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Hecke Algebras and Automorphic Forms

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Abstract. The goal of this paper is to carry out some explicit calculations of the actions of Hecke operators on spaces of algebraic modular forms on certain simple groups. In order to do this, we give the coset decomposition for the supports of these operators. We present the results of our calculations along with interpretations concerning the lifting of forms. The data we have obtained is of interest both from the point of view of number theory and of representation theory. For example, our data, together with a conjecture of Gross, predicts the existence of a Galois extension of $\mathbb{Q}$ with Galois group $G_2(\mathbb{F}_5)$ which is ramified only at the prime 5. We also provide evidence of the existence of the symmetric cube lifting from PGL_2 to PGSp_4.

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1. Introduction

Let G be a connected reductive group over $\mathbb{Q}$ with $G(\mathbb{R})$ compact. We will keep this assumption on all groups over $\mathbb{Q}$ in this paper. We denote by $\hat{\mathbb{Q}} = \mathbb{Q} \otimes \hat{\mathbb{Z}}$ the ring of finite adèles of $\mathbb{Q}$.

We will be studying certain spaces of modular forms for G. The *weight* of the forms will be an algebraic representation W of G over a number field E and the *level* will be an open compact subgroup K of $G(\hat{\mathbb{Q}})$. Following [10, 13] we define the space of modular forms of weight W and level K on G to be the E-vector space

$$M(W, K) = \{F: G(\hat{\mathbb{Q}})/K \to W(E) : F(\gamma g) = \gamma F(g), \text{ for all } \gamma \in G(\mathbb{Q})\},$$

When $W = \mathbb{Q}$ is the trivial representation this is simply the space of $\mathbb{Q}$-valued functions on the (finite) double coset space $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$.

Let $K_p \subset G(\mathbb{Q}_p)$ be an open compact subgroup and let dg be the Haar measure giving K_p volume 1. The Hecke algebra, $\mathcal{H}(G(\mathbb{Q}_p), K_p)$, is the convolution algebra with respect to dg of compactly supported $\mathbb{Q}$-valued functions on $G(\mathbb{Q}_p)$ which are bi-invariant by K_p. When there is no confusion, we will denote this algebra

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$\mathcal{H}_{K_p}$. If K_p is hyperspecial maximal compact, $\mathcal{H}_{K_p}$ is often called the *spherical* Hecke algebra.

If the level K is a product of local factors $K = \prod_p K_p$ then there is an action of each $\mathcal{H}_{K_p}$ on $M(W, K)$. If $T \in \mathcal{H}_{K_p}$ and $f \in M(W, K)$ we have $Tf \in M(W, K)$ given by

$$Tf(h) = \int_{G(\mathbb{Q}_p)} T(g)f(hg)dg.$$

This integral is actually a finite sum, taken over the cosets of K_p contained in the support of T. Indeed, if the support of T is $\bigcup_i a_i K_p$ then

$$Tf(h) = \sum_i T(a_i)f(ha_i). \tag{1}$$

Although we will not make direct use of it, it is worth pointing out the close connection between these modular forms and automorphic representations. If W is absolutely irreducible, it follows from [10, Prop 8.5] that the irreducible $\mathcal{H}_K$-submodules of $M(W, K) \otimes \mathbb{C}$ correspond to the irreducible automorphic representations $\pi = \pi_\infty \otimes \hat{\pi}$ with $\pi_\infty \cong W \otimes \mathbb{C}$ and $\hat{\pi}^K \neq 0$. In fact if the irreducible submodule N corresponds to the automorphic representation π, then N and $\hat{\pi}^K$ are isomorphic as Hecke modules. Thus knowing the action of the Hecke algebra $\mathcal{H}_{K_p}$ on the irreducible pieces of $M(W, K) \otimes \mathbb{C}$ allows us to identify the local components of the automorphic representations with infinite component W and having K-fixed vectors.

The goal of this paper is to carry out some explicit calculations of the action of local Hecke algebras on certain spaces of modular forms. In Section 4 we discuss some aspects of modular forms that can be read off from our data. In Section 5 we present the results of our calculations along with interpretations of these results in light of the discussion in Section 4. The data we have obtained are of interest both from the point of view of number theory and of representation theory. For example, our data, together with a conjecture of Gross, predicts the existence of a Galois extension of $\mathbb{Q}$ with Galois group $G_2(\mathbb{F}_5)$ which is ramified only at the prime 5 [11, §2, §5]. We also provide evidence of the existence of the symmetric cube lifting from PGL_2 to PGSp_4 (see Section 4.3).

The first step in making these calculations is purely local. We determine coset representatives a_i of the various cosets of K_p in the support of an operator $T \in \mathcal{H}_{K_p}$. We work this out in Section 2, for G split over $\mathbb{Q}_p$ and K_p either a hyperspecial maximal compact or an Iwahori subgroup. The analysis there closely follows [17]. We also handle the case where G is a form of PGSp_4 not split over $\mathbb{Q}_p$ and K_p is the Iwahori subgroup.

We then give an overview of the global aspects of our algorithm, especially the issue of finding double coset representatives for $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$. This, along with some comments on the reliability of the computer calculations, appears in Section 3.2.

We performed our calculations for compact forms of G_2 and PGSp_4, but the methods are fairly general and can be applied in other cases, subject to constraints on computer speed and memory.

2. Double Cosets in Groups over Local Fields

2.1. PRELIMINARIES

Throughout this section, G will denote a connected semisimple algebraic group that is split over a non-Archimedean local field F with ring of integers $\mathcal{O}_F$ and prime ideal $\mathfrak{p}$. (We will apply the results of this section to the $\mathbb{Q}_p$ points of various rational algebraic groups.) Let π in $\mathfrak{p}$ denote a uniformizing parameter, let k be the residue field $\mathcal{O}_F/\mathfrak{p}$, and let $R \subset \mathcal{O}_F$ be a set of representatives for k containing 0. Let q denote the cardinality of k.

We select a hyperspecial maximal compact subgroup K of $G(F)$. The group K gives rise to a Chevalley group scheme $\underline{G}$ over $\mathcal{O}_F$ such that $K = \underline{G}(\mathcal{O}_F) \subset \underline{G}(F) = G(F)$ (cf. [23, 3.4.1,3.8.1]) and such that the special fiber $\overline{G}$ of $\underline{G}$ is semisimple of the same type as G.

Let $\underline{T} \subset \underline{G}$ be a split maximal torus scheme, and let T be its general fiber. We define N_T to be the normalizer of T in G. Denote by $X^*(T)$ the character module $\mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$ of T and by $X_*(T)$ the co-character module $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, T)$ of T. Let $\Phi \subset X^*(T)$ be the set of roots of T, $\Phi^+ \subset \Phi$ a subset of positive roots, and $\Delta \subset \Phi^+$ the corresponding set of simple roots. Also, let $\Phi^\vee \subset X_*(T)$ be the coroots of T and $\alpha \mapsto \alpha^\vee$ the standard bijection between Φ and $\Phi^\vee$.

For each $\alpha \in \Phi$ let $\underline{U}_\alpha$ be the one-dimensional unipotent subgroup scheme of $\underline{G}$ corresponding to α. Denote the general fiber of $\underline{U}_\alpha$ by U_α. We choose for each α an isomorphism $x_\alpha\colon \mathbb{G}_{\mathrm{a}} \longrightarrow \underline{U}_\alpha$. When considered as a map $F \to \underline{U}_\alpha(F)$, x_α restricts to an isomorphism of $\mathcal{O}_F$ with $\underline{U}_\alpha(\mathcal{O}_F) = U_\alpha(F) \cap K$.

Let W be the Weyl group $N_T/T = (N_T(F) \cap K)/\underline{T}(\mathcal{O}_F)$ of G and $\tilde{W}$ the extended affine Weyl group $N_T(F)/\underline{T}(\mathcal{O}_F)$. Then W and $\tilde{W}$ act as groups of affine transformations on the space $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The stabilizer in $\tilde{W}$ of $0 \in X_*(T) \otimes \mathbb{R}$ is W, and there is an isomorphism $\tilde{W} \cong X_*(T) \rtimes W$, where $X_*(T)$ is embedded in $\tilde{W}$ as a group of translations on $X_*(T) \otimes \mathbb{R}$. We denote by e the identity element of $\tilde{W}$ and by $t(\lambda)$ the element of $\tilde{W}$ corresponding to λ in $X_*(T)$ We can and will choose the above isomorphism so that the image of $\lambda(\pi)$ is $t(\lambda)$. Observe that in this notation $wt(\lambda)w^{-1} = t(w\lambda)$. We let $\langle\,,\rangle\colon X^*(T) \times X_*(T) \to \mathbb{Z}$ be the usual W-invariant pairing, and we define $X_+ \subset X_*(T)$ to be the set of all co-characters λ such that $\langle \alpha, \lambda \rangle \geq 0$ for all α in Φ^+.

Denote by w_α the reflection in W through the vanishing hyperplane in $X_*(T) \otimes \mathbb{R}$ of the root α. Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_m$ be the decomposition of Φ into irreducible root systems. (Each Φ_i corresponds to the root system of an almost simple normal subgroup of G.) Also, let $\Delta_i = \Delta \cap \Phi_i$, and put $l_i = \#\Delta_i$. Then $l_1 + \cdots + l_m = l$, the dimension of T, i.e., the rank of G. Let $\alpha_{0,i}$ be the highest root of Φ_i with respect

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to the basis of simple roots Δ_i. Then the Coxeter group with set of involutive generators

$$\tilde{S} = \{w_\alpha | \alpha \in \Delta\} \cup \left\{ w_{\alpha_{0,i}} t(\alpha_{0,i}^\vee) | 1 \leqslant i \leqslant m \right\}$$

is isomorphic to the affine Weyl group W_{af} of Φ ([23, Prop. 1.1]). Via this isomorphism, we will view W_{af} as a subgroup of $\tilde{W}$.

Let I be the Iwahori subgroup of K generated by $\underline{T}(\mathcal{O}_F)$, the subgroups $x_\alpha(\mathcal{O}_F) = \underline{U}_\alpha(\mathcal{O}_F)$ for all α in Φ^+, and the subgroups $x_\alpha(\mathfrak{p})$ for all α in Φ^-. If we denote by $\overline{G}$ the semisimple algebraic group over k obtained by taking the special fiber of $\underline{G}$ then (as in [23, §3.5]) the reduction mod $\mathfrak{p}$ map $K \longrightarrow \overline{G}(k)$ is surjective, and I is the inverse image in K of the Borel subgroup of $\overline{G}(k)$ corresponding to Φ^+. The triple $(G(F), I, N_T(F))$ is a generalized Tits system in the sense of [16], a fact which will be used in 2.3 to study the structure of $G(F)$.

Denote the normalizer of I in $G(F)$ by $\tilde{I}$ and let $\Omega \subset \tilde{W}$ be the group $(N_T(F) \cap \tilde{I})/\underline{T}(\mathcal{O}_F)$. The group Ω is finite Abelian and canonically isomorphic to $X_*(T)/\Lambda_r$, where Λ_r is the submodule of $X_*(T)$ generated by $\Phi^\vee$ (cf. [16, §2]). Moreover, Ω normalizes W_{af} and there is an isomorphism $\tilde{W} \cong W_{\mathrm{af}} \rtimes \Omega$.

For w in $\tilde{W}$, let $l(w)$ denote the standard combinatorial length of w with respect to the set $\tilde{S}$. If $w' \in \tilde{W}$ then we can write $w' = w_1 \cdots w_d \rho$ for some $w_1, \ldots, w_d$ in $\tilde{S}$ and ρ in Ω, and we say that the expression $w' = w_1 \cdots w_d \rho$ is reduced if $l(w) = d$. (Under this definition, the expression $e = e$ is also to be considered reduced.)

2.2. THE GROUPS W AND $\tilde{W}$

Let W' be a subgroup of W which is generated by a subset of involutions $S' \subset S$. We will refer to such a subgroup as a *special subgroup* of W. Note that the stabilizer $W^\lambda = \{w \in W \mid w(\lambda) = \lambda\}$ of λ in W is special. For any special W', define $[W/W']$ to be the set

$$\{w \in W | l(ww') = l(w) + l(w') \ \text{ for all } w' \in W'\}.$$

The elements of $[W/W']$ are the representatives for W/W' of minimal length (cf. [5, §2.5]).

For the coset decomposition of 2.4, we will need a result on the additivity of lengths of certain elements of the extended affine Weyl group similar to that of Howlett in [6, §2.7]. This lemma will follow from the geometric interpretation of $\tilde{W}$ as a group of affine transformations on the space $X_*(T) \otimes \mathbb{R}$. The key idea is the connection between the length of an element σ of $\tilde{W}$ and the inner products of the translation part of σ with certain roots as given in [17, §1.9]. We summarize the relevant facts in the following propositions.

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PROPOSITION 2.1. *For all λ in $X_*(T)$ and w in W*

$$l(t(\lambda)w) = \sum_{\alpha\in\Phi^+\cap w\Phi^+} |\langle\alpha, \lambda\rangle| + \sum_{\alpha\in\Phi^+\cap w\Phi^-} |\langle\alpha, \lambda\rangle - 1|.$$

In particular, $l(w) = \#(\Phi^+ \cap w\Phi^-)$.

PROPOSITION 2.2. *Let $\lambda \in X_*(T)$. Then, there is a unique element σ_λ in W such that $l(t(\lambda)\sigma_\lambda) = \min_{w\in W} t(\lambda)w$, and, in fact, $l(t(\lambda)\sigma_\lambda w) = l(t(\lambda)\sigma_\lambda) + l(w)$ for all w in W. Moreover, if we put*

$$\Phi_1 = \{\alpha \in \Phi^+ \mid \langle\alpha, \lambda\rangle \leqslant 0\}, \qquad \Phi_2 = \{\alpha \in \Phi^+ \mid \langle\alpha, \lambda\rangle > 0\},$$

then

$$l(t(\lambda)\sigma_\lambda) = \sum_{\alpha\in\Phi_1} |\langle\alpha, \lambda\rangle| + \sum_{\alpha\in\Phi_2} (\langle\alpha, \lambda\rangle - 1). \tag{2}$$

Let λ be an element of X_+, and let σ_λ be as in Proposition 2. The length additivity result that we wish to prove is the following: for all w in W and τ in $[W/W^\lambda]$, $l(\tau t(\lambda)\sigma_\lambda w) = l(\tau) + l(t(\lambda)\sigma_\lambda) + l(w)$.

We will need the following auxiliary lemma on $[W/W^\lambda]$ for the proof of the additivity result.

LEMMA 2.3. *Let $\lambda \in X_+$, $\tau \in [W/W^\lambda]$, and $\beta \in \Phi^+ \cap \tau^{-1}\Phi^-$. Then $\langle\beta, \lambda\rangle > 0$.*

Proof. Since $\lambda \in X_+$, we have that $\langle\beta, \lambda\rangle \geqslant 0$. Thus, we need only rule out $\langle\beta, \lambda\rangle = 0$. If this is the case, then $w_\beta(\lambda) = \lambda$ so $w_\beta \in W^\lambda$. Let $J \subset \Delta$ be the set of simple roots α such that $\langle\alpha, \lambda\rangle = 0$. Then the special subgroup W^λ equals $\langle w_\alpha | \alpha \in J\rangle$. As shown in [5, §2.5], $\tau \in [W/W^\lambda]$ if and only if $\tau(\alpha) \in \Phi^+$ for all α in J. Furthermore, since $w_\beta \in W^\lambda$, β is a sum of roots in J, and therefore $\tau(\beta) \in \Phi^+$. This contradicts $\beta \in \tau^{-1}\Phi^-$. □

We now state and prove the length additivity lemma.

LEMMA 2.4. *Suppose $\lambda \in X_+$, $w \in W$, and $\tau \in [W/W^\lambda]$, then*

$$l(\tau t(\lambda)\sigma_\lambda w) = l(\tau) + l(t(\lambda)\sigma_\lambda) + l(w).$$

Proof. By Proposition 2.2, we have that $l(t(\lambda)\sigma_\lambda w) = l(t(\lambda)\sigma_\lambda) + l(w)$. Therefore, it suffices to show that

$$l(\tau t(\lambda)\sigma) = l(\tau) + l(t(\lambda)\sigma) \tag{3}$$

for any σ in W.

By Proposition 2.1 we have

$$l(\tau) + l(t(\lambda)\sigma) = \sum_{\beta\in\Phi^+\cap\tau\Phi^-} 1 + \sum_{\beta\in\Phi^+\cap\sigma\Phi^+} |\langle\beta, \lambda\rangle| + \sum_{\beta\in\Phi^+\cap\sigma\Phi^-} |\langle\beta, \lambda\rangle - 1|. \tag{4}$$

On the other hand, we also have

$$\begin{aligned} l(\tau t(\lambda)\sigma) &= l(t(\tau(\lambda))\tau\sigma) \\ &= \sum_{\alpha\in\Phi^+\cap\tau\sigma\Phi^+} |\langle\alpha, \tau(\lambda)\rangle| + \sum_{\alpha\in\Phi^+\cap\tau\sigma\Phi^-} |\langle\alpha, \tau(\lambda)\rangle - 1| \\ &= \sum_{\alpha\in\Phi^+\cap\tau\sigma\Phi^+} |\langle\tau^{-1}(\alpha), \lambda\rangle| + \sum_{\alpha\in\Phi^+\cap\tau\sigma\Phi^-} |\langle\tau^{-1}(\alpha), \lambda\rangle - 1| \\ &= \sum_{\beta\in\tau^{-1}\Phi^+\cap\sigma\Phi^+} |\langle\beta, \lambda\rangle| + \sum_{\beta\in\tau^{-1}\Phi^+\cap\sigma\Phi^-} |\langle\beta, \lambda\rangle - 1| \end{aligned} \tag{5}$$

In order to show that (5) equals (4), we will break up the sets over which the sums in (5) are taken and rearrange the resulting sums.

First, we note that

$$\begin{aligned} &(\tau^{-1}\Phi^+ \cap \sigma\Phi^\pm) - (\Phi^+ \cap \sigma\Phi^\pm) \\ &\quad = \tau^{-1}\Phi^+ \cap \sigma\Phi^\pm \cap \Phi^- = \Phi^- \cap \tau^{-1}\Phi^+ \cap \sigma\Phi^\pm, \\ &(\Phi^+ \cap \sigma\Phi^\pm) - (\tau^{-1}\Phi^+ \cap \sigma\Phi^\pm) \\ &\quad = \Phi^+ \cap \sigma\Phi^\pm \cap \tau^{-1}\Phi^- = \Phi^+ \cap \tau^{-1}\Phi^- \cap \sigma\Phi^\pm. \end{aligned}$$

It follows that the first term of (5) is

$$\begin{aligned} \sum_{\tau^{-1}\Phi^+\cap\sigma\Phi^+} |\langle\beta, \lambda\rangle| = &\sum_{\Phi^+\cap\sigma\Phi^+} |\langle\beta, \lambda\rangle| \quad + \sum_{\Phi^-\cap\tau^{-1}\Phi^+\cap\sigma\Phi^+} |\langle\beta, \lambda\rangle| - \\ &- \sum_{\Phi^+\cap\tau^{-1}\Phi^-\cap\sigma\Phi^+} |\langle\beta, \lambda\rangle| \end{aligned} \tag{6}$$

while the second term of (5) is

$$\begin{aligned} &\sum_{\tau^{-1}\Phi^+\cap\sigma\Phi^-} |\langle\beta, \lambda\rangle - 1| \\ &\quad = \sum_{\Phi^+\cap\sigma\Phi^-} |\langle\beta, \lambda\rangle - 1| \quad + \sum_{\Phi^-\cap\tau^{-1}\Phi^+\cap\sigma\Phi^-} |\langle\beta, \lambda\rangle - 1| - \\ &\quad - \sum_{\Phi^+\cap\tau^{-1}\Phi^-\cap\sigma\Phi^-} |\langle\beta, \lambda\rangle - 1|. \end{aligned} \tag{7}$$

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Adding (6) and (7), we obtain

$$\begin{aligned}
&\sum_{\tau^{-1}\Phi^+\cap\sigma\Phi^+} |\langle\beta,\lambda\rangle| + \sum_{\tau^{-1}\Phi^+\cap\sigma\Phi^-} |\langle\beta,\lambda\rangle - 1| \\
&\quad = \sum_{\Phi^+\cap\sigma\Phi^+} |\langle\beta,\lambda\rangle| + \sum_{\Phi^+\cap\sigma\Phi^-} |\langle\beta,\lambda\rangle - 1| + \\
&\qquad + \sum_{\Phi^-\cap\tau^{-1}\Phi^+\cap\sigma\Phi^+} |\langle\beta,\lambda\rangle| - \sum_{\Phi^+\cap\tau^{-1}\Phi^-\cap\sigma\Phi^+} |\langle\beta,\lambda\rangle| + \\
&\qquad + \sum_{\Phi^-\cap\tau^{-1}\Phi^+\cap\sigma\Phi^-} |\langle\beta,\lambda\rangle - 1| - \sum_{\Phi^+\cap\tau^{-1}\Phi^-\cap\sigma\Phi^-} |\langle\beta,\lambda\rangle - 1|.
\end{aligned} \tag{8}$$

Therefore, in order to show that (4) equals (5), we must prove that the sum of the last four terms of (8) is $\sum_{\Phi^+\cap\tau\Phi^-} 1$.

Replacing β with $-\beta$ in the third and fifth terms, we find that the sum of the last four terms of (8) is

$$\sum_{\Phi^+\cap\tau^{-1}\Phi^-\cap\sigma\Phi^-} (|\langle\beta,\lambda\rangle| - |\langle\beta,\lambda\rangle - 1|) \quad + \sum_{\Phi^+\cap\tau^{-1}\Phi^-\cap\sigma\Phi^+} (|\langle\beta,\lambda\rangle + 1| - |\langle\beta,\lambda\rangle|).$$

By Lemma 2.3, the set of β in $\Phi^+ \cap \tau^{-1}\Phi^-$ which satisfy $\langle\beta,\lambda\rangle = 0$ is empty. Thus for $\beta \in \Phi^+ \cap \tau^{-1}\Phi^- \cap \sigma\Phi^-$, $\langle\beta,\lambda\rangle > 0$ so that $|\langle\beta,\lambda\rangle| - |\langle\beta,\lambda\rangle - 1| = 1$ and $|\langle\beta,\lambda\rangle + 1| - |\langle\beta,\lambda\rangle| = 1$.

This means that the sum of the last four terms of (8) equals

$$\sum_{\Phi^+\cap\tau^{-1}\Phi^-\cap\sigma\Phi^-} 1 \;+ \sum_{\Phi^+\cap\tau^{-1}\Phi^-\cap\sigma\Phi^+} 1 \;= \sum_{\Phi^+\cap\tau^{-1}\Phi^-} 1 \;= \sum_{\Phi^+\cap\tau\Phi^-} 1,$$

since $l(\tau) = l(\tau^{-1})$. This gives (3) and completes the proof. □

The following corollary follows easily from Lemma 2.4

COROLLARY 2.5. *If $\lambda \in X_+$ then the unique element of shortest length in the double coset $Wt(\lambda)W$ is $t(\lambda)\sigma_\lambda$.*

2.3. DOUBLE COSET DECOMPOSITION FOR IWAHORI SUBGROUPS

We will now give a summary of the aspects of the structure of $G(F)$ which stem from the fact that the triple $(G(F), I, N_T(F))$ is a generalized Tits system (as defined in [16]). We also state a result of Iwahori and Matsumoto ([17, Cor. 2.7]) which gives a set of representatives for the left cosets of I contained in an arbitrary double coset of I.

We first state a result of Iwahori and Matsumoto ([17, Prop. 2.34]) concerning double cosets of subgroups of $G(F)$ containing I. For any such subgroup P, we denote by W_P the subgroup $(N_T(F) \cap P)/\underline{T}(\mathcal{O}_F)$ of $\tilde{W}$.

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PROPOSITION 2.6. *Let P, P_1 and P_2 be subgroups of $G(F)$ containing I. Then,*

(i) $$P = IW_PI = \coprod_{w\in W_P} IwI.$$

(ii) *If $\Sigma_{P_1,P_2} \subset \tilde{W}$ is a set of representatives for $W_{P_1}\backslash\tilde{W}/W_{P_2}$ then*

$$G(F) = \coprod_{\sigma\in\Sigma_{P_1,P_2}} P_1\sigma P_2.$$

In particular, if $\gamma, \gamma' \in \tilde{W}$ then $P_1\gamma P_2 = P_1\gamma' P_2$ if and only if $W_{P_1}\gamma W_{P_2} = W_{P_1}\gamma' W_{P_2}$.

Note that if we take $P = G(F)$ in Proposition 2.6(i), we obtain the affine Bruhat decomposition $G(F) = \coprod_{w\in\tilde{W}} IwI$. Thus the Iwahori–Hecke algebra $\mathcal{H}(G(F), I)$ is spanned by the characteristic functions char_{IwI} ($w \in \tilde{W}$) so in order to decompose the support of a Hecke operator into left cosets of I, it suffices to obtain a decomposition for each double coset IwI.

The following proposition summarizes the structure of $I\backslash G(F)/I$ ([17, Prop 2.8, Theorem 3.3]).

PROPOSITION 2.7. *Let w, w' be elements of $\tilde{W}$. Then*

(i) *For all $s \in \tilde{S}$*
  (a) *$IsIwI = IswI$ if $l(sw) > l(w)$,*
  (b) *$IsIwI = IswI \cup IwI$ if $l(sw) < l(w)$.*

(ii) *If $l(ww') = l(w) + l(w')$ then*

$$IwIw'I = Iww'I. \tag{9}$$

In particular, if $s_1, \ldots, s_d \in \tilde{S}$, $\rho \in \Omega$ and $w = s_1 \cdots s_d\rho$ is a reduced expression, then

$$Is_1I \cdots Is_dI\rho I = IwI. \tag{10}$$

In addition to the information resulting from the fact that the triple $(G(F), I, N_T(F))$ is a generalized Tits system, we will also need the following statement (cf. [17, Cor. 2.7]) concerning representatives for the left cosets of I inside certain double cosets of I (namely, those corresponding to the elements of $\tilde{S}$). Recall that R is a set of representatives in $\mathcal{O}_F$ for k containing 0.

PROPOSITION 2.8. *Suppose $\alpha \in \Delta$ and $i \in \{1, \ldots, m\}$, where m is the number of irreducible root systems into which Φ decomposes. Then*

(i) $Iw_\alpha I = \coprod_{v\in R} x_\alpha(v)w_\alpha I$,
(ii) $Iw_{\alpha_{0,i}}t(\alpha_{0,i}^\vee)I = \coprod_{v\in R} x_{-\alpha_{0,i}}(\pi v)w_{\alpha_{0,i}}t(\alpha_{0,i}^\vee)I$.

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Remark 2.9. While we have restricted our attention thus far to split groups, a result similar to Corollary 2.8 holds even when G is not split. We will need this result for forms of PGSp_4 over $\mathbb{Q}_p$ of split rank one. In this case, the corresponding Iwahori double cosets will contain either p or p^2 (easily enumerated) single cosets.

We now develop notation which will allow us to derive a formula for representatives of the left cosets of I in an arbitrary double coset in $I\backslash G(F)/I$. This formula will follow easily from the above results. For each s in $\tilde{S}$, we fix a lifting $\bar{s}$ of s to $N_T(F)$. We define elements $g_s(v) \in G(F)$ for all s in $\tilde{S}$ and v in R by setting

$$g_s(v) = \begin{cases} x_\alpha(v)\bar{s} & \text{if } s = w_\alpha \text{ for some } \alpha \text{ in } \Delta, \\ x_{-\alpha_{0,i}}(\pi v)\bar{s} & \text{if } s = t(\alpha_{0,i}^\vee)w_{\alpha_{0,i}} \text{ for some } i \text{ in } \{1, \ldots, m\}. \end{cases}$$

In this notation, Proposition 2.8 says that for each $s \in \tilde{S}$, $IsI = \coprod_{v \in R} g_s(v)I$. For each ρ in Ω we also choose some lifting $\bar{\rho}$ of ρ to $N_T(F)$.

For each w in $\tilde{W}$ we fix an $(l(w)+1)$-tuple $e(w) = (s_{w,1}, \ldots, s_{w,l(w)}, \rho_w)$ in $\tilde{S}^{l(w)} \times \Omega$ such that $w = s_{w,1} \cdots s_{w,l(w)}\rho_w$. We define $g_w\colon R^{l(w)} \longrightarrow G(F)$ to be the function which assigns to each $(v_1, \ldots, v_{l(w)})$ in $R^{l(w)}$ the element $g_{s_{w,1}}(v_1) \cdots g_{s_{w,l(w)}}(v_{l(w)})\overline{\rho_w}$, using the notation of the previous paragraph. Then we have the following fact concerning the coset space IwI/I.

COROLLARY 2.10. *Suppose that $w \in \tilde{W}$ and that $w = s_1 \cdots s_d\rho$ is a reduced expression (i.e., $d = l(w)$), where $s_1, \ldots, s_d \in \tilde{S}$ and $\rho \in \Omega$. Then the index $[IwI : I]$ is $q^{l(w)}$. In fact,*

$$IwI = \coprod_{v_i \in R} g_{s_1}(v_1) \cdots g_{s_d}(v_d)\bar{\rho}I = \coprod_{v \in R^{l(w)}} g_w(v)I.$$

Proof. By Proposition 2.7(ii), $IwI = Is_1I \cdots Is_dI\rho I$. It follows (cf. [7, §3.5]) that

$$[IwI : I] = [Is_1I : I] \cdots [Is_dI : I][I\rho I : I] = q^{l(w)}.$$

To complete the proof it suffices to show that the union of the $q^{l(w)}$ cosets given above is all of IwI. This also follows from Propositions 2.7 and 2.8 since

$$\begin{aligned} IwI = Is_1s_2 \cdots s_d\rho I &= Is_1Is_2I \cdots Is_dI\rho I \\ &= \bigcup_{v_1 \in R} g_{s_1}(v_1)Is_2I \cdots Is_dI\rho I \\ &= \bigcup_{v_1, \ldots, v_d \in R} g_{s_1}(v_1)g_{s_2}(v_2) \cdots g_{s_d}(v_d)\bar{\rho}I. \end{aligned}$$ □

2.4. DOUBLE COSET DECOMPOSITION FOR K

As stated earlier, the explicit determination of the action of a spherical Hecke operator on a modular form necessitates the decomposition of the support of that operator into left cosets of K. The Cartan decomposition states that

$G(F) = \coprod_{\lambda \in X_+} K\lambda(\pi)K$. Therefore, the local Hecke algebra $\mathcal{H}(G(F), K)$ is spanned by the characteristic functions $\mathrm{char}(K\lambda(\pi)K)$ ($\lambda \in X_+$). As a result, we now concentrate on decomposing a given double coset $K\lambda(\pi)K = Kt(\lambda)K$ into a union of left cosets of K.

Fix λ in X_+. We begin by considering the decomposition of $K\lambda(\pi)K$ into cosets IgK.

LEMMA 2.11. *The double coset $K\lambda(\pi)K = Kt(\lambda)K$ is the disjoint union of the cosets $I\tau t(\lambda)K$ as τ ranges over $[W/W^\lambda]$.*

Proof. Since $W_K = (N_T(F) \cap K)/\underline{T}(\mathcal{O}_F) = W$ we have by Proposition 2.6(i) that $K = \coprod_{w \in W} IwI$. (This is simply the lifting of the Bruhat decomposition for the group $\overline{G}(k)$ to K.) It follows that $Kt(\lambda)K = \bigcup_{w \in W} IwIt(\lambda)K$. We will show that this last expression is equal to $\bigcup_{w \in W} Iwt(\lambda)K$.

By Equation (10) in Proposition 2.7, if we write w' in W as a reduced expression $w' = s_1 \cdots s_d$ where $s_1, \ldots, s_d \in S$, we have that

$$Iw'It(\lambda)I = Is_1 \cdots s_d It(\lambda)I = Is_1 I \cdots Is_d It(\lambda)I.$$

By Proposition 2.7(i) and induction on d, it follows that

$$Iw'It(\lambda)I = Is_1 I \cdots Is_d It(\lambda)I \supset Is_1 \cdots s_d t(\lambda)I = Iw't(\lambda)I$$

and hence that

$$\bigcup_{w \in W} IwIt(\lambda)K = \bigcup_{w \in W} IwIt(\lambda)IK \supset \bigcup_{w \in W} Iwt(\lambda)IK = \bigcup_{w \in W} Iwt(\lambda)K.$$

On the other hand, we know by Proposition 2.7 that for any s in S and γ in $\tilde{W}$,

$$IsI\gamma I \subset I\gamma I \cup Is\gamma I \subset \bigcup_{w \in W} Iw\gamma I.$$

Thus if $w' = s_1 \cdots s_d \in W$ is a reduced expression, we have, by induction on d again,

$$Iw'It(\lambda)I = Is_1 I \cdots Is_d It(\lambda)I \subset \bigcup_{w \in W} Iwt(\lambda)I.$$

Hence, it follows that

$$\bigcup_{w \in W} IwIt(\lambda)K = \bigcup_{w \in W} IwIt(\lambda)IK \subset \bigcup_{w \in W} Iwt(\lambda)IK = \bigcup_{w \in W} Iwt(\lambda)K$$

so that

$$\bigcup_{w \in W} Iwt(\lambda)K = \bigcup_{w \in W} IwIt(\lambda)K.$$

Thus

$$Kt(\lambda)K = \bigcup_{w \in W} Iwt(\lambda)K. \tag{11}$$

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We must now determine which of the terms in the above union are the same. To this end, we apply Proposition 2.6 to the subgroups I and K of $G(F)$. Since $W_I = \langle e \rangle$ and $W_K = W$, it follows that for any w, w' in W, $Iwt(\lambda)K = Iw't(\lambda)K$ if and only if $wt(\lambda) \equiv w't(\lambda)(\mathrm{mod}\, W)$, that is, if and only if $t(w(\lambda)) = t(w'(\lambda))$. Thus the two double cosets are equal if and only if $w \equiv w'(\mathrm{mod}\, W^{\lambda})$. Therefore, to obtain a disjoint union in (11) we take the union over the set of representatives $[W/W^{\lambda}]$. We therefore obtain

$$Kt(\lambda)K = \coprod_{\tau \in [W/W^{\lambda}]} I\tau t(\lambda)K. \qquad \square \tag{12}$$

It remains now to express each coset $I\tau t(\lambda)K$ as a union of distinct left cosets of K.

THEOREM 2.12. *Let $\lambda \in X_+$ and let σ_λ be as in Section 2.2. Then the double coset $Kt(\lambda)K$ is equal to the disjoint union*

$$\coprod_{\tau \in [W/W^{\lambda}]} \coprod_{v \in R^{l(\tau t(\lambda)\sigma_\lambda)}} g_{\tau t(\lambda)\sigma_\lambda}(v)K,$$

where R is a set of representatives for $\mathcal{O}_F/\mathfrak{p}$ containing 0.

Proof. Let $\tau \in [W/W^{\lambda}]$. Clearly, $I\tau t(\lambda)K = I\tau t(\lambda)\sigma_\lambda K = I\tau t(\lambda)\sigma_\lambda IK$ and by Corollary 2.10, this is equal to

$$\left(\coprod_{v \in R^{l(\tau t(\lambda)\sigma_\lambda)}} g_{\tau t(\lambda)\sigma_\lambda}(v)I \right) K = \bigcup_{v \in R^{l(\tau t(\lambda)\sigma_\lambda)}} g_{\tau t(\lambda)\sigma_\lambda}(v)K. \tag{13}$$

Because of Lemma 2.11, the theorem will follow if we show that the cosets in the union (13) are distinct for distinct v. So suppose that $g_{\tau t(\lambda)\sigma_\lambda}(v)K = g_{\tau t(\lambda)\sigma_\lambda}(v')K$ for some $\tau \in [W/W^{\lambda}]$ and $v, v' \in R^{l(\tau t(\lambda)\sigma_\lambda)}$. We will show that $v = v'$. The main idea of the argument is to transfer the problem from K-cosets in $G(F)$ to W-cosets in $\tilde{W}$ and then to bring to bear our results on Coxeter groups from Section 2.2.

First, we note that by Proposition 2.6(i) and Corollary 2.10,

$$\begin{aligned} g_{\tau t(\lambda)\sigma_\lambda}(v)K &= \coprod_{w \in W} g_{\tau t(\lambda)\sigma_\lambda}(v)IwI \\ &= \coprod_{w \in W} \coprod_{v'' \in R^{l(w)}} g_{\tau t(\lambda)\sigma_\lambda}(v)g_w(v'')I \end{aligned} \tag{14}$$

and similarly

$$g_{\tau t(\lambda)\sigma_\lambda}(v')K = \coprod_{w \in W} \coprod_{v'' \in R^{l(w)}} g_{\tau t(\lambda)\sigma_\lambda}(v')g_w(v'')I. \tag{15}$$

Since these two K-cosets are equal, each I-coset in (14) must also appear in (15). In particular, $g_{\tau t(\lambda)\sigma_\lambda}(v)g_e(0)I = g_{\tau t(\lambda)\sigma_\lambda}(v)I$ and $g_{\tau t(\lambda)\sigma_\lambda}(v')g_w(v'')I$ must be equal for some w in W and v'' in $R^{l(w)}$. We will show that this equality can only hold if $w = e$. Then

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we will have that $g_{\tau t(\lambda)\sigma_\lambda}(v)I = g_{\tau t(\lambda)\sigma_\lambda}(v')I$, which immediately implies that $v = v'$ by Corollary 2.10.

So suppose that $g_{\tau t(\lambda)\sigma_\lambda}(v)I = g_{\tau t(\lambda)\sigma_\lambda}(v')g_w(v'')I$, where $w \in W$ and $v'' \in R^{l(w)}$. By the definition of $g_{\tau t(\lambda)\sigma_\lambda}(v)$, we have that

$$g_{\tau t(\lambda)\sigma_\lambda}(v)I \subset I\tau t(\lambda)\sigma_\lambda I. \tag{16}$$

Similarly, for each v'' in $R^{l(w)}$,

$$g_{\tau t(\lambda)\sigma_\lambda}(v')g_w(v'')I \subset I\tau t(\lambda)\sigma_\lambda IwI.$$

We are now able to use Section 2.2 since $\lambda \in X_+$, $\tau \in [W/W^\lambda]$ and $w \in W$. By Theorem 2.4, we conclude that $l(\tau t(\lambda)\sigma_\lambda w) = l(\tau t(\lambda)\sigma_\lambda) + l(w)$. This implies via equation (20) in Proposition 2.7 that $I\tau t(\lambda)\sigma_\lambda IwI = I\tau t(\lambda)\sigma_\lambda wI$. Hence,

$$g_{\tau t(\lambda)\sigma_\lambda}(v')g_w(v'')I \subset I\tau t(\lambda)\sigma_\lambda wI. \tag{17}$$

Since the double cosets in (16) and (17) both contain the left coset

$$g_{\tau t(\lambda)\sigma_\lambda}(v)I = g_{\tau t(\lambda)\sigma_\lambda}(v')g_w(v'')I,$$

we conclude that they must be equal. But $I\tau t(\lambda)\sigma_\lambda I = I\tau t(\lambda)\sigma_\lambda wI$ implies $w = e$ since $I\backslash G(F)/I$ is represented by $\tilde{W}$ (Proposition 2.6). □

The following corollary follows easily from Theorem 2.12 and Lemma 2.4.

COROLLARY 2.13. *The number of left (or right) cosets of K in $K\lambda(\pi)K$ is*

$$q^{\min_{w\in W} l(t(\lambda)w)} \sum_{\tau\in[W/W^\lambda]} q^{l(\tau)} = \sum_{\gamma\in Wt(\lambda)W} q^{l(\gamma)} \Big/ \sum_{w\in W} q^{l(w)}.$$

2.5. AN EXAMPLE

In later sections, we determine the actions of local Hecke algebras on spaces of modular forms on the compact form of G_2 over $\mathbb{Q}$ and on certain compact forms of PGSp_4 over $\mathbb{Q}$. To do this we need to know the coset representatives a_i appearing in the sum (1) in the introduction. Below we illustrate the use of Theorem 2.12 to compute these representatives for the split group G_2 over an arbitrary non-Archimedean local field F.

The rank of G_2 is 2, and a set of simple roots consists of a long root α_1 and a short root α_2. We let α_0 be the corresponding highest root. The Weyl group W, which is dihedral of order 12, is generated by the reflections $w_1 = w_{\alpha_1}$ and $w_2 = w_{\alpha_2}$, while W_{af} is generated by these reflections and $w_0 = w_{\alpha_0}t(\alpha_0^\vee)$. Since G_2 is both simply connected and adjoint, Ω is trivial and it follows that $\tilde{W} = W_{\mathrm{af}}$. Let $\check{\omega}_1, \check{\omega}_2$ be the fundamental co-characters (i.e., those satisfying $\langle\alpha_i, \check{\omega}_j\rangle = \delta_{ij}$).

We have that $W^{\check{\omega}_1} = \langle w_2\rangle$ and $[W/W^{\check{\omega}_1}]$ is the set $\{e, w_1, w_2w_1, w_1w_2w_1, w_2w_1w_2w_1, w_1w_2w_1w_2w_1\}$. Also, $t(\check{\omega}_1)$ can be shown to have the reduced expression

$w_0w_1w_2w_1w_2w_1$, while a reduced expression for $t(\check{\omega}_1)\sigma_{\check{\omega}_1}$ is w_0. Hence, it follows from Theorem 2.12 that $K\check{\omega}_1(\pi^{-1})K$ is the union of the

$$q + q^2 + q^3 + q^4 + q^5 + q^6 = q \cdot \frac{q^6 - 1}{q - 1}$$

left cosets

$$\begin{array}{ll} g_{w_0}(v_1)K & (v_1 \in R) \\ g_{w_1w_0}(v_2)K & (v_2 \in R^2) \\ g_{w_2w_1w_0}(v_3)K & (v_3 \in R^3) \\ g_{w_1w_2w_1w_0}(v_4)K & (v_4 \in R^4) \\ g_{w_2w_1w_2w_1w_0}(v_5)K & (v_5 \in R^5) \\ g_{w_1w_2w_1w_2w_1w_0}(v_6)K & (v_6 \in R^6). \end{array}$$

For the short co-weight $\check{\omega}_2$, we have $W^{\check{\omega}_2} = \langle w_1 \rangle$ and $[W/W^{\check{\omega}_2}] = \{e, w_2, w_1w_2, w_2w_1w_2, w_1w_2w_1w_2, w_2w_1w_2w_1w_2\}$. A reduced expression for $t(\check{\omega}_2)$ is $w_0w_1w_2w_1w_0$ $w_2w_1w_2w_1w_2$, and for $t(\check{\omega}_2)\sigma_{\check{\omega}_2}$ is $w_0w_2w_1w_2w_0$. Hence, the double coset $K\check{\omega}_2(\pi^{-1})K$ is the disjoint union of the

$$q^5 + q^6 + q^7 + q^8 + q^9 + q^{10} = q^5 \cdot \frac{q^6 - 1}{q - 1}$$

cosets

$$\begin{array}{ll} g_{w_0w_2w_1w_2w_0}(v_1)K & (v_1 \in R) \\ g_{w_2w_0w_2w_1w_2w_0}(v_2)K & (v_2 \in R^2) \\ g_{w_1w_2w_0w_2w_1w_2w_0}(v_3)K & (v_3 \in R^3) \\ g_{w_2w_1w_2w_0w_2w_1w_2w_0}(v_4)K & (v_4 \in R^4) \\ g_{w_1w_2w_1w_2w_0w_2w_1w_2w_0}(v_5)K & (v_5 \in R^5) \\ g_{w_2w_1w_2w_1w_2w_0w_2w_1w_2w_0}(v_6)K & (v_6 \in R^6). \end{array}$$

3. The Calculations

We now return to the global setting. We take G to be a connected reductive group over $\mathbb{Q}$ and as above we assume that $G(\mathbb{R})$ is compact. We let $K = \prod K_p$ where each K_p is a parahoric subgroup of $G(\mathbb{Q}_p)$, with all but finitely many K_p hyperspecial maximal compact. We also let W be an algebraic representation of G over a number field E.

Our goal is to compute the actions of various Hecke operators on the space

$$M(W, K) = \{F\colon G(\hat{\mathbb{Q}})/K \to W(E) : F(\gamma g) = \gamma F(g),\ \text{for all } \gamma \in G(\mathbb{Q})\}$$

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of modular forms of weight W and level K on G. We begin with an overview of these calculations and then indicate how to carry out some of the steps for the particular G in which we are interested.

3.1. OVERVIEW

A function $f \in M(W, K)$ is determined by its values on a system of representatives of $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$. We fix such a system $\{g_\alpha\}$. Note that for $f \in M(W, K)$ each $f(g_\alpha)$ lies in W^{Γ_α} where Γ_α is the finite subgroup $G(\mathbb{Q}) \cap g_\alpha K g_\alpha^{-1}$ of $G(\mathbb{Q})$ stabilizing $g_\alpha K$. Conversely, any function $f: \{g_\alpha\} \to W$ with $f(g_\alpha) \in W^{\Gamma_\alpha}$ for all α extends uniquely to an element of $M(W, K)$. We pick bases $\{v_{\alpha,k}\}$ of the W^{Γ_α} and define δ_α^k to be the modular form such that

$$\delta_\alpha^k(g_\beta) = \begin{cases} v_{\alpha,k} & \text{if} \quad \beta = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then the δ_α^k form a basis of $M(W, K)$. We will do all of our calculations with respect to this basis.

Now consider the action of a Hecke operator $T \in \mathcal{H}(G(\mathbb{Q}_p), K_p)$. We wish to compute $T\delta_\alpha^k(g_\nu)$. Using Equation (1) and writing the support of T as a disjoint union of cosets $a_l K_p$ we see

$$T\delta_\alpha^k(g_\nu) = \sum_l T(a_l)\delta_\alpha^k(g_\nu a_l). \tag{18}$$

Note that $T\delta_\alpha^k(g_\nu) \in W^{\Gamma_\nu}$, so once we have computed it we can write

$$T\delta_\alpha^k(g_\nu) = \sum_l m_{\alpha\nu}^{kl} v_{\nu,l},$$

so that

$$T\delta_\alpha^k = \sum_{\nu,l} m_{\alpha\nu}^{kl} \delta_\nu^l. \tag{19}$$

These $m_{\alpha\nu}^{kl}$ are the entries in the matrix for T with respect to the basis δ_α^k of $M(W, K)$. So once we have seen how to carry out each step of this outline, we will be able to compute matrices for the actions of our Hecke operators.

We discuss our computation of the g_α in 3.2, pointing out aspects particular to some individual examples. Once we have the g_α, computing the groups Γ_α and the fixed spaces W^{Γ_α} is straightforward, and gives us our explicit basis $\{\delta_\alpha^k\}$.

We will only be interested in Hecke operators T supported at primes where K_p is Iwahori or $G(\mathbb{Q}_p)$ is split and K_p is hyperspecial and so Corollary 2.10, Remark 2.9 and Theorem 2.12 allow us to find the a_i we need for decomposition of the support of T.

All that remains then to complete the calculation is to evaluate the δ_α^k at various points h. To do so, it suffices to write h as a product $h = rg_\alpha k$ with $r \in G(\mathbb{Q})$,

$k \in K$, and g_α one of our fixed coset representatives. Our method for finding this product decomposition is essentially brute force. In each case it amounts to analyzing the congruence constraints placed on r by insisting that $g_\alpha^{-1} r^{-1} h$ be integral and searching through the solution space of these congruences.

3.2. GLOBAL DOUBLE COSETS

We will begin with a general description of the space $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$ which is finite since K is open and $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})$ is compact [2].

Suppose first that G is the general fiber of a group scheme $\underline{G}$ over $\mathbb{Z}$ and each K_p is the group $\underline{G}(\mathbb{Z}_p)$ of $\mathbb{Z}_p$ points of $\underline{G}$. In this case, the size h of $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$ is called the class number of $\underline{G}$. In [9] Gross compiles a table of some values of the class number when G is simply connected and the K_p are all hyperspecial.

If $K' = \prod K'_p \subset K$ is any deeper level, the following easy proposition relates $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$ to $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K'$.

PROPOSITION 3.1. *Let G be a connected reductive group over $\mathbb{Q}$ arising as the general fiber of a group scheme $\underline{G}$ over $\mathbb{Z}$. Suppose that $G(\mathbb{R})$ is compact and that $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/\underline{G}(\hat{\mathbb{Z}})$ has representatives $g_1, \ldots, g_h$. For each i, let Γ'_i be the finite group $g_i^{-1} G(\mathbb{Q}) g_i \cap \underline{G}(\hat{\mathbb{Z}})$.*

For each p, let K_p be a parahoric subgroup of $\underline{G}(\mathbb{Z}_p)$ with $K_p = \underline{G}(\mathbb{Z}_p)$ for p outside a finite set S. Then there is a natural bijection between $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$ and

$$\coprod_i \left(\Gamma'_i \backslash \prod_{p \in S} \mathcal{F}_p(\mathbb{F}_p) \right)$$

where $\mathcal{F}_p$ is the flag variety of parabolics of the same type as the reduction of K_p in $\underline{G}(\mathbb{F}_p)$.

Proposition 3.1 reduces our problem of finding systems of representatives for $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$ for various K to first doing so once with K the integral points on a model of G and then simply enumerating flag varieties over finite fields. We will now discuss each of these issues in two examples.

3.2.1. *Double Cosets for the Group G_2*

There are only two forms of G_2 over $\mathbb{Q}$, one is split and the other is compact at $\mathbb{R}$ [22]. We describe the compact form below. If we replace the Cayley octonions with the split octonions in the construction, we would obtain the split group.

Let $\mathbb{O}$ be a maximal order in the Cayley octonions over $\mathbb{Q}$. $\mathbb{O}$ has an anti-involution given by $\overline{e_0} = e_0$ and $\overline{e_i} = -e_i$ for $1 \leqslant i \leqslant 7$. We have the trace given by $\mathrm{Tr}\, x = x + \bar{x}$ and the norm given by $\mathrm{N} x = x\bar{x}$. The trace allows us to define an inner product $\langle x, y \rangle = \mathrm{Tr}\, x\bar{y}$. Note that the norm, trace and inner product all take

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integral values on $\mathbb{O}$ and so can be extended to $\mathbb{O} \otimes R$ for any ring R. Note also that R embeds in $\mathbb{O} \otimes R$ as Re_0.

We define a group G over $\mathbb{Z}$ by letting $G(R)$ be the group of R-algebra automorphisms of $\mathbb{O} \otimes R$. Then G is a model of G_2 over $\mathbb{Z}$ in the sense of [9]. In particular for all p, $G(\mathbb{Q}_p)$ is $G_2(\mathbb{Q}_p)$, the split group of type G_2 over $\mathbb{Q}_p$, and $G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ is a hyperspecial maximal compact subgroup. Further $G(\mathbb{R})$ is compact, and so $G(\mathbb{Z})$ is finite, being a discrete subgroup of $G(\mathbb{R})$. In fact $G(\mathbb{Z})$ is the group $G_2(\mathbb{F}_2)$, and has size 12096. Additionally, G has class number one, that is $G(\hat{\mathbb{Q}}) = G(\mathbb{Q})G(\hat{\mathbb{Z}})$ [9].

If $M = \mathbb{O} \otimes k$ for k a field of characteristic not 2, we define M_0 to be the orthogonal complement of e_0 in M, namely the elements of trace 0 in M. Automorphisms of M preserve M_0, as well as both the norm and trace [18, §3].

Suppose G is split over k with the characteristic of k still not 2. Then a Borel subgroup of $G(k)$ is the stabilizer of a flag $0 \subset V_1 \subset V_2$ in the 7-dimensional space M_0 with V_1 spanned by a vector of norm, and hence square, 0 (a null line) and V_2 such that $xy = 0$ for all $x, y \in V_2$ (a null plane). Note that any such flag can be extended uniquely to a complete null flag by setting

$$V_3 = \{y\colon\ xy = 0 \text{ for all } x \in V_2\}$$
$$V_i = V_{7-i}^{\perp} \text{ for } 4 \leqslant \mathrm{i} \leqslant 7,$$

so that a Borel is the stabilizer of a complete flag.

The other two types of parabolics stabilize partial flags. They are P_1, the stabilizer of a null line, and P_2, the stabilizer of a null plane. The collection of null flags (resp. lines, resp. planes) exactly parametrizes the space of Borel subgroups (resp. parabolics of type P_1, resp. parabolics of type P_2). If k is the finite field $\mathbb{F}_p$, there are $(p^6 - 1)/(p - 1)$ parabolic subgroups of types P_1 and P_2 and $(p^6 - 1)(p + 1)/(p - 1)$ Borel subgroups in $G(k)$.

3.2.2. *Double Cosets for the Groups* PGSp_4^H

Let H be a quaternion algebra over $\mathbb{Q}$, and let $u \mapsto \bar{u}$ be its canonical involution. Then we define an algebraic group G^H over $\mathbb{Q}$ whose R points are

$$\{g \in M_2(H \otimes_{\mathbb{Q}} R)\colon g\bar{g}^t = \nu(g)I, \nu(g) \in R^*\}$$

for any commutative $\mathbb{Q}$-algebra R. If H splits over R, then G^H is isomorphic to GSp_4 over R. Thus G^H is a form of GSp_4 over $\mathbb{Q}$. We will henceforth denote G^H by GSp_4^H. We then let PGSp_4^H be the quotient of GSp_4^H by its center. If H is ramified at ∞ then $\mathrm{PGSp}_4^H(\mathbb{R})$ is compact. We will assume this is the case from here on.

If we choose a maximal order M in H we can give GSp_4^H and PGSp_4^H the structure of algebraic groups over $\mathbb{Z}$ as we did for G_2. If H is split at p, so that PGSp_4^H is split over $\mathbb{Q}_p$, then $\mathrm{PGSp}_4^H(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup of $\mathrm{PGSp}_4^H(\mathbb{Q}_p) = \mathrm{PGSp}_4(\mathbb{Q}_p)$. If H is not split at p, then PGSp_4^H has split rank 1 over

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$\mathbb{Q}_p$. $K_p = \mathrm{PGSp}_4^H(\mathbb{Z}_p)$ is still maximal compact, but is not hyperspecial ($\mathrm{PGSp}_4^H(\mathbb{Q}_p)$ has no hyperspecials).

The global situation is more complicated for PGSp_4^H than for G_2, in that PGSp_4^H need not have class number 1 (with respect to the integral structure given by a maximal order M). Following Shimura [21] we consider the action of $\mathrm{PGSp}_4^H(\hat{\mathbb{Q}})$ on the set of M-lattices in H^2. We see that the stabilizer of M^2 is $\mathrm{PGSp}_4^H(\hat{\mathbb{Z}})$, so that $\mathrm{PGSp}_4^H(\hat{\mathbb{Q}})/\mathrm{PGSp}_4^H(\hat{\mathbb{Z}})$ is the orbit of M^2, called the *principal genus* of M-lattices. $\mathrm{PGSp}_4^H(\mathbb{Q})$ acts on the principal genus, its orbits are called classes. So the class number of PGSp_4^H is the number of classes in the principal genus.

In [14, Theorem 2] Hashimoto and Ibukiyama give a formula for this class number and they tabulate some small values in their §5-3. In §6-1 they give an algorithm for finding representative of the various classes in the principal genus, which amounts to finding representatives for

$$\mathrm{PGSp}_4^H(\mathbb{Q})\backslash\mathrm{PGSp}_4^H(\hat{\mathbb{Q}})/\mathrm{PGSp}_4^H(\hat{\mathbb{Z}}).$$

If H is ramified only at 2 and ∞ or only at 3 and ∞, the class number of PGSp_4^H is 1. If H is ramified at 5 and ∞ the class number is 2. We represent this H by the algebra $\mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ with $i^2 = -5, j^2 = -2$, and i,j,k satisfying the usual product formulas. We take here the maximal order spanned by

$$1,\ j,\ \frac{2+j-k}{4},\ \frac{2i+j+k}{4}$$

and then a choice of representative for the nontrivial double coset is

$$\begin{pmatrix} 2+k & -2i-9j \\ -2i-9j & 2+k \end{pmatrix}.$$

We now fix a quaternion algebra H ramified at ∞ and at some finite set S_0 of primes and choose a maximal order M in H. We let $G = \mathrm{PGSp}_4^H$ with the $\mathbb{Z}$-structure induced by M.

We must again look at the flag varieties $G(\mathbb{F}_p)/B_p$. For an unramified prime (i.e. $p \notin S_0$), G is isomorphic to the split group PGSp_4 over $\mathbb{F}_p$. The Borel subgroups are again parametrized by flags consisting of a null line contained in a null plane (now with respect to the symplectic inner product on ${\mathbb{F}_p}^4$), and the parabolics by partial flags. For a ramified prime, the Borels are parametrized by the projective line over $\mathbb{F}_p$ in a computable way. That is, $G(\mathbb{F}_p)$ acts transitively on $\mathbb{P}^1$, with stabilizer B_p.

3.3. COMPUTING AND RELIABILITY

So far we have presented the algorithms for carrying out our calculations, but have said nothing about how they are implemented. A few words on that topic are in order.

Most of the calculations were carried out by programs we wrote in C++ and compiled with gcc running under SunOS on a Sun SPARCserver-1000 and separately under egcs running under Linux on an Intel Pentium 2. Some parts (especially processing the matrices obtained as the actions of the Hecke operators) were done using Mathematica [19], Gap [20], and Pari [8]. All calculations were done using exact data types (i.e. integers or rationals, rather than reals). Any p-adic numbers were only needed modulo a fixed power of p, and so could be represented by an integer. When dealing with data produced by large programs, one hopes for as many checks as possible to verify that the program has worked correctly. Fortunately, our data allows for a great many strong checks. For example, we know that the spherical Hecke operators all commute as elements of $\mathcal{H}_K$. So when we compute the actions of these Hecke operators on a space of forms, the matrices we obtain ought to commute. This is the case in every example we have computed. Since these matrices can have several hundred rows and columns, this is a rather meaningful check.

We mention two other checks that we will come across later. In Section 4.3 we discuss some conjectured 'liftings' of modular forms from PGL_2. We have been able to identify many of these lifts. Finally in an upcoming paper the second author and S. Padowitz use some of our data, along with the stable trace formula, to work out an explicit formula for the dimension of a certain subspace of $M(W, K)$. Our data over-determines the explicit formula, and the fact that the formula fits all of our data points is a very strong indication of the validity of our calculations.

4. Interpretations of the Data

4.1. THE STEINBERG SPACE

Let π be the Steinberg representation of $G(\mathbb{Q}_p)$. Then π has a one-dimensional space of fixed vectors for an Iwahori subgroup I. So π^I induces a character of the Iwahori–Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p), I_p)$ called the Steinberg character.

To describe this character concretely we adopt the notation of Section 2 and let $s \in \tilde{S}$ be a standard involutive generator of W_{af}. Then the Steinberg character sends the Hecke operator $U_s = \mathrm{char}_{IsI}$ to -1. Any irreducible representation of $\mathcal{H}_{I_p}$ which has a vector on which each U_s acts by -1 is in fact 1-dimensional. These representations are called special characters.

If $K = I_S = \prod_{p \notin S} K_p \prod_{p \in S} I_p$ with K_p hyperspecial, we call the subspace of $M(W, K)$ on which each $\mathcal{H}_{I_p}$ acts by a special character the Steinberg subspace and denote it $M(W, K)^{St}$. Note that this is an abuse of language as we allow any special character, not just the Steinberg. This subspace is of particular number theoretic interest (cf. [10, §12]). Our calculations of the action of the Iwahori–Hecke algebra on various spaces of forms allows us to identify the Steinberg subspaces. We have tabulated some dimensions of these Steinberg subspaces in Section 5.

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4.2. SATAKE AND LANGLANDS PARAMETERS

4.2.1. *Satake Parameters*

Given an eigenform f for the Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p), K_p)$ we get a complex character $\theta_f : \mathcal{H}(G(\mathbb{Q}_p, K_p)) \to \mathbb{C}^*$. We will describe a convenient indexing of such characters, for K_p hyperspecial.

Let $\hat{G}$ be the complex dual group of G, and let $\hat{T}$ be a maximal torus of $\hat{G}$. Then G and $\hat{G}$ have the same Weyl group W, which acts on the character group $X^*(\hat{T})$, and hence on the group algebra $\mathbb{Z}[X^*(\hat{T})]$. Then the *representation ring* of $\hat{G}$ is the ring of formal sums of characters of representations of $\hat{G}(\mathbb{C})$: $R(\hat{G}) = \mathbb{Z}[X^*(\hat{T})]^W$. The Satake transform gives an isomorphism [12, Prop. 3.6]

$$\mathcal{H}(G(\mathbb{Q}_p), K_p) \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}] \cong R(\hat{G}) \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}].$$

So our character θ_f induces a character on $R(\hat{G}) \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}]$, and hence on $R(\hat{G}) \otimes \mathbb{C}$. Such characters are parametrized by the semi-simple conjugacy classes in $\hat{G}(\mathbb{C})$ [12, §6]. In particular, if s is such a conjugacy class we define a character ω_s of $R(\hat{G}) \otimes \mathbb{C}$ which sends $\chi = \sum a_\lambda \cdot \lambda$ to

$$\omega_s(\chi) = \sum a_\lambda \cdot \lambda(s_0)$$

where s_0 is any element of $s \cap \hat{T}$. Since χ is W-invariant, this sum is independent of our choice of s_0.

Thus we associate to θ_f, and hence to f, a semi-simple conjugacy class $s_p(f) \in \hat{G}(\mathbb{C})$. We call this class the *Satake parameter* of f. To compute $s_p(f)$ it suffices to know the eigenvalues $\theta_f(T_i(p))$ of the generators of $\mathcal{H}(G(\mathbb{Q}_p), K_p)$ on f, and the images of the $T_i(p)$ under the Satake transform.

In [12] Gross works out the Satake transform in several cases. For $G = G_2$, $\hat{G}(\mathbb{C}) = G_2(\mathbb{C})$ and Gross finds

$$\mathrm{Tr}(V_7) = \frac{S(T_1(p)) + 1}{p^3}$$
$$\mathrm{Tr}(V_{14}) = \frac{S(T_1(p)) + S(T_2(p)) + (1 + p^4)}{p^5}$$

where, for V a representations of $\hat{G}(\mathbb{C})$, $\mathrm{Tr}(V) \in \hat{G}(\mathbb{C})$ is the formal sum of the weights that appear in V. A conjugacy class s in $G_2(\mathbb{C})$ is determined by its traces on V_7 and V_{14}. Since $\wedge^2 V_7 \cong V_{14} \oplus V_7$ this information is encoded in the characteristic polynomial of s on V_7. Using also that $\wedge^3 V_7 \oplus V_{14} \cong V_7 \otimes V_7$ and that $G_2(\mathbb{C})$ acts orthogonally on V, we can work out the characteristic polynomial of $s_p(f)$ on V_7 given the Hecke eigenvalues $\alpha_i = \theta_f(T_i(p))$ of f [12, cf. Eq. 6.10].

For $G = \mathrm{PGSp}_4$, $\hat{G}(\mathbb{C}) = \mathrm{Sp}_4(\mathbb{C})$ and Gross gets

$$\mathrm{Tr}(V_4) = \frac{S(T_1(p))}{p^{\frac{3}{2}}}$$
$$\mathrm{Tr}(V_5) = \frac{S(T_2(p)) + 1}{p^2}$$

Once again, this information determines a conjugacy class s in $\mathrm{Sp}_4(\mathbb{C})$, and is equivalent to the information contained in the characteristic polynomial of s on V_4.

4.2.2. *Archimedean Parameters*

If $G(\mathbb{R})$ is compact, an irreducible representation π of $G(\mathbb{R})$ is classically parametrized by a dominant weight $\chi \in X^*(T)$ where T is a maximal torus of G and $X^*(T)$ is the character module over $\mathbb{C}$. Note that $X^*(T) \cong X_*(\hat{T})$ where $\hat{T}$ is the corresponding maximal torus of $\hat{G}$, so we may view χ as an element of $X_*(\hat{T})$. Let $\mu = \chi + \rho$, where ρ is half the sum of the positive weights. Then, viewing $\mu \in X_*(\hat{T}) \otimes \mathbb{R}$ we have a map $\mathbb{C}^* \to \hat{T}(\mathbb{C}) \subset \hat{G}(\mathbb{C})$ given by $z \longmapsto z^{\mu}\bar{z}^{-\mu}$ (see [3, §9.1, §10.5] for details).

This map is the Archimedean Langlands parameter of π. If f is a modular form for G of weight W we denote the Langlands parameter of W by $\phi_\infty(f)$.

4.3. LIFTINGS

If G and G' are reductive groups over $\mathbb{Q}$ and $\rho\colon \hat{G}(\mathbb{C}) \to \hat{G}'(\mathbb{C})$, then we can use ρ to 'lift' Satake and Archimedean parameters from $\hat{G}(\mathbb{C})$ to $\hat{G}'(\mathbb{C})$. If f is a Hecke eigenform for G, then we get a collection of Satake parameters $\{\rho(s_p(f))\}$ and a Langlands parameter $\rho \circ \phi_\infty(f)$ for G'. It is natural to ask if these (or at least all but finitely many of these) arise as the parameters of a Hecke eigenform f' for G'. If G' is quasi-split, then Langlands functoriality conjectures that the answer will be yes [1, pg. 12]. Such an f', if it exists, is called a lift of f. If G' is not quasi-split, not all maps ρ are expected to yield lifts, and even when they do not all f are expected to lift. In this section we discuss some maps of dual groups, and analyze the corresponding lifts of modular forms.

First we look at lifts from PGL_2 to G_2. Recall that the dual group of PGL_2 is SL_2, while G_2 is its own dual. There are 4 non-trivial conjugacy classes of unipotents in G_2 [15, pg. 132] and so there are 4 conjugacy classes of non-trivial maps $\mathrm{SL}_2(\mathbb{C}) \to G_2(\mathbb{C})$. The images of these are the long root, short root, principal and subregular SL_2's. Of these all but the principal lie in a proper parabolic. This implies (c.f. [4, §8.2]) that of these maps, only the principal homomorphism provides a lift from modular forms on PGL_2 to modular forms on the anisotropic form of G_2. (In particular, under the other maps the real components of the corresponding automorphic representations do not transfer to representations on the compact form of $G_2(\mathbb{R})$.) However, the long and short root embeddings can be chosen to have commuting images and so piece together to give a map from $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ to $G_2(\mathbb{C})$. The connected centralizer of the subregular SL_2 is unipotent so we can not make a similar construction.

We first discuss the map $\rho\colon SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to G_2(\mathbb{C})$ given by the short root embedding in the first coordinate, and the long root in the second. Suppose that ϕ_p and ψ_p are Satake parameters for $PGL_2(\mathbb{Q}_p)$ (i.e. conjugacy classes in $SL_2(\mathbb{C})$) given by Satake parameters with characteristic polynomials $x^2 - a_p x + 1$ and $x^2 - b_p x + 1$. Then (ϕ_p, ψ_p) lifts to the Satake parameter for G_2 whose characteristic polynomial in the 7-dimensional representation is

$$(x-1)(x^2 - (a_p^2 - 2)x + 1)(x^4 - a_p b_p x^3 + (a_p^2 + b_p^2 - 2)x^2 - a_p b_p x + 1).$$

From this we can obtain the traces of the Satake parameter on the 7 and 14-dimensional representations and then use the Satake isomorphism to compute the eigenvalues λ_1 and λ_2 of the Hecke operators $T_1(p)$ and $T_2(p)$ on the corresponding representation of the Hecke algebra. We find

$$\begin{aligned}
\lambda_1 &= p^3(a_p b_p + a_p^2 - 1) - 1,\\
\lambda_2 &= p^5(a_p^3 b_p + a_p^2 - 2a_p b_p + b^2 - 2) - p^3(a_p b_p + a_p^2 - 1) - p^4.
\end{aligned}$$

We now look at the Archimedean parameters. Suppose ϕ and ψ are the Archimedean Langlands parameters of the real components of automorphic forms corresponding to classical eigenforms of weights k and j. If $k = j$ or $3k - 2 = j$ then (ϕ, ψ) does not lift to an admissible parameter for the compact form of G_2. Otherwise it does and the lifted parameter corresponds to the representation of highest weight in the Weyl orbit of:

$$\begin{cases}
\dfrac{j-2}{2}\omega_1 + \dfrac{k-4}{2}\omega_2, & \text{if } j < k,\\
(j-5)\omega_1 + \dfrac{k-j+2}{2}\omega_2, & \text{if } k < j < 3k-2,\\
(j-6)\omega_1 + \dfrac{k-j+4}{2}\omega_2, & \text{if } 3k-2 < j.
\end{cases}$$

We will consider lifting systems of parameters ϕ and ψ that arise from classical eigenforms or from the trivial representation of an anisotropic form H of PGL_2. In the former case, if f is a normalized eigenform of weight k with Fourier expansion $\sum a_n q^n$ then $s_p(f)$ has characteristic polynomial $x^2 - p^{\frac{1-k}{2}} a_p + 1$. In the later case, at primes p where the group H is split the Satake parameter has characteristic polynomial $x^2 - (p+1)x + 1$ while the representation at the real place has the same parameter as a weight two cusp form.

For example, there is a unique cusp form f on PGL_2 of weight 4 and level $\Gamma_0(5)$, given by

$$f = q(\eta(z)\eta(5z))^4 = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 + \cdots$$

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There is also a unique cusp form g on PGL_2 of weight 6 and level $\Gamma_0(5)$, given by

$$g = q + 2q^2 - 4q^3 - 28q^4 + 25q^5 - 8q^6 + 192q^7 + \cdots$$

We let ϕ_f and ϕ_g be the corresponding parameters, and let ϕ_τ be the parameter of the trivial representation of an anisotropic form of PGL_2 (say the one ramified at 5 and ∞).

Lifting (ϕ_f, ϕ_τ) to G_2 we expect to find a form of trivial weight for which the short and long Hecke operators at $p = 2$ have eigenvalues -17 and 144, while the Hecke operators at $p = 3$ have eigenvalues 0 and 364. Indeed, we have seen there is such a form of level K^5. If we lift (ϕ_τ, ϕ_g) we are then looking for a form on G_2 of trivial weight with Hecke eigenvalues 33 and 94 for $p = 2$ and 100 and 164 for $p = 3$. Once again, we have seen such a form of level K^5. Finally if we lift (ϕ_f, ϕ_g) we are looking for a form with weight the 7-dimensional representation with Hecke eigenvalues 3 and -16 for $p = 2$ and $-80/3$ and $-4004/9$ for $p = 3$. There is in fact such a form, of level K^5 (see Table V).

Now consider the map $\mathrm{SL}_2(\mathbb{C}) \to G_2(\mathbb{C})$ corresponding to the principal unipotent. Again let ϕ_p be a Satake parameter for $\mathrm{PGL}_2(\mathbb{Q}_p)$ having characteristic polynomial $x^2 - a_p x + 1$. Then ϕ_p lifts to the Satake parameter for $G_2(\mathbb{Q}_p)$ corresponding to the character of Hecke algebra for which the $T_i(p)$ have eigenvalues

$$\lambda_1 = p^3(a^6 - 5a^4 + 6a^2 - 1),$$
$$\lambda_2 = p^5(a^{10} - 9a^8 + 18a^6 - 35a^4 + 16a^2 - 2) - p^3(a^6 - 5a^4 + 6a^2 - 1) - p^4.$$

The Archimedean Langlands parameter of the real component of an automorphic representation corresponding to a classical eigenform of weight k now lifts to the parameter of the representation of $G_2(\mathbb{R})$ with highest weight $(k-2)\rho$, where ρ as usual is half the sum of the positive roots.

For example, if f is the unique cusp form of weight 2 and level $\Gamma_0(11)$ given by

$$f = q(\eta(z)\eta(11z))^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \cdots$$

then we are looking for a form on G_2 with weight $\mathbb{C}$ and Hecke eigenvalues -9 and 56 for $p = 2$. Indeed, we do find such a form.

We now look at lifts to PGSp_4. Here there are three conjugacy classes of unipotents, corresponding to the long root, short root, and principal SL_2's. Once again we do not get lifts from the root SL_2's directly, but we can choose two commuting long root SL_2's and get a map $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$. The reductive part of the connected centralizer of the short root SL_2 is a torus, and so does not enable us to construct a map of dual groups whose image does not lie in the a proper parabolic.

First consider the map $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ arising from two commuting long root SL_2's. Again let ϕ_p and ψ_p be Satake parameters for $\mathrm{PGL}_2(\mathbb{Q}_p)$ with characteristic polynomial $x^2 - a_p x + 1$ and $x^2 - b_p x + 1$. Then (ϕ_p, ψ_p) lifts to the Satake parameter whose characteristic polynomial in the 4-dimensional representation is

$$x^4 - (a_p + b_p)x^3 + (a_p b_p + 2)x^2 - (a_p + b_p)x + 1.$$

We then find that the eigenvalues λ_1 and λ_2 of $T_1(p)$ and $T_2(p)$ on the corresponding representation of the Hecke algebra are

$$\lambda_1 = p^{\frac{3}{2}}(a_p + b_p), \qquad \lambda_2 = p^2(a_p b_p + 1) - 1.$$

If ϕ and ψ are the Langlands parameters for the real component of an automorphic representation corresponding to classical cusp forms of weights k and j then (ϕ, ψ) lifts to a parameter of a representation of compact $\mathrm{PGSp}_4(\mathbb{R})$ if and only if $k \neq j$. If so, we may assume without loss of generality that $k > j$. In that case the lifted parameter corresponds to the representation of highest weight $((k-j-2)/2)\omega_2 + (j-2)\omega_1$, where ω_1 and ω_2 are the long and short fundamental weights.

Now we consider lifts via the principal homomorphism, $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$. Again let ϕ_p be the Satake parameter for $\mathrm{PGL}_2(\mathbb{Q}_p)$ having characteristic polynomial $x^2 - a_p x + 1$. Then we find that ϕ_p lifts to the parameter whose characteristic polynomial in the 4-dimensional representation is

$$x^4 - (a_p^3 - 2a_p)x^3 + (a_p^4 - 3a_p^2 + 1)x^2 - (a_p^3 - 2a_p)x + 1.$$

We find that the corresponding Hecke eigenvalues are

$$\lambda_1 = p^{\frac{3}{2}}(a_p^3 - 2a_p), \qquad \lambda_2 = p^2(a_p^4 - 3a^2 + 1) - 1$$

The Langlands parameter corresponding to a classical cusp form of weight k now lifts to the parameter of the representation of PGSp_4 of highest weight $(k-2)\rho$ where ρ is again half the sum of the positive roots. We should point out that the principal homomorphism here is the symmetric cube map $\mathrm{SL}_2 \to \mathrm{Sp}_4$.

5. Data

In this chapter we present some of the data from our calculations. More data, including the matrices giving the action of the Hecke operators, can be found at the second author's web page at www.math.ohio-state.edu/~pollack.

In Tables I–III we tabulate the dimensions of the various spaces of modular forms we have calculated. The entry in the row corresponding to S and the column corresponding to W is

$$\dim(M(W, K^S)), \ \dim(M(W, K^S)^{St}).$$

Recall that $K^S = \prod_{p \notin S} K_p \prod_{p \in S} I_p$ with each K_p hyperspecial and each I_p an Iwahori subgroup.

The next three tables (Tables IV–VI) give the decompositions of $M(W, K^S)$ into irreducible representations of $\mathcal{H}(G(\hat{\mathbb{Q}}), K^S)$ for certain W and S. We only include

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Table I. The dimensions of $M(W, K^S)$ and $M(W, K^S)^{St}$ for G_2

	W_1	W_7	W_{14}	W_{27}	W_{64}
{2}	1, 0	0, 0	0, 0	2, 0	1, 1
{3}	3, 0	0, 0	1, 0	7, 2	
{5}	7, 1	13, 7	26, 11	63, 31	
{7}	29, 13	82, 54	194, 120		
{11}	187, 134				
{13}	523, 385				
{2, 3}	43, 1				
{2, 7}	2532, 252				
{3, 5}	2956				

Table II. The dimensions of $M(W, K^S)$ and $M(W, K^S)^{St}$ for $\mathrm{PGSp_4}^{H_2}$

	W_1	W_5
{2}	1, 0	0, 0
{2, 3}	3, 0	3, 1
{2, 5}	11, 2	
{2, 7}	28, 5	
{2, 11}	99, 34	

Table III. The dimensions of $M(W, K^S)$ and $M(W, K^S)^{St}$ for $\mathrm{PGSp_4}^{H_5}$

	W_1
{2, 5}	13, 2
{3, 5}	36, 9

those cases where $M(W, K^S)$ has relatively low dimension, but in each case we include we do give the complete decomposition into irreducibles. We now give a guide to reading these tables.

Each row in the table corresponds to an irreducible Hecke-submodule N of a space of forms $M(W, K^S)$. The first two columns specify the level K^S and the weight W.

Let $\mathcal{H}_{\mathrm{I}} = \otimes_{p\in S}\mathcal{H}_{I_p}$ be the tensor product of the Iwahori–Hecke algebras and $\mathcal{H}_{\mathrm{HS}} = \tilde{\otimes}_{p\notin S}\mathcal{H}_{K_p}$ be the restricted tensor product of the spherical Hecke algebras. Then N is a module for $\mathcal{H}_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{HS}}$ and as such a tensor product $N_I \otimes N_{\mathrm{HS}}$ of two irreducible representations. Note that N_I is absolutely irreducible, while N_{HS} decomposes over $\overline{\mathbb{Q}}$ as a sum of characters.

The third column (labeled H_I) in the table gives the dimension of N_I. If N_I is a special representation of $\mathcal{H}_{I_p}$ for each $p \in S$ then the third column contains the entry '*St*' rather than 1.

The next columns in the table regard the spherical Hecke operators. If N_{HS} is one-dimensional then the column labeled $T_i(p)$ contains the (unique) eigenvalue of $T_i(p)$ on N_{HS}. Otherwise, some of the $T_i(p)$ have eigenvalues that aren't rational

Table IV. Irreducible Hecke submodules for $PGSp_4^{H_5}$

Level	Weight	$\mathcal{H}_I$	$T_1(3)$	$T_2(3)$	$T_1(7)$	$T_2(7)$	Lifted?
$K^{\{2,5\}}$	$\mathbb{Q}$	1	40	120	400	2800	**
		2	4	–24	52	16	$(S_4(\Gamma_0(10)), \tau)$
		4	–4	–8	–4	–96	
		4	14	16	62	96	$(S_4(\Gamma_0(5)), \tau)$
		St	1	3	10	–86	
		St	7	3	–26	58	

Table V. Irreducible Hecke submodules for G_2

Level	Weight	$\mathcal{H}_I$	$T_1(2)$	$T_2(2)$	$T_1(3)$	$T_2(3)$	$T_1(5)$	Lifted?
K^2	$\mathbb{Q}$	1	·	·	1092	88452	19530	**
	W_{27}	2	·	·	$\frac{140}{3}$	$\frac{-9884}{9}$	$\frac{4914}{5}$	$(\tau, 10)$
	W_{64}	St	·	·	-28	$\frac{1988}{9}$	$\frac{1638}{25}$	
K^3	$\mathbb{Q}$	1	126	2016	·	·	19530	**
		2	9	-90	·	·	810	$(\tau, 6)$
	W_{14}	1	-18	180	·	·	$-\frac{2214}{25}$	$(6, \tau)$
	W_{27}	2	$\frac{81}{2}$	$\frac{711}{4}$	·	·	·	$(\tau, 10)$
		3	0	-126	·	·	·	$(\tau, 10)$
		St	N_{HS} is 2-dimensional					
K^5	$\mathbb{Q}$	1	126	2016	1092	88452	·	**
		St	-3	-38	28	-196	·	
		2	33	94	100	164	·	$(\tau, 6)$
		3	-17	144	0	364	·	$(4, \tau)$
	W.7	2	$-\frac{19}{2}$	$\frac{111}{4}$	-10	$-\frac{143}{3}$	·	
		3	3	-16	$-\frac{80}{3}$	$-\frac{4004}{9}$	·	(4,6)
		1	6	-104	52	-988	·	$(\tau, 8)$
		St	N_{HS} is 7-dimensional					
	W_{14}	1	-2	24	$-\frac{380}{9}$	$\frac{28748}{27}$	·	$(6, \tau)$
		3	$-\frac{33}{4}$	$-\frac{133}{8}$	$\frac{65}{3}$	$-\frac{1850}{9}$	·	
		3	21	11	$-\frac{104}{3}$	$-\frac{572}{9}$	·	(4,8)
		4	N_{HS} is 2-dimensional			(4,8)		
		St	N_{HS} is 2-dimensional					
		St	N_{HS} is 9-dimensional					
K–7	$\mathbb{Q}$	1	126	2016	1092	88452	19530	**
		3	-14	126	-48	1212	610	$(4, \tau)$
		3	-3	-134	60	-816	438	$(\tau, 6)$
		6	-3	-6	-4	-240	-138	
		2	N_{HS} is 2-dimensional					$(\tau, 6)$
		St	N_{HS} is 2-dimensional					
		St	N_{HS} is 10-dimensional					
K^{11}	$\mathbb{Q}$	The full space of forms here is 187-dimensional. We will only comment here that there is a form in the Steinberg space here that seems to be lifted from $S_2(\Gamma_0(11))$ via the principal SL_2. We have only checked that the Satake parameter at 2 is what the lift predicts.						

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Table VI. Irreducible Hecke submodules for $\mathrm{PGSp}_4^{H_2}$

Level	Weight	$\mathcal{H}_\mathrm{I}$	$T_1(3)$	$T_2(3)$	$T_1(5)$	$T_2(5)$	$T_1(7)$	$T_2(7)$	$T_1(11)$	$T_2(11)$	Lifted?
$K^{\{2\}}$	$\mathbb{Q}$	1	40	120	156	780	400	2800	1464	16104	**
$K^{\{2,3\}}$	$\mathbb{Q}$	1	·	·	156	780	400	2800	1464	16104	**
		2	·	·	36	60	40	−80	144	264	$(S_4(\Gamma_0(6)), \tau)$
	W_5	St	·	·	$\frac{-84}{5}$	$\frac{156}{5}$	$\frac{-80}{7}$	−80	24	$\frac{-1416}{11}$	
		2	·	·	$\frac{156}{5}$	$\frac{156}{5}$	$\frac{352}{7}$	$\frac{16}{7}$	$\frac{888}{11}$	$\frac{-5448}{11}$	$(S_6(\Gamma_0(3)), \tau)$
$K^{\{2,5\}}$	$\mathbb{Q}$	1	40	120	·	·	400	2800	1464	16104	**
		2	4	−24	·	·	52	16	144	264	$(S_4(\Gamma_0(10)), \tau)$
		2	−4	−8	·	·	−4	−96	−16	−120	
		4	14	16	·	·	62	96	164	504	$(S_4(\Gamma_0(5)), \tau)$
		St	1	3	·	·	10	−86	−18	−42	
		St	7	3	·	·	−26	58	6	−186	
$K^{\{2,7\}}$	$\mathbb{Q}$	1	40	120	156	780	·	·	1464	16104	**
		2	0	−20	−4	−40	·	·	−16	−136	
		2	20	40	16	−60	·	·	104	−216	$(S_4(\Gamma_0(14)), \tau)$
		2	10	0	18	−48	·	·	180	696	$(S_4(\Gamma_0(14)), \tau)$
		4	10	0	46	120	·	·	124	24	$(S_4(\Gamma_0(7)), \tau)$
		4	−8	12	16	24	·	·	−8	120	
		4	−8	12	−12	24	·	·	48	120	
		St	0	4	−16	8	·	·	−64	104	
		St	0	0	16	20	·	·	24	−136	
		St	−8	6	−6	−12	·	·	−30	84	
		St	4	−12	0	24	·	·	0	120	$S_2(\Gamma_0(14))$, princ.
		St	0	−26	−10	20	·	·	50	20	
		2	N_{HS} is 2-dimensional								

and we simply record the dimension of N_{HS}. In some cases the characteristic polynomials of the $T_i(p)$ on N_{HS} can be found on the second author's web page.

Finally, the last column regards apparent lifting of forms. Note that in no cases have we *proven* that a form is lifted from a smaller group. Rather we have located forms that *appear* to be lifts, in that the Satake parameters we have computed agree with ones predicted by a lift. In the lifting column of the PGSp_4 tables there are three types of entries. An ordered pair $(S_k(\Gamma_0(N)), \tau)$ indicates that the forms in the corresponding irreducible Hecke submodule appear to be lifted, via the $\mathrm{SL}_2 \times \mathrm{SL}_2$ embedding discussed in Section 4.3, from a classical cusp form of weight k and level N together with the parameter, τ, of the trivial representation of a compact form of PGL_2. Next, we mark the forms corresponding to the trivial automorphic representation with '**'. These forms are lifts in two ways. Namely they arise from the pair (E_4, τ) via the $\mathrm{SL}_2 \times \mathrm{SL}_2$ lift and from τ via the principal lift, where E_4 is the classical Eisenstein series of weight 4. Finally one entry indicates that the forms are lifted via the principal SL_2.

In the G_2 tables (Table V) we use similar notation. Recall that the $\mathrm{SL}_2 \times \mathrm{SL}_2$ embedding has the short root in the first coordinate and the long root in the second. We record only the weight of the cusp forms being lifted; in each case the level

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is the prime appearing in the level of the forms on G_2. Again we mark the forms corresponding to the trivial representation with '**'. Here these forms are lifts in three ways: from (E_4, τ) and (τ, E_6) under the $SL_2 \times SL_2$ lift and from τ under the principal lift.

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