ON THE REPRESENTATION OF FUNCTIONS AS FOURIER TRANSFORMS

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If $f \in L_p(-\infty, \infty)$, $1 < p \leqslant 2$, then f has a Fourier-Plancherel transform $F \in L_q(-\infty, \infty)$ where $p^{-1} + q^{-1} = 1$. Also if $|x|^{1-2/q} f(x) \in L_q(-\infty, \infty)$, $q \geqslant 2$, then f has a Fourier-Plancherel transform $F \in L_q(-\infty, \infty)$. These results can be found in (**2**, Theorems 74 and 79). In neither case, however, does the collection of transforms cover L_q, except when $p = q = 2$, and in neither case, with the same exception, has the collection of transforms been characterized.

Further, if $f \in L_p(-\infty, \infty)$, $1 < p \leqslant 2$, then its transform F has the property $|x|^{1-2/p} F(x) \in L_p(-\infty, \infty)$ (see **2**, Theorem 80) but, except when $p = 2$, the collection of transforms does not cover the set of functions with this property, and again, except when $p = 2$, the collection of transforms has not been characterized.

Our object here is to find such characterizations, and this is done for the various cases in Theorems 1, 2, and 3 below. This characterization is given in terms of an operator

$$\mathfrak{F}_{k,t}[F] = \frac{(-ik/t)^{k+1}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{(x - ik/t)^{k+1}} F(x)dx, \qquad k = 1, 2, \ldots.$$

It transpires that this operator is an inversion operator for the Fourier transform, and its inversion theory will be the subject of another paper.

THEOREM 1. *A necessary and sufficient condition that a function* $F \in L_q(-\infty, \infty)$, $q \geqslant 2$, *be the Fourier transform of a function in* $L_p(-\infty, \infty)$, *with* $p^{-1} + q^{-1} = 1$, *is that there exist a constant* M *such that*

$$\int_{-\infty}^{\infty} |\mathfrak{F}_{k,t}[F]|^p dt \leqslant M, \qquad k = 1, 2, \ldots.$$

Proof of necessity. Suppose F is the Fourier transform of $f \in L_p(-\infty, \infty)$. Now an easy calculation shows that for $k = 1, 2, \ldots$,

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{(x - ik/t)^{k+1}} e^{-ixy} dx = \begin{cases} -(2\pi)^{\frac{1}{2}}(-i)^{k+1} y^k e^{ky/t}/k!, & y < 0, t > 0, \\ (2\pi)^{\frac{1}{2}}(-i)^{k+1} y^k e^{ky/t}/k!, & y > 0, t < 0, \\ 0 \quad , & yt > 0. \end{cases}$$

Hence, since for each $t \neq 0$ and each $k = 1, 2, \ldots$, $(x - ik/t)^{-(k+1)} \in L_p(-\infty, \infty)$, we have from (**2**, Theorem 75) that

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$$\mathfrak{F}_{k,t}[F] = \frac{(-ik/t)^{k+1}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{(x-ik/t)^{k+1}} F(x)dx$$

$$= \begin{cases} (k/t)^{k+1}(k!)^{-1} \int_0^{\infty} e^{-ky/t} y^k f(y)dy, & t > 0 \\ (k/|t|)^{k+1}(k!)^{-1} \int_{-\infty}^{0} e^{-ky/t} |y|^k f(y)dy, & t < 0. \end{cases}$$

Thus, using Hölder's inequality, we have for $t > 0$

$$|\mathfrak{F}_{k,t}[F]| \leqslant (k/t)^{k+1}(k!)^{-1} \left\{\int_0^{\infty} e^{-ky/t} y^k |f(y)|^p dy\right\}^{1/p} \left\{\int_0^{\infty} e^{-ky/t} y^k dy\right\}^{1/q}$$

$$= \left\{(k/t)^{k+1}(k!)^{-1} \int_0^{\infty} e^{-ky/t} |f(y)|^p dy\right\}^{1/p},$$

and consequently,

$$\int_0^{\infty} |\mathfrak{F}_{k,t}[F]|^p dt \leqslant \frac{k^{k+1}}{k!} \int_0^{\infty} t^{-(k+1)}\, dt \int_0^{\infty} e^{-ky/t} y^k |f(y)|^p dy$$

$$= \frac{k^{k+1}}{k!} \int_0^{\infty} y^k |f(y)|^p dy \int_0^{\infty} t^{-(k+1)} e^{-ky/t} dt = \int_0^{\infty} |f(y)|^p dy.$$

A similar calculation for $t < 0$ shows that

$$\int_{-\infty}^{0} |\mathfrak{F}_{k,t}[F]|^p dt \leqslant \int_{-\infty}^{0} |f(y)|^p dy,$$

and hence

$$\int_{-\infty}^{\infty} |\mathfrak{F}_{k,t}[F]|^p dt \leqslant \int_{-\infty}^{\infty} |f(y)|^p dy = M.$$

Proof of sufficiency. For $s > 0$ let

$$g_+(s) = -(2\pi)^{-\frac{1}{2}} i \int_{-\infty}^{\infty} \frac{1}{x - is} F(x)dx,$$

and

$$g_-(s) = (2\pi)^{-\frac{1}{2}} i \int_{-\infty}^{\infty} \frac{1}{x + is} F(x)dx,$$

and denote by $L_{k,t}$ the Widder-Post inversion operator for the Laplace transformation; that is

$$L_{k,t}[g] = (-1)^k (k/t)^{k+1} g^{(k)}(k/t)/k!, \qquad k = 1, 2, \ldots .$$

Now if $s \geqslant \delta > 0$, and $k = 1, 2, \ldots$, then

$$|(x \pm is)^{-(k+1)} F(x)| \leqslant (x^2 + \delta^2)^{-(k+1)/2} |F(x)| \in L_1(-\infty, \infty),$$

since from Hölder's inequality

$$\int_{-\infty}^{\infty} (x^2 + \delta^2)^{-(k+1)/2} |F(x)| dx$$

$$\leqslant \left\{\int_{-\infty}^{\infty} (x^2 + \delta^2)^{-p(k+1)/2} dx\right\}^{1/p} \cdot \left\{\int_{-\infty}^{\infty} |F(x)|^q dx\right\}^{1/q} < \infty.$$

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Hence by (**1**, Corollary 39.2), $g_\pm(s)$ has derivatives of all orders in $0 < s < \infty$, and these derivatives can be calculated by differentiating under the integral sign. Thus for $t > 0$,

$$L_{k,t}[g_+] = \frac{(-ik/t)^{k+1}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{(x - ik/t)^{k+1}} F(x)dx = \mathfrak{F}_{k,t}[F],$$

and

$$L_{k,t}[g_-] = \frac{(ik/t)^{k+1}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{(x + ik/t)^{k+1}} F(x)dx = \mathfrak{F}_{k,-t}[F],$$

so that

$$\int_0^\infty |L_{k,t}[g_+]|^p dt = \int_0^\infty |\mathfrak{F}_{k,t}[F]|^p dt \leqslant M, \qquad k = 1, 2, \ldots,$$

and

$$\int_0^\infty |L_{k,t}[g_-]|^p dt = \int_0^\infty |\mathfrak{F}_{k,-t}[F]|^p dt \leqslant M, \qquad k = 1, 2, \ldots.$$

Further $g_\pm(s) \to 0$ as $s \to \infty$. For from Hölder's inequality we have

$$|g_\pm(s)| \leqslant (2\pi)^{-\frac{1}{2}} \left\{\int_{-\infty}^{\infty} (x^2 + s^2)^{-p/2} dx\right\}^{1/p} \left\{\int_{-\infty}^{\infty} |F(x)|^q dx\right\}^{1/q} = 0(s^{-1/q}).$$

Hence by (**3**, Chapter 7, Theorem 15a) there are functions f_+ and f_- in $L_p(0, \infty)$ such that

$$g_+(s) = \int_0^\infty e^{-st} f_+(t)dt, \qquad s > 0,$$

and

$$g_-(s) = \int_0^\infty e^{-st} f_-(t)dt, \qquad s > 0.$$

Let

$$f(t) = \begin{cases} f_+(t), & t > 0, \\ f_-(-t), & t < 0. \end{cases}$$

Then clearly $f \in L_p(-\infty, \infty)$ and hence by (**2**, Theorem 74) f has a Fourier transform $F^* \in L_q(-\infty, \infty)$. We now show $F = F^*$ a.e.

Let

$$g_+^*(s) = -(2\pi)^{-\frac{1}{2}} i \int_{-\infty}^{\infty} \frac{1}{x - is} F^*(x)dx, \qquad s > 0,$$

and

$$g_-^*(s) = (2\pi)^{-\frac{1}{2}} i \int_{-\infty}^{\infty} \frac{1}{x + is} F^*(x)dx, \qquad s > 0.$$

Then since for each $s > 0$, $(x - is)^{-1} \in L_p(-\infty, \infty)$, and

$$(2\pi)^{-\frac{1}{2}}(P) \int_{-\infty}^{\infty} \frac{1}{(x - is)} e^{-ixy} dx = \begin{cases} (2\pi)^{\frac{1}{2}}\, i\, e^{sy}, & y < 0, s > 0, \\ -(2\pi)^{\frac{1}{2}}\, i\, e^{sy}, & y > 0, s < 0, \\ 0, & sy > 0, \end{cases}$$

we have from (2, Theorem 75) for $s > 0$,

$$\begin{aligned} g_+^*(s) &= -(2\pi)^{-\frac{1}{2}}\, i \int_{-\infty}^{\infty} \frac{1}{x - is} F^*(x)dx \\ &= \int_0^{\infty} e^{-sy} f(y)dy = \int_0^{\infty} e^{-sy} f_+(y) = g_+(s), \end{aligned}$$

and

$$\begin{aligned} g_-^*(s) &= (2\pi)^{-\frac{1}{2}}\, i \int_{-\infty}^{\infty} \frac{1}{x + is} F^*(x)dx \\ &= \int_{-\infty}^{0} e^{sy} f(y)dy = \int_0^{\infty} e^{-sy} f_-(y)dy = g_-(s). \end{aligned}$$

Consequently, for $s > 0$

$$\int_{-\infty}^{\infty} \frac{1}{x - is} (F(x) - F^*(x))dx = 0$$

and

$$\int_{-\infty}^{\infty} \frac{1}{x + is} (F(x) - F^*(x))dx = 0.$$

Letting $\phi(x) = F(x) - F^*(x)$, the last two equations yield

$$\int_{-\infty}^{\infty} \frac{1}{x + is} \phi(x)dx = 0, \qquad s \neq 0.$$

Then denoting the even and odd parts of ϕ by ϕ_e and ϕ_0 respectively, we have for $s \neq 0$

$$\int_{-\infty}^{\infty} \frac{1}{x + is} \phi_e(x)dx = -\int_{-\infty}^{\infty} \frac{1}{x + is} \phi_0(x)dx.$$

But the function on the left of this equation is an odd function of s while the function on the right is even. Hence each is zero, so that for $s \neq 0$

$$\int_0^{\infty} \frac{1}{x^2 + s^2} \phi_e(x)dx = -\frac{1}{2is} \int_{-\infty}^{\infty} \frac{1}{x + is} \phi_e(x)dx = 0,$$

and

$$\int_0^{\infty} \frac{x}{x^2 + s^2} \phi_0(x)dx = -\tfrac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x + is} \phi_0(x)dx = 0.$$

Thus for each $s > 0$,

$$\int_0^{\infty} \frac{1}{x + s} x^{-\frac{1}{2}} \phi_e(x^{\frac{1}{2}})dx = 2\int_0^{\infty} \frac{1}{x^2 + s} \phi_e(x)dx = 0,$$

and

$$\int_0^{\infty} \frac{1}{x + s} \phi_0(x^{\frac{1}{2}})dx = 2\int_0^{\infty} \frac{x}{x^2 + s} \phi_0(x)dx = 0,$$

and hence by the uniqueness theorem for the Stieltjes transformation (3, chapter 8, Theorem 5a) ϕ_e and ϕ_0 are zero almost everywhere. Thus ϕ is zero almost everywhere so that $F = F^*$ almost everywhere, and F has the prescribed representation.

For Theorems 2 and 3 let us denote by $\mathscr{L}_r(-\infty, \infty)$ the collection of functions f such that $|x|^{1-2/r} f(x) \in L_r(-\infty, \infty)$.

THEOREM 2. *A necessary and sufficient condition that a function $F \in L_q(-\infty, \infty)$, $q \geqslant 2$, be the Fourier transform of a function in $\mathscr{L}_q(-\infty, \infty)$, $q \geqslant 2$, is that there exist a constant M such that*

$$\int_{-\infty}^{\infty} |t|^{q-2} |\mathfrak{F}_{k,t}[F]|^q dt \leqslant M, \qquad k > q - 2.$$

Proof of necessity. Suppose F is the Fourier transform of $f \in \mathscr{L}_q(-\infty, \infty)$. Then as in the proof of Theorem 1, for $t > 0$ and $k > q - 2$

$$|\mathfrak{F}_{k,t}[F]| \leqslant \left\{(k/t)^{k+1}(k!)^{-1} \int_0^{\infty} e^{-ky/t} y^k |f(y)|^q dy\right\}^{1/q}$$

and consequently if $k > q - 2$

$$\int_0^{\infty} t^{q-2} |\mathfrak{F}_{k,t}[F]|^q dt \leqslant \frac{k^{k+1}}{k!} \int_0^{\infty} t^{q-k-3} dt \int_0^{\infty} e^{-ky/t} y^k |f(y)|^q dy$$
$$= \frac{k^{k+1}}{k!} \int_0^{\infty} y^k |f(y)|^q dy \int_0^{\infty} e^{-ky/t}\, t^{q-k-3} dt$$
$$= K(k) \int_0^{\infty} y^{q-2} |f(y)|^q dy,$$

where $K(k) = k^{q-1}\,\Gamma(k - q + 2)/k!$ Similarly

$$\int_{-\infty}^{0} |t|^{q-2} |\mathfrak{F}_{k,t}[F]|^q dt \leqslant K(k) \int_{-\infty}^{0} |y|^{q-2} |f(y)|^q dy,$$

so that

$$\int_{-\infty}^{\infty} |t|^{q-2} |\mathfrak{F}_{k,t}[F]|^q dt \leqslant K(k) \int_{-\infty}^{\infty} |y|^{q-2} |f(y)|^q dy.$$

But from Stirling's formula,

$$\lim_{k\to\infty} K(k) = 1,$$

so that $K(k)$ is bounded for $k > q - 2$. Hence there is an M such that

$$\int_{-\infty}^{\infty} |t|^{q-2} |\mathfrak{F}_{k,t}[F]|^q dt \leqslant M, \qquad k > q - 2.$$

Proof of sufficiency. Let g_+ and g_- be defined as in the proof of Theorem 1. Then as in that proof, for $t > 0$

$$L_{k,t}[g_+] = \mathfrak{F}_{k,t}[F],$$

and

$$L_{k,t}[g_-] = \mathfrak{F}_{k,-t}[F],$$

and hence

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$$\int_0^\infty t^{q-2}|L_{k,t}[g_+]|^q dt = \int_0^\infty t^{q-2}|\mathfrak{F}_{k,t}[F]|^q dt \leqslant M, \qquad k > q-2$$

and

$$\int_0^\infty t^{q-2}|L_{k,t}[g_-]|^q dt = \int_0^\infty t^{q-2}|\mathfrak{F}_{k,t}[F]|^q dt \leqslant M, \qquad k > q-2.$$

Consider first g_+. By (**3**, chapter 1, Theorem 17a), with $\alpha_k(t) = t^{1-2/q}L_{k,t}[g_+]$, there is a function f_+ with $t^{1-2/q}f_+(t) \in L_q\ (0, \infty)$, and an increasing unbounded sequence of integers $\{k_i\}$ such that for any function $\beta(t) \in L_p\ (0, \infty)$,

$$\lim_{i\to\infty} \int_0^\infty \beta(t)\, t^{1-2/q}\, L_{k_i,t}[g_+]dt = \int_0^\infty \beta(t)\, t^{1-2/q} f_+(t)dt.$$

But for each $s > 0$, $t^{-(1-2/q)}\, e^{-st} \in L_p\ (0, \infty)$, and hence choosing this as our $\beta(t)$ we have for $s > 0$

$$\lim_{i\to\infty} \int_0^\infty e^{-st}\, L_{k_i,t}[g_+]dt = \int_0^\infty e^{-st} f_+(t)dt.$$

However, for $x > 0$,

$$\int_0^x |L_{k,t}[g_+]|dt \leqslant \left\{\int_0^x t^{p-2}dt\right\}^{1/p}\left\{\int_0^x t^{q-2}\, |L_{k,t}[g_+]|^q dt\right\}^{1/q}$$
$$\leqslant (p-1)^{-1/p}\, M\, x^{1/q} = O(x) \qquad \text{as} \qquad x \to \infty,$$

and as in the proof of Theorem 1, $g_+(s) \to 0$ as $s \to \infty$. Hence by (**3**, chapter 7, Theorem 11b),

$$\lim_{i\to\infty} \int_0^\infty e^{-st}\, L_{k_i,t}[g_+]dt = g_+(s), \qquad s > 0,$$

and thus

$$g_+(s) = \int_0^\infty e^{-st} f_+(t)dt, \qquad s > 0.$$

Similarly f_- exists with $t^{1-2/q}f_-(t) \in L_q\ (0, \infty)$ such that

$$g_-(s) = \int_0^\infty e^{-st} f_-(t)dt, \qquad s > 0.$$

Let

$$f(t) = \begin{cases} f_+(t), & t > 0, \\ f_-(-t), & t < 0. \end{cases}$$

Then clearly $f \in \mathscr{L}_q\ (-\infty, \infty)$, and hence by (**2**, Theorem 79) f has a Fourier transform $F^* \in L_q\ (-\infty, \infty)$. It remains to show $F = F^*$ a.e., which now follows as in Theorem 1.

THEOREM 3. *A necessary and sufficient condition that a function* $F \in \mathscr{L}_p\ (-\infty, \infty)$, $1 < p \leqslant 2$, *be the Fourier transform of a function in* $L_p\ (-\infty, \infty)$ *is that there exist a constant* M *such that*

$$\int_{-\infty}^\infty |\mathfrak{F}_{k,t}[F]|^p dt \leqslant M, \qquad k = 1, 2, \ldots.$$

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Proof of necessity. If $F \in \mathscr{L}_p(-\infty, \infty)$ is the Fourier transform of $f \in L_p(-\infty, \infty)$ then by (**2**, Theorem 74), $F \in L_q(-\infty, \infty)$, and hence by Theorem 1, there is a constant M so that

$$\int_{-\infty}^{\infty} |\mathfrak{F}_{k,t}[F]|^p dt \leqslant M, \qquad k = 1, 2, \ldots .$$

Proof of sufficiency. Let $g_+(s)$ and $g_-(s)$ be defined as in Theorem 1. Then as in that theorem,

$$\int_0^{\infty} |L_{k,t}[g_+(s)]|^p dt \leqslant M, \qquad k = 1, 2, \ldots ,$$

and

$$\int_0^{\infty} |L_{k,t}[g_-(s)]|^p dt \leqslant M, \qquad k = 1, 2, \ldots .$$

Further $g_{\pm}(s) \to 0$ as $s \to \infty$. For from Hölder's inequality we have for $s > 0$

$$|g_{\pm}(s)| \leqslant \left\{\int_{-\infty}^{\infty} \frac{|x|^{q-2}}{(x^2+s^2)^{q/2}} dx\right\}^{1/q} \left\{\int_{-\infty}^{\infty} |x|^{p-2} |F(x)|^p dx\right\}^{1/p} = 0(s^{-1/q}).$$

Hence by (**3**, chapter 7, Theorem 15a), there are functions f_+ and f_- in $L_p(0, \infty)$ such that

$$g_+(s) = \int_0^{\infty} e^{-st} f_+(t) dt, \qquad s > 0$$

and

$$g_-(s) = \int_0^{\infty} e^{-st} f_-(t) dt, \qquad s > 0.$$

Let

$$f(t) = \begin{cases} f_+(t), & t > 0, \\ f_-(-t), & t < 0. \end{cases}$$

Then clearly $f \in L_p(-\infty, \infty)$ and hence by (**2**, Theorems 75 and 80) f has a Fourier transform $F^* \in \mathscr{L}_p(-\infty, \infty)$. It remains to show that $F = F^*$ a.e., and this follows as in Theorem 1.

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