A)$ is, in general, just an embedding of inverse monoids which preserves infima and suprema of idempotents. $\square$

EXAMPLE 5.2. Consider the group of integers $\mathbb{Z}$ and the automorphism $\mu : n \mapsto -n$. The smallest equivalence β on $\mathbb{Z}$ generated from the graph $\alpha = \{(n, -n) \mid n \in \mathbb{Z}\}$ has equivalence classes of the form $\{n, -n\}$. The congruence γ on $\mathbb{Z}$ generated from α has the cosets $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ as its congruence classes. Hence $f[\alpha] = \beta$ in $\mathscr{I}^*(\mathbf{Set}, \mathbb{Z})$, while $f[\alpha] = \gamma$ in $\mathscr{I}^*(\mathbf{Ab}, \mathbb{Z})$.

In the terms of the proof above, the co-equalizer in **Set** of μ and the identity map $\iota_{\mathbb{Z}}$ is the induced map $\mathbb{Z} \to \mathbb{Z}/\beta$; the co-equalizer of μ and $\iota_{\mathbb{Z}}$ in the variety of groups is the induced epimorphism $\mathbb{Z} \to \mathbb{Z}/\gamma$.

We turn to representations of inverse monoids and algebras by partial automorphisms and bicongruences, starting with a variant of Theorem 4.1. (See also [11].)

THEOREM 5.3. *Given a universal algebra* $\mathbf{A} = (A, \Omega)$ *and a representation* $\varphi : S \to End(\mathbf{A})$ *of an inverse monoid* S *by endomorphisms of* $\mathbf{A}$, *the derived representations* χ *and* ψ *of* S *have as their respective codomains the inverse algebras* $\mathscr{I}_{\mathbf{A}}$ *and* $\mathscr{I}^*_{\mathbf{A}}$ *of partial automorphisms and bicongruences of the algebra* $\mathbf{A}$.

PROOF. Since $s\varphi$ is an endomorphism for each $s \in S$, (the graphs of) $s\varphi$ and $(s^{-1}\varphi)^{-1}$ are subalgebras of $\mathbf{A} \times \mathbf{A}$ and so too is $s\chi$; it follows it is a partial automorphism. Using the description of $s\psi$ given in Lemma 4.2, one may see that $s\psi$ is a subalgebra of $\mathbf{A} \times \mathbf{A}$, and thus is a bicongruence of $\mathbf{A}$. $\square$

A classic instance of Theorem 5.3 is the Munn representation. Let φ be the representation of S by endomorphisms of $E(S)$ given by conjugation: $e(s\varphi) = s^{-1}es$ for all $e \in E(S)$. In this case we obtain the Munn representation χ_E of S in $\mathscr{I}_{E(S)}$, and the *dual Munn representation* ψ_E of S in $\mathscr{I}^*_{E(S)}$. Since $S\chi_E \cong S\psi_E$, ψ_E also induces on S its maximum idempotent-separating congruence.

Given a representation $\varphi : S \to \mathscr{T}_X$ and a nontrivial variety $\mathscr{V}$ of algebras, a representation $\varphi^{\mathscr{V}}$ of S into the endomorphism monoid of the free algebra $F(X)$ on X is induced by freely extending the S-action on X to all of $F(X)$ (thus applying Theorem 1.5 to the functor F creating the free algebra). By Theorem 5.3, $\varphi^{\mathscr{V}}$ induces in turn representations $\chi^{\mathscr{V}} : S \to \mathscr{I}_{F(X)}$ and $\psi^{\mathscr{V}} : S \to \mathscr{I}^*_{F(X)}$. If φ is faithful, so are $\chi^{\mathscr{V}}$ and $\psi^{\mathscr{V}}$. In particular, let φ be the standard representation $S \to \mathscr{T}_S$. Then the embeddings $\chi^{\mathscr{V}} : S \to \mathscr{I}_{F(S)}$ and $\psi^{\mathscr{V}} : S \to \mathscr{I}^*_{F(S)}$ yield the $\mathscr{V}$-versions of the Wagner-Preston Theorem and its dual.

COROLLARY 5.4. *Given a nontrivial variety* $\mathscr{V}$ *of algebras, every inverse monoid has a faithful representation by* [*dual*] *partial automorphisms of an algebra in* $\mathscr{V}$, *which may be chosen as the free algebra on the carrier set of the monoid.*

Recall that the category R-**Mod** of all left R-modules (for R a fixed nontrivial ring) may be regarded as a variety of algebras each of which consists of an abelian group enriched by a family of unary operations ρ_r, one for each element $r \in R$, written as scalar multiplication: $\rho_r a = ra$. The module identities, stated for all possible choices of R-scalars (for instance, $r_1a + r_2a = (r_1 + r_2)a$ for all choices of r_1 and r_2) become the defining identities for the variety. In this case, $F(S)$ is the free R-module on S, $R(S) = \bigoplus_{x \in S} Rx$, and the action $\varphi^{\oplus}$ induced from the right regular representation $s\varphi : x \mapsto xs$ takes the form $(\sum r_x x)s\varphi^{\oplus} = \sum r_x(xs)$. By Theorem 5.3, $\varphi^{\oplus}$ induces in turn representations $\chi^{\oplus} : S \to \mathscr{I}_{R(S)}$ and $\psi^{\oplus} : S \to \mathscr{I}^*_{R(S)}$. Since φ is faithful, so are $\varphi^{\oplus}$, $\chi^{\oplus}$ and $\psi^{\oplus}$. We thus obtain module versions of the Wagner-Preston Theorem and its dual: *For each nontrivial ring R, every inverse monoid S has a faithful representation by* [*dual*] *partial automorphisms of the free* (*left*) *R-module on the carrier set of S.*

When $R = \mathbb{Z}$ (so that R-**Mod** is essentially the category **Ab** of abelian groups) we recover Fichtner-Schultz's theorem [2], [3] that *every inverse monoid may be faithfully represented by partial automorphisms of some abelian group.* If $R = \mathbb{Z}_2$, then the above description of $\chi^{\oplus}$ is essentially Schein's short proof of this theorem given in [13]. With $\psi^{\oplus}$ we also obtain Schein's dualization of this theorem, which is the major content of [13].

Suppose that S is an inverse algebra. When are $\chi^{\oplus} : S \to \mathscr{I}_{R(S)}$ and $\psi^{\oplus} : S \to \mathscr{I}^*_{R(S)}$ algebraic, that is, yield embeddings of inverse algebras? A mild modification of the argument in Theorem 4.3 shows that $\psi^{\oplus}$ is algebraic precisely when S is aperiodic. In general, and in contrast to the Wagner-Preston embedding, $\chi^{\oplus}$ need not preserve natural meets and fixed point idempotents.

To describe when $\chi^{\oplus}$ is algebraic, first recall that an inverse monoid is *torsion-free* if it has no nontrivial finite subgroups. It was shown in [7, Corollary 1.16] that torsion free inverse algebras form a subvariety of inverse algebras characterized by the identities $\{f[x^n] = f[x] \mid n \geq 1\}$.

THEOREM 5.5. *If* S *is an inverse algebra, then $\chi^{\oplus} : S \to \mathscr{I}_{R(S)}$ is algebraic if and only if* S *is torsion free.*

PROOF. In this proof, let us abbreviate $s\chi^{\oplus}$ by χ_s. First note that, for $s \in S$, χ_s is the isomorphism of $\bigoplus\{Rx \mid x \in Ss^{-1}\}$ to $\bigoplus\{Rx \mid x \in Ss\}$ defined by $(\sum r_x x)\chi_s = \sum r_x(xs)$. Thus $\chi_{f[s]}$ is the identity map on $M = \bigoplus\{Rx \mid x \in Sf[s]\}$, while $f[\chi_s]$ is the restriction of χ_s to its set N of fixed points. It is always the case that $M \subseteq N$, so the theorem asserts that $N \setminus M$ is non-empty (for some $s \in S$) if, and only if, S has a nontrivial finite subgroup.

Suppose then that $s \in S$ is a subgroup element of finite order $n > 1$, and take $r \neq 0$ in R. Then $rs + rs^2 + \cdots + rs^n \in N \setminus M$. Conversely, let $\sigma = \sum r_x x \in N \setminus M$, so that σ has a nonzero summand $r_x x$ such that $x \notin Sf[s]$. Since $\sigma\chi_s = \sigma$, postmultiplication by s induces a permutation of the terms of σ, and since these are finite in number, the orbit of x under this permutation is finite. Thus there exists an integer n such that $xs^n = x$, or equivalently $x^{-1}x \leq f[s^n]$. Now if $f[s^n] = f[s^n]s$, there would follow $f[s^n] = f[s]$ and hence $x^{-1}x \leq f[s]$, that is, $x \in Sf[s]$, contradicting our assumption on x. Hence $f[s^n]s \neq f[s^n]$. But then [7, Proposition 1.11(h)] implies that $f[s^n]s$ generates a nontrivial finite subgroup. $\square$

When $R = \mathbb{Z}$, so R-**Mod** = **Ab**, we also have the following results.

THEOREM 5.6. *A given inverse algebra S has an algebraic embedding into the partial automorphism algebra* $\mathscr{I}_A$ *of some abelian group A if and only if it has an algebraic embedding into the dual partial automorphism algebra* $\mathscr{I}^*_B$ *of some abelian group B.*

PROOF. It is standard knowledge (see, for example, [5] and [9]) that the contravariant hom-functor $J = \hom(\_, A)$ of **Ab** to **Ab** is always left exact, and is also right exact precisely when A is injective. Let A be so chosen, for example as $\mathbb{Q}/\mathbb{Z}$. Then the same map J, regarded as a covariant functor of **Ab** to $\mathbf{Ab}^{op}$, preserves finite limits, and preserves and reflects monics and epis, by exactness. In particular, it reflects isomorphisms. In short, J satisfies the conditions of Theorem 1.5, and so induces an inverse algebra embedding of $\mathscr{I}_A$ into $\mathscr{I}^*_{J(A)}$.

Likewise, $J : \mathbf{Ab}^{op} \to \mathbf{Ab}$ induces an inverse algebra embedding of $\mathscr{I}^*_A$ into $\mathscr{I}_{J(A)}$; indeed, this embedding is as a *complete* subalgebra, since the contravariant J converts *arbitrary* colimits to limits. □

COROLLARY 5.7. *Every torsion free inverse algebra has an algebraic embedding into the dual partial automorphism algebra of some abelian group.*

Are there inverse algebras which cannot be so embedded? To answer this question, we begin with an analogue of Proposition 3.5 holding for the [dual] symmetric inverse algebras of abelian groups.

LEMMA 5.8. *Let* μ *and* ν *be* [*dual*] *partial automorphisms of an abelian group A such that* $\mu^3 = \mu$ *and* $\mu \geq \nu \geq f[\mu]$. *Then also* $\nu^3 = \nu$.

PROOF. Suppose first that μ: $A \to A$ is an automorphism of order 2. Let A_μ be the fixed point subset of μ and let B be any subgroup containing A_μ. Then $b + b\mu \in A_\mu \subseteq B$ for each $b \in B$, whence $b\mu \in B$. Thus $\nu = \mu \mid_B$ is an automorphism of B. Since $\mu^3 = \mu$, we have $\nu^3 = \nu$ also. In general, if $\mu^3 = \mu$ in $\mathscr{I}_A$, then μ^2 is idempotent, and μ restricts to an automorphism of order 2 of $C = A\mu^2$. Then $\nu = \nu^3$ in $\mathscr{I}_C$, which is embedded as a local submonoid in $\mathscr{I}_A$. Finally, that the conclusion also holds for $\mu \in \mathscr{I}^*_A$ follows from Theorem 5.6. □

COROLLARY 5.9. *Let X be a set with* $|X| \geq 2$. *Then* (i) *there is no algebraic embedding of* $\mathscr{I}_X$ *into the* [*dual*] *partial automorphism algebra of any abelian group; however,* (ii) *every dual symmetric inverse algebra* $\mathscr{I}^*_X$ *can be algebraically embedded in the* [*dual*] *partial automorphism algebra of some abelian group.*

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PROOF. The transposition argument of Corollary 3.6 shows that $\mathscr{I}_X$ cannot be embedded into either $\mathscr{I}_A$ or $\mathscr{I}_A^*$ for any abelian group A. The assertion about $\mathscr{I}_X^*$ follows from the fact that, given a nontrivial abelian group A, the contravariant functor $\hom(\_, A) : \mathbf{Set} \to \mathbf{Ab}$ is faithful, exact, reflects isomorphisms and sends colimits to limits, and so induces, by Theorem 1.5, an embedding $\mathscr{I}_X^* \hookrightarrow \mathscr{I}_{\hom(X,A)}$ of complete inverse algebras. □

That the converse to Corollary 5.9(ii) is false is shown by the following example of a group whose [dual] partial automorphism algebra has no algebraic embedding into the dual algebra $\mathscr{I}_X^*$ of any set X.

EXAMPLE 5.10. Let $\mathbb{Z}_5$ denote as usual the additive cyclic group on $\{0, 1, 2, 3, 4\}$. Its automorphism group is cyclic, being generated by the 4-cycle $\sigma = (1243)$ fixing 0. Thus every nonidentity automorphism of $\mathbb{Z}_5$ has $\{0\}$ as its fixed point subgroup, and since the only subgroups of $\mathbb{Z}_5$ are itself and $\{0\}$, it follows that $\mathscr{I}_{\mathbb{Z}_5} \cong \mathbb{Z}_4^0$. In particular, $f[\sigma] = f[\sigma^2]$; so if $\theta : \mathscr{I}_{\mathbb{Z}_5} \to \mathscr{I}_X^*$ is an embedding of monoids for some set X, Proposition 3.2(iii) implies that θ does not preserve the fixed point idempotent of σ.

The results of the last two sections distinguish three classes of inverse algebras on the basis of algebraic embedding properties: the class of algebras which may be embedded into $\mathscr{I}_X$ for some set X, that is, the class of all inverse algebras; the smaller class of those algebras which may be embedded into the dual symmetric inverse algebra $\mathscr{I}_X^*$ of some set X; and finally, a properly intermediate class consisting of those algebras which may be embedded into the partial automorphism algebra $\mathscr{I}_A$ of some abelian group A.

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