13

The baryonic spectrum of multiflavor QCD_2 in the strong coupling limit

We are now going to compute the baryonic spectrum of QCD_2, for which as it turns out the bosonic formulation is very convenient. The mesonic spectrum was found earlier, using large N with quark fields as variables in Chapter 10, as well as using currents as building blocks in Section 11.3. For the baryon spectrum, however, the large N limit, in terms of fermionic fields, is not the natural framework to use since in such a picture the baryon is a bound state of a large number N of constituents. Instead, it will be shown in this chapter that the bosonized version of QCD_2 in the strong coupling limit provides an effective description of the baryons.[1] We will start by deriving the effective action at the strong coupling limit. It will be argued that for the purpose of extracting the low-lying baryons, one can in fact use the product scheme instead of the $U(N_c \times N_c)$ scheme, with the former being more suitable for our purposes. Once the effective action is written down we will search for soliton solutions that carry a baryon number. It will be shown that for a static configuration the effective action reduces to a sum of sine-Gordon actions. Using the knowledge acquired on solitons, in Chapter 5, it will be easy to write down the classical baryonic configuration. We will then semi-classically quantize these solitons. This problem will be mapped into a quantum mechanical model on a $CP^{(N_f-1)}$ manifold. The energy and charges of the quantized soliton can be derived and thus the spectrum of the baryons is determined. We then analyze the quark flavor content of the baryons and discuss multi-baryon states. Finally, we include meson-baryon scattering, this time also for the case of any coupling.

13.1 The strong coupling limit

It turns out that the mass term plays an essential role in the determination of classical soliton solutions in 1+1 space-time dimensions. It is therefore required to switch on this term before deducing the low energy effective action. As was explained in Chapter 6, we know how to do this rigorously only in the scheme of $U(N_{\text{F}} N_{\text{C}})$. It will turn out, however, that the product scheme can be used for the low mass states in the strong coupling limit.

[1] The spectrum of baryons of two-dimensional QCD extracted in the strong coupling limit was derived in [75].

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Our starting point is the last equation of Chapter 9. In the strong coupling limit $\frac{e_c}{m_q} \to \infty$, the fields in $\tilde{h}$ which contribute to $\tilde{H}$ will become infinitely heavy. The sector $\tilde{g}l \subset \frac{SU(N_F N_C)}{SU(N_C)}$, however, will not acquire mass from the gauge interaction term. Since we are interested only in the light particles we can, in the strong coupling limit, ignore the heavy fields, if we first normal order the heavy fields at the mass scale $\tilde{\mu} = \frac{e_c\sqrt{N_F}}{\sqrt{2\pi}}$. Using the relation, for a given operator O,

$$\left(\frac{\tilde{\mu}}{\tilde{m}}\right)^{\Delta} N_{\tilde{\mu}} O = N_{\tilde{m}} O, \tag{13.1}$$

to perform the change in the scale of normal ordering, and then substituting $h^a_b = \delta^a_b$, we get for the low energy effective action,

$$\begin{aligned} S_{\text{eff}}[u] &= S[\tilde{g}] + S[l] + \frac{1}{2}\int \mathrm{d}^2x \partial_\mu\phi\partial^\mu\phi \\ &+ cm_q\tilde{\mu}N_{\tilde{\mu}}\int \mathrm{d}^2x \mathrm{Tr}(e^{-i\sqrt{\frac{4\pi}{N_C N_F}}\phi}\tilde{g}l + \mathrm{e}^{+i\sqrt{\frac{4\pi}{N_C N_F}}\phi}l^\dagger\tilde{g}^\dagger). \end{aligned} \tag{13.2}$$

We can now replace the two mass scales m_q and $\tilde{\mu}$ by a single scale, by normal ordering at a certain m so the final form of the effective action becomes,

$$\begin{aligned} S_{\text{eff}}[u] &= S[\tilde{g}] + S[l] + \frac{1}{2}\int \mathrm{d}^2x \partial_\mu\phi\partial^\mu\phi \\ &+ \frac{m^2}{N_C}N_m\int \mathrm{d}^2x \mathrm{Tr}(\mathrm{e}^{-i\sqrt{\frac{4\pi}{N_C N_F}}\phi}\tilde{g}l + \mathrm{e}^{+i\sqrt{\frac{4\pi}{N_C N_F}}\phi}\tilde{l}^\dagger g^\dagger), \end{aligned} \tag{13.3}$$

with m given by,

$$m = \left[N_C c m_q \left(\frac{e_c\sqrt{N_F}}{\sqrt{2\pi}}\right)^{\Delta_C}\right]^{\frac{1}{1+\Delta_C}}, \tag{13.4}$$

here Δ_C, the dimension of $\tilde{h}$, is $\frac{N_C^2-1}{N_C(N_C+N_F)}$. For the $l=1$ sector, defining $g' = \tilde{g}\mathrm{e}^{-i\sqrt{\frac{4\pi}{N_C N_F}}\phi} \subset U(N_F)$ one gets the effective action,

$$S_{\text{eff}}[g'] = N_C S[g'] + m^2 N_m \int \mathrm{d}^2x \mathrm{Tr}_F(g' + g'^\dagger). \tag{13.5}$$

Thus, the low energy effective action in the $l=1$ sector coincides with the result of the "naive" approach of the product scheme.

In the strong coupling limit $e_c/m_q \to \infty$ the low energy effective action reads,[2]

$$S[g] = N_C S[g] + m^2 N_m \int \mathrm{d}^2x(\mathrm{Tr}g + \mathrm{Tr}g^\dagger), \tag{13.6}$$

with g in $U(N_F)$. Note that the analog of our strong coupling to the case of 3+1 space-time, would be that of light current quarks compared to the QCD scale Λ_{QCD}.

[2] From here on we omit the prime from g' so we denote $g \in U(N_F)$.

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13.2 Classical soliton solutions

We now look for static solutions of the classical action. For a static field configuration, the WZ term does not contribute. One way to see this is by noting that the variation of the WZ term can be written as,

$$\delta WZ \propto \int d^2x \varepsilon^{ij} \mathrm{Tr}(\delta g) g^\dagger (\partial_i g)(\partial_j g^\dagger), \tag{13.7}$$

and for g that has only spatial dependence $\delta WZ = 0$. Without loss of generality we may take, for the lowest energy, a diagonal $g(x)$,

$$g(x) = \left(e^{-i\sqrt{\frac{4\pi}{N_C}}\varphi_1}, \ldots, e^{-i\sqrt{\frac{4\pi}{N_C}}\varphi_{N_F}} \right). \tag{13.8}$$

For this ansatz and with a redefinition of the constant term, the action density reduces to,

$$\tilde{S}_d[g] = -\int dx \sum_{i=1}^{N_F} \left[\frac{1}{2}\left(\frac{d\varphi_i}{dx}\right)^2 - 2m^2 \left(\cos\sqrt{\frac{4\pi}{N_C}}\varphi_i - 1 \right) \right]. \tag{13.9}$$

This is a sum of decoupled standard sine-Gordon actions for each φ_i. The well-known solutions of the associated equations of motion are,

$$\varphi_i(x) = \sqrt{\frac{4N_C}{\pi}} \mathrm{arctg}\left[e^{(\sqrt{\frac{8\pi}{N_C}} m x)} \right], \tag{13.10}$$

with the corresponding classical energy,

$$E_i = 4m\sqrt{\frac{2N_C}{\pi}}, \quad i = 1, \ldots, N_F. \tag{13.11}$$

Clearly the minimum energy configuration for this class is when only one of the φ_i is nonzero, for example,

$$g_\circ(x) = \mathrm{Diag}\left(1, 1, \ldots, e^{-i\sqrt{\frac{4\pi}{N_C}}\varphi(x)}\right) \tag{13.12}$$

Conserved charges, corresponding to the vector current, can be computed using the definition,

$$Q^A[g(x)] = \frac{1}{2}\int dx \mathrm{Tr}(J_0 T^A), \tag{13.13}$$

where $\frac{1}{2}T^A$ are the $SU(N_F)$ generators and the $U(1)$ baryon number is generated by the unit matrix. This follows from $J_\mu = J^A_\mu T^A$, and in the fermionic basis $J^A_\mu = \bar{\psi}\gamma_\mu \frac{1}{2} T^A \psi$.

In particular, for eqn. (13.10), we get charges different from zero only for Q_B and Q_Y corresponding to baryon number and "hypercharge", respectively,

$$Q^\circ_B = N_C, \quad Q^\circ_Y = -\frac{1}{2}\sqrt{\frac{2(N_F-1)}{N_F}} N_C, \tag{13.14}$$

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these charges are determined solely by the boundary values of $\varphi(x)$, which are,

$$\sqrt{\frac{4\pi}{N_{\rm C}}}\varphi(\infty) = 2\pi, \quad \sqrt{\frac{4\pi}{N_{\rm C}}}\varphi(-\infty) = 0. \tag{13.15}$$

Under a general $\mathrm{U}_V(N_{\rm F})$ global transformation $g_\circ(x) \to \tilde{g}_\circ(x) = Ag_\circ(x)A^{-1}$ the energy of the soliton is obviously unchanged, but charges other than Q_B and Q_Y will be turned on. Let us introduce a parametrization of A that will be useful later,

$$A = \begin{pmatrix} & & & & z_1 \\ & & A_{ij} & & \vdots \\ & & & & \vdots \\ & & & & z_{(N_{\rm F}-1)} \\ Y_1 & \dots & \dots & Y_{(N_{\rm F}-1)} & z_{N_{\rm F}} \end{pmatrix}. \tag{13.16}$$

Now,

$$\tilde{g}_\circ = 1 + (\mathrm{e}^{-i\sqrt{\frac{4\pi}{N_{\rm C}}}\varphi} - 1)\mathrm{z}, \tag{13.17}$$

where $(\mathrm{z})_{\alpha\beta} = z_\alpha z^*_\beta$, and from unitarity $\sum_{\alpha=i}^{N_{\rm F}} z_\alpha z^*_\alpha = 1$. The charges with $\tilde{g}_\circ(x)$ are,

$$(\tilde{Q}^\circ)^A = \frac{1}{2}N_{\rm C}\mathrm{Tr}(T^A\mathrm{z}). \tag{13.18}$$

Only the baryon number is unchanged. The discussion of the possible $U(N_{\rm F})$ representations cannot be done yet, since we are dealing so far with a classical system. We will return to the question of possible representations after quantizing the system.

13.3 Semi-classical quantization and the baryons

The next step in the semi-classical analysis is to consider configurations of the form,

$$g(x,t) = A(t)g_\circ(x)A^{-1}(t), \quad A(t) \in U(N_{\rm F}), \tag{13.19}$$

and to derive the effective action for A(t).[3] Quantization of this action corresponds to doing the functional integral over $\mathrm{g}(x,t)$ of the above form. The effective action for A(t) is derived by substituting $g(x,t) = A(t)g_\circ(x)A^{-1}(t)$ in the original action. Here we use the following property of the WZ action,

$$S\left[AgB^{-1}\right] = S\left[AB^{-1}\right] + S\left[g, \tilde{A}_\mu\right], \tag{13.20}$$

[3] The semi-classical quantization makes use of the Polyakov–Wiegmann formula [179].

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where $S[g]$ is the WZW action and $S[g,\tilde{A}]$ is given by (9.25), respectively, with the gauge field $\tilde{A}_\mu$ given as,

$$i\tilde{A}_+ = A^{-1}\partial_+ A, \quad i\tilde{A}_- = B^{-1}\partial_- B; \quad A, B \in U(N_\mathrm{F}). \tag{13.21}$$

Using the above formula for A = B, noting that S(1) = 0, and taking $A = A(t)$,

$$\partial_+ A = \partial_- A = \frac{\dot{A}}{\sqrt{2}}, \tag{13.22}$$

we get,

$$\begin{aligned}\tilde{S}\left[A(t)g_\circ(x)A^{-1}(t)\right] - \tilde{S}[g_\circ] &= \frac{N_\mathrm{C}}{8\pi}\int \mathrm{d}^2x\mathrm{Tr}\left\{[A^{-1}\dot{A}, g_\circ][A^{-1}\dot{A}, g_\circ^\dagger]\right\}\\ &+ \frac{N_\mathrm{C}}{2\pi}\int \mathrm{d}^2x\mathrm{Tr}\left\{(A^{-1}\dot{A})(g_\circ^\dagger\partial_1 g_\circ)\right\}.\end{aligned} \tag{13.23}$$

This action is invariant under global $U(N_\mathrm{F})$ transformations $A \to UA$, where $U \in G = U(N_\mathrm{F})$. This corresponds to the invariance of the original action under $g \to UgU^{-1}$. On top of this it is also invariant under the local changes $A(t) \to A(t)V(t)$, where $V(t) \in H = SU(N_\mathrm{F}-1) \times U_B(1) \times U_Y(1)$, with the last two $U(1)$ factors corresponding to baryon number and hypercharge, respectively. This subgroup H of G is nothing but the invariance group of $g_\circ(x)$. In terms of $g_\circ(x)$ and $A(t)$ the charges associated with the global $U(N_\mathrm{F})$ symmetry, eqn. (13.13), have the form,

$$Q^B = i\frac{N_\mathrm{C}}{8\pi}\int \mathrm{d}x\mathrm{Tr}\left\{T^B A\left(\left(g_\circ^\dagger\partial_1 g_\circ - g_\circ\partial_1 g_\circ^\dagger\right) + \left[g_\circ, \left[A^{-1}\dot{A}, g_\circ^\dagger\right]\right]\right)A^{-1}\right\}. \tag{13.24}$$

The effective action, eqn. (13.23), is an action for the coordinates describing the coset space,

$$\begin{aligned}G/H &= SU(N_\mathrm{F}) \times U_B(1)/SU(N_\mathrm{F}-1) \times U_Y(1) \times U_B(1)\\ &= SU(N_\mathrm{F})/SU(N_\mathrm{F}-1) \times U_Y(1) = CP^N.\end{aligned} \tag{13.25}$$

To see this explicitly we define the Lie algebra valued variables q^A through $A^{-1}\dot{A} = i\sum T^A\dot{q}^A$. In terms of these variables (13.23) takes the form (the part that depends on q^A),

$$\begin{aligned}S_q &= \int \mathrm{d}t\left[\frac{1}{2M}\sum_{A=1}^{2(N_\mathrm{F}-1)}(\dot{q}^A)^2 - N_\mathrm{C}\sqrt{\frac{2(N_\mathrm{F}-1)}{N_\mathrm{F}}}\dot{q}^Y\right]\\ \frac{1}{2M} &= \frac{N_\mathrm{C}}{2\pi}\int_{-\infty}^{\infty}(1-\cos\sqrt{\frac{4\pi}{N_\mathrm{C}}}\varphi)\mathrm{d}x = \frac{\sqrt{2}}{m}(\frac{N_\mathrm{C}}{\pi})^{3/2}.\end{aligned} \tag{13.26}$$

The sum is over those q^A which correspond to the G/H generators and q^Y is associated with the hypercharge generator. Although the q^A seem to be a "natural" choice of variables for the action (13.23), which depends only on the combination $A^{-1}\dot{A}$, they are not a convenient choice of variables. The reason for that is the explicit dependence of the charges (13.24) on $A^{-1}(t)$ and $A(t)$ as well as on $A^{-1}\dot{A}(t)$.

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Instead we found that a convenient parametrization is that of (13.16). One can rewrite the action (13.23), as well as the charges (13.24), in terms of the $z_1,\ldots,z_{N_\mathrm{F}}$ variables, which however are subject to the constraint $\sum_{\alpha=1}^{N_\mathrm{F}} z_\alpha z_\alpha^* = 1$. Thus,

$$\tilde{S}\left[A(t)g_\circ A^{-1}(t)\right] - \tilde{S}[g_\circ] = S[z_\alpha(t), \varphi(x)], \tag{13.27}$$

where,

$$\begin{aligned} S\left[z_\alpha(t), \varphi(x)\right] &= \tfrac{N_\mathrm{C}}{2\pi}\int \mathrm{d}^2x\{(1-\cos\sqrt{\tfrac{4\pi}{N_\mathrm{C}}}\varphi)[\dot{z}_\alpha^*\dot{z}_\alpha \\ &\quad -(z_\gamma^*\dot{z}_\gamma)(\dot{z}_\beta^* z_\beta)] - i\sqrt{\tfrac{4\pi}{N_\mathrm{C}}}\varphi' z_\alpha^*\dot{z}_\alpha\}. \end{aligned} \tag{13.28}$$

We can do the integral over x and rewrite (13.28) as,

$$S[z_\alpha(t)] = \frac{1}{2M}\int \mathrm{d}t[\dot{z}_\alpha^*\dot{z}_\alpha - (z_\gamma^*\dot{z}_\gamma)(\dot{z}_\beta^* z_\beta)] - i\frac{N_\mathrm{C}}{2}\int \mathrm{d}t(z_\alpha^*\dot{z}_\alpha - \dot{z}_\alpha^* z_\alpha), \tag{13.29}$$

where 1/M is defined in eqn. (13.26). The first term in (13.29) is the usual $\mathrm{CP}^{(N_\mathrm{F}-1)}$ quantum mechanical action, while the second term is a modification due to the WZ term.

Similarly we express the $U(N_\mathrm{F})$ charges in terms of the z variables, using eqn. (13.24),

$$\begin{aligned} Q^C &= \frac{1}{2}T^C_{\beta\alpha}Q_{\alpha\beta} \\ Q_{\alpha\beta} &= N_\mathrm{C} z_\alpha z_\beta^* + \frac{i}{2M}[z_\alpha z_\beta^*(z_\gamma^*\dot{z}_\gamma - \dot{z}_\gamma^* z_\gamma) + z_\alpha\dot{z}_\beta^* - z_\beta^*\dot{z}_\alpha]. \end{aligned} \tag{13.30}$$

Of course the symmetries of $S[z]$ are the global $U(N_\mathrm{F})$ group under which,

$$z_\alpha \to z'_\alpha = U_{\alpha\beta}z_\beta, \quad U \in U(N_\mathrm{F}), \tag{13.31}$$

and a local $U(1)$ subgroup of H under which,

$$z_\alpha \to z'_\alpha = \mathrm{e}^{i\delta(t)}z_\alpha. \tag{13.32}$$

As a consequence of the gauge invariance one can rewrite the action in a covariant form,

$$S[z_\alpha] = \frac{1}{2M}\int \mathrm{d}t\mathrm{Tr}(Dz)^\dagger Dz + iN_\mathrm{C}\int \mathrm{d}t\mathrm{Tr}\dot{z}^\dagger z, \tag{13.33}$$

where,

$$(Dz)_\alpha = \dot{z}_\alpha + z_\alpha(\dot{z}_\beta^\star z_\beta). \tag{13.34}$$

Constructing Noether charges of the $U(N_\mathrm{F})$ global invariance of (13.31) out of the action (13.33 leads to expressions identical with (13.30)). Note that in eqn. (13.34) we can view $\dot{z}_\beta^\star z_\beta = ia(t)$ as a composite $U(1)$ gauge potential.

Now let us count the degrees of freedom. The local $U(1)$ symmetry allows us to take one of the zs to be real, and the constraint $\sum_\alpha z_\alpha z_\alpha^* = 1$ removes one more degree of freedom, so altogether we are left with $2N_\mathrm{F} - 2 = 2(N_\mathrm{F} - 1)$

physical degrees of freedom. This is exactly the dimension of the coset space $\frac{SU(N_F)}{SU(N_F-1)\times U(1)}$. The corresponding phase space should have a real dimension of $4(N_F-1)$. Naively, however, we have a phase space of $4N_F$ dimensions and, therefore, we expect four constraints.

There are several methods of quantizing systems with constraints. Here we choose to eliminate the redundancy in the z variables and then invoke the canonical quantization procedure.[4]

But before following these lines let us briefly describe another method, through the use of Dirac's brackets. We outline the classical case. The quantum case is obtained by replacing $\{\ ,\ \}$ with $i[\ ,\ ]$.

The first step in this prescription is to add to the Lagrangian a term of the form $\lambda(\sum_\alpha z_\alpha z_\alpha^* - 1)$, in which case the conjugate momentum π_λ of the Lagrange multiplier vanishes. By requiring that this condition be preserved in time one gets the secondary constraint $\Phi_1 = (\sum_\alpha z_\alpha z_\alpha^* - 1) = 0$. Further imposing $\dot{\Phi}_1 = \{\Phi_1, H\}_P = 0$, where $\{\ \}_P$ denotes a Poisson bracket, one finds another second-class constraint $\Phi_2 = \Pi\cdot z + z^\dagger\cdot\Pi^\dagger$. In addition there is a first-class constraint $\Phi_3 = \Pi\cdot z - z^\dagger\cdot\Pi^\dagger$, which corresponds to the local $U(1)$ invariance of the model. Fixing this symmetry one gets an additional constraint Φ_4. For instance one can choose the unitary gauge $\Phi_4 = z_{N_F} - z^*_{N_F}$. The next step is to compute the constraint matrix $\{\Phi_i, \Phi_j\}_P = c_{ij}$. In the constrained theory, the brackets between F and G are replaced by the Dirac brackets of those operators, given by

$$\{F, G\}_D = \{F, G\}_P - \{F, \Phi_i\}_P (c_{ij}^{-1})\{\Phi_j, G\}_P, \tag{13.35}$$

where c_{ij}^{-1} is the inverse of the constraint matrix. Imposing the constraints as operator relations it is easy to see that z_{N_F}, Π_{N_F} and their complex conjugates can be eliminated. The brackets for the rest of the fields coincide with the results we derive below, when eliminating the constraints explicitly.

We now describe in some detail the quantization of the system using unconstrained variables. We want to choose a set of new variables so that the constraint $\sum_{\alpha=1}^{N_F} z_\alpha z_\alpha^* = 1$ is automatically fulfilled. There is a standard choice of such variables, namely (for $i = 1, \ldots, N_F - 1$),

$$z_i = \frac{k_i}{\sqrt{1+X}}, \quad z_i^* = \frac{k_i^*}{\sqrt{1+X}}, \quad z_{N_F} = \frac{e^{i\chi}}{\sqrt{1+X}}$$
$$X = \sum_{i=1}^{N_F-1} k_i^* k_i. \tag{13.36}$$

The k_i, k_i^* and χ are $2N_F - 1$ real variables with no constraints on them. The phase space will now have dimension $2(2N_F - 1)$ and we still have two extra

[4] The quantization of the system including its constraint was done in [75]. For an alternative procedure of quantization in the presence of constraints see [181].

constraints. After some straightforward algebra we can write,

$$
\begin{aligned}
S[k, k^*, \chi] &= \int \mathrm{d}t L(k, k^*, \chi) \\
L(k, k^*, \chi) &= \frac{1}{2M}\dot{k}_i^* h_{ij}\dot{k}_j - i\frac{N_\mathrm{C}}{2}\,\frac{k_i^*\dot{k}_i - \dot{k}_i^* k_i}{1+X} \\
&\quad + \frac{1}{2M}\frac{X}{(1+X)^2}\dot{\chi}^2 + \dot{\chi}\left\{\frac{i}{2M}\,\frac{k_i^*\dot{k}_i - \dot{k}_i^* k_i}{(1+X)^2} + \frac{N_\mathrm{C}}{1+X}\right\}, \qquad (13.37)
\end{aligned}
$$

where,

$$
h_{ij} = \frac{\delta_{ij}}{1+X} - \frac{k_i k_j^*}{(1+X)^2}. \qquad (13.38)
$$

The local $U(1)$ transformations of the z variables transcribe into the transformations,

$$
\delta\chi = \epsilon(t), \quad \delta k_i = i\epsilon(t)k_i, \quad \delta k_i^* = -i\epsilon(t)k_i^*, \qquad (13.39)
$$

and $\delta L = -N_\mathrm{C}\dot{\epsilon}$ just as in terms of the z variables. This local $U(1)$ symmetry can be made manifest by defining the covariant derivatives,

$$
Dk_i = \dot{k}_i - i\dot{\chi}k_i \quad Dk_i^* = \dot{k}_i^* + i\dot{\chi}k_i^*. \qquad (13.40)
$$

The Lagrangian can then be recast in a manifestly gauge-invariant form,

$$
L(k, k^*, x) = \frac{1}{2M}Dk_i^* h_{ij} Dk_j - i\frac{N_\mathrm{C}}{2}\,\frac{k_i^* Dk_i - (Dk_i^*)k_i}{1+X} + N_\mathrm{C}\dot{\chi}. \qquad (13.41)
$$

Although one can now fix the gauge, for instance $\dot{\chi} = 0$, we will continue to work with (13.41). The conjugate momenta are given by,

$$
\begin{aligned}
\pi_i &= \frac{\partial L}{\partial \dot{k}_i} = \frac{1}{2M}Dk_j^* h_{ji} - i\frac{N_\mathrm{C}}{2}\frac{k_i^*}{1+X} \\
\pi_i^* &= \frac{\partial L}{\partial \dot{k}_i^*} = \frac{1}{2M}h_{ij}\ Dk_j + i\frac{N_\mathrm{C}}{2}\frac{k_i}{1+X} \\
\pi_\chi &= \frac{\partial L}{\partial \dot{\chi}} = \frac{i}{2M}(k_i^* h_{ij} Dk_j - Dk_i^* h_{ij} k_j) + N_\mathrm{C}\frac{1}{1+X}. \qquad (13.42)
\end{aligned}
$$

Since h_{ij} is invertible we can solve for Dk_i^*, Dk_i in term of the phase space variables,

$$
\begin{aligned}
Dk_i^* &= 2M\left[\pi_j + i\frac{N_\mathrm{C}}{2}\frac{k_j^*}{1+X}\right]h_{ji}^{-1} \\
Dk_i &= 2M h_{ij}^{-1}\left[\pi_j^* - i\frac{N_\mathrm{C}}{2}\frac{k_j}{1+X}\right], \qquad (13.43)
\end{aligned}
$$

where,

$$
h_{ij}^{-1} = (1+X)(\delta_{ij} + k_i k_j^*). \qquad (13.44)
$$

Also,

$$
\pi_\chi = i(k_i^*\pi_i^* - \pi_i k_i) + N_\mathrm{C}, \qquad (13.45)
$$

giving the constraint equation,

$$\psi = \pi_\chi - i(k_i^* \pi_i^* - \pi_i k_i) - N_{\rm C} = 0. \tag{13.46}$$

The canonical Hamiltonian is given by,

$$\begin{aligned} H_c &= \pi_i \dot{k}_i + \pi_i^* \dot{k}_i^* + \pi_\chi \dot{\chi} - L \\ &= 2M \left[\pi_i + i \frac{N_{\rm C} k_i^*}{2(1+X)} \right] h_{ij}^{-1} \left[\pi_j^* - i \frac{N_{\rm C} k_j}{2(1+X)} \right] \\ &\quad + \dot{\chi}[\pi_\chi - i(\pi_i^* k_i^* - \pi_i k_i) - N_{\rm C}], \end{aligned} \tag{13.47}$$

and this can be further simplified to,

$$\begin{aligned} H_c = 2M(1+X) \bigg[&\pi_i \pi_i^* + (\pi_i k_i)(\pi_i^* k_i^*) \\ &- i \frac{N_{\rm C}}{2} (\pi_i k_i - \pi_i^* k_i^*) + \frac{1}{4} \frac{N_{\rm C}^2 X}{(1+X)} \bigg] + \dot{\chi}\psi. \end{aligned} \tag{13.48}$$

Here H_c is obtained explicitly in terms of the canonical variables $k_i, k_i^*, \pi_i, \pi_i^*$. The $\dot{\chi}\psi$ term indicates that $\dot{\chi}$ also behaves as a Lagrange multiplier since, following the Dirac procedure, we should define,

$$H_T = H_c + \lambda(t)\psi, \tag{13.49}$$

where λ is a priori an arbitrary function of t. We could absorb the $\dot{\chi}$ in λ.

Quantization of this Hamiltonian is now essentially straightforward. Let us first consider the symmetry generators $Q_{\alpha\beta}$, which in terms of the new canonical variables take the form,

$$\begin{aligned} Q_{ij} &= i(k_i \pi_j - \pi_i^* k_j^*) \\ Q_{i,N_{\rm F}} &= \mathrm{e}^{-i\chi} \left[\frac{N_{\rm C} k_i}{2} - i(\pi_i^* + k_i \pi_j k_j) \right] \\ Q_{N_{\rm F},i} &= \mathrm{e}^{i\chi} \left[\frac{N_{\rm C} k_i^*}{2} + i(\pi_i + k_j^* \pi_j^* k_i^*) \right] = Q_{i,N_{\rm F}}^* \\ Q_{N_{\rm F},N_{\rm F}} &= N_{\rm C} - i(\pi_i k_i - \pi_i^* k_i^*). \end{aligned} \tag{13.50}$$

We will now show that the H_T can be expressed in terms of the second Casimir operator of the $SU(N_{\rm F})$ group.

The second $U(N_{\rm F})$ Casimir operator is related to charge matrix elements $Q_{\alpha\beta}$ as,

$$Q_A Q^A = \frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha}. \tag{13.51}$$

A straightforward substitution gives,

$$\begin{aligned} \frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha} = (1+X)[&\pi_i^* \pi_i + \pi_i k_i \pi_j^* k_j^* \\ &- i \frac{N_{\rm C}}{2} (\pi_i k_i - \pi_i^* k_i^*)] + \frac{1}{2} N_{\rm C}^2 \left(1 + \frac{X}{2} \right). \end{aligned} \tag{13.52}$$

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Therefore, the Hamiltonian is,

$$H_T = 2M\left[Q^A Q^A - \frac{N_{\rm C}^2}{2}\right] + \lambda(t)\psi. \tag{13.53}$$

Denoting the $SU(N_{\rm F})$ second Casimir operator by C_2, and using $Q_A Q^A = C_2 + \frac{1}{2N_{\rm F}}(Q_B)^2$ we get (also applying the constraint $\psi = 0$),

$$H_T = 2M\left[C_2 - N_{\rm C}^2\frac{(N_{\rm F}-1)}{2N_{\rm F}}\right]. \tag{13.54}$$

The fact that H_T is, up to a constant, the second Casimir operator, is another way to show that the charges $Q_{\alpha\beta}$ are conserved. These conserved charges will generate symmetry transformations via,

$$\begin{aligned}\delta k_i &= i[\mathrm{Tr}(\epsilon Q), k_i], \quad \delta k_i^* = i[Tr(\epsilon Q), k_i^*)]\\ \delta\chi &= i[\mathrm{Tr}(\epsilon Q), \chi],\end{aligned} \tag{13.55}$$

and similar equations for the momenta π_i, π_i^*, π_χ. Here $\epsilon_{ij} = \frac{1}{2}\epsilon^A T_{ij}^A$ is the matrix of parameters. The transformation laws are derived using the constraint equation $\psi = 0$ after performing the commutator calculations. Notice that Q_{ij} and $Q_{N_{\rm F},N_{\rm F}}$ are linear in coordinates and momenta and therefore the $SU(N_{\rm F}-1)\times U_Y(1)$ transformations they generate are linear. The $Q_{N_{\rm F},i}$ and $Q_{i,N_{\rm F}}$ charges, on the other hand, have cubic terms as well (quadratic in coordinates), so that the coset-space transformations of $\frac{SU(N_{\rm F})}{SU(N_{\rm F}-1)\times U(1)}$ are non-linear. This is a well-known property of CP^n models. Substitution of $Q_{\alpha\beta}$ in eqn (13.55) gives,

$$\delta k_l = i[\epsilon_{ji}k_i\delta_{jl} + \mathrm{e}^{i\chi}\epsilon_{iN_{\rm F}}\delta_{il} - \mathrm{e}^{-i\chi}\epsilon_{N_{\rm F}i}k_ik_l - \epsilon_{N_{\rm F}N_{\rm F}}k_l], \tag{13.56}$$

where we use $[k,\pi] = i$.

Inversely, starting with these transformation laws it is easy to verify the invariance of the action. The standard Noether procedure then gives the charges $Q_{\alpha\beta}$ in terms of the coordinates and velocities, which (not suprisingly) coincide with those given in eqn. (13.50). One could also deduce these transformation laws by making the change of variables $z_\alpha, z_\alpha^* \to k_i, k_i^*, \chi$ in (13.30) directly.

One can verify that,

$$[Q^A, Q^B] = if^{ABC}Q^C, \tag{13.57}$$

where f^{ABC} are the structure constants of the $U(N_{\rm F})$ group.

Do we have further restrictions on the physical states? We shall see now that in fact we do have. Remember that our Lagrangian (13.41) includes an auxiliary gauge field $A_\circ \equiv \dot\chi$ and thus has to obey the associated Gauss law,

$$\frac{\partial L}{\partial A_\circ} = \frac{\partial L}{\partial\dot\chi} = \pi_\chi = N_{\rm C} - i(\pi_i k_i - \pi_i^* k_i^*) = 0. \tag{13.58}$$

Since π_χ is a linear combination of Q_B and Q_Y, and the first is constrained to be $Q_B = N_{\rm C}$, the Q_Y is restricted as well. More specifically, $Q_Y = \bar Q_Y$, with,

$$\bar Q_Y = \frac{1}{2}\sqrt{\frac{2}{(N_{\rm F}-1)N_{\rm F}}}N_{\rm C}. \tag{13.59}$$

13.4 The baryonic spectrum

The masses of the baryons (13.11) and (13.54), and the two constraints on the multiplets of the physical states, namely $Q_B = N_\mathrm{C}$ and that the multiplets contain $Q_Y = \bar{Q}_Y = \frac{1}{2}\sqrt{\frac{2}{(N_\mathrm{F}-1)N_\mathrm{F}}}N_\mathrm{C}$, are the main results of the last section. All states of the multiplet with $Q_Y \neq \bar{Q}_Y$ will be generated from the state $Q_Y = \bar{Q}_Y$ by $SU(N_\mathrm{F})$ transformations as in (13.19). Using the above constraints we can investigate now what possible representations will appear in the low energy baryon sector. Considering states with quarks only (no anti-quarks), the requirement of $Q_B = N_\mathrm{C}$ implies that only representations described by Young tableaux with N_C boxes appear. The extra constraint $Q_Y = \bar{Q}_Y$ implies that all N_C quarks are from $SU(N_\mathrm{F}-1)$, not involving the N_Fth. These are automatically obeyed in the totally symmetric representation of N_C boxes. In fact, this is the only representation possible for flavor space, since the states have to be constructed out of the components of one complex vector z as $\prod_{i=1}^{N_\mathrm{F}} z_i^{n_i}$ with $\sum_i n_i = N_\mathrm{C}$. See also more detailed discussion in the next section. For another way of deriving this result see Section 13.7.

Thus for $N_\mathrm{C} = 3, N_\mathrm{F} = 3$ we get only 10 of $SU(3)$. This is understandable, since there is no physical spin in two dimensions.

What about the masses of the baryons? The total mass of a baryons is given by the sum of (13.11) and (13.54), namely,

$$E = 4m\sqrt{\frac{2N_\mathrm{C}}{\pi}} + m\sqrt{2}\sqrt{\left(\frac{\pi}{N_\mathrm{C}}\right)^3}\left[C_2 - N_\mathrm{C}^2\frac{(N_\mathrm{F}-1)}{2N_\mathrm{F}}\right]. \tag{13.60}$$

For large N_C, the classical term behaves like N_C, while the quantum correction like 1. This will be worked out in Section 13.7.

That the total mass goes like N_C for large N_C, and that the quantum fluctuations are $\frac{1}{N_\mathrm{C}}$ of the classical result, is in accord with general considerations.

13.5 Quark flavor content of the baryons

A measure of the quark content of a given flavor q_i in a baryon state $|B\rangle$ is given by[5]

$$\langle\bar{q}_i q_i\rangle_B = \int \mathrm{d}x\,\langle g_{ii}\rangle_B - \int \mathrm{d}x\,\langle g_{ii}\rangle_0 \tag{13.61}$$

$$= \int \mathrm{d}x z_i^* z_i \left\langle\left[\mathrm{e}^{-i\sqrt{\frac{4\pi}{N_\mathrm{C}}}\phi_c} - 1\right]\right\rangle_B \tag{13.62}$$

$$= \text{const.}\,\langle z_i^* z_i\rangle_B. \tag{13.63}$$

In order to make contact with the real world, we take here $N_\mathrm{C} = 3$ and $N_\mathrm{F} = 3$, getting the baryons in the 10 representation of flavor. Similarly, for $SU_\mathrm{F}(2)$ there

[5] Quark solitons as constituents of hadrons were discussed in [86]. Following that, the flavor content of the baryons was discussed in [97].

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is only the isospin $\frac{3}{2}$ representation. This is what we would expect from naïve quark model considerations. The total wave function must be antisymmetric. Baryon is a color singlet, so the wave function is antisymmetric in color and it must be symmetric in all other degrees of freedom. There is no spin, so the baryon must be in a totally symmetric representation of the flavor group, a 10 for three flavors. Therefore, strictly speaking there is no state analogous to the proton. On the other hand, there is a state which is the analog of the Δ^+, namely the charge 1 state in the 10 representation, $z_1^2 z_2$.

The 10 is the lowest baryon multiplet in QCD_2. In the following we shall be dealing with the relative weight of a given flavor in some baryon state. Thus, $\langle \bar{q}q \rangle_B$ will henceforth stand for the ratio,

$$\frac{\langle \bar{q}q \rangle_B}{\langle \bar{u}u + \bar{d}d + \bar{s}s \rangle_B}. \tag{13.64}$$

For $\Delta^+ \sim z_1^2 z_2$ we obtain

$$\langle \bar{s}s \rangle_{\Delta^+} = \frac{\int (\mathrm{d}^2 z_1)(\mathrm{d}^2 z_2)|z_3|^2 (z_1^2 z_2)(z_1^2 z_2)^*}{\int (\mathrm{d}^2 z_1)(\mathrm{d}^2 z_2)(z_1^2 z_2)(z_1^2 z_2)^*} = \frac{1}{6}, \tag{13.65}$$

as well as,

$$\langle \bar{u}u \rangle_{\Delta^+} = \frac{1}{2}, \qquad \langle \bar{d}d \rangle_{\Delta^+} = \frac{1}{3}. \tag{13.66}$$

In evaluating the integral in the numerator in eqn. (13.65) we have used $|z_3|^2 = 1 - |z_1|^2 - |z_2|^2$, which follows from the unitarity of the matrix A in (13.19). Similarly, for $\Delta^{++} \sim z_1^3$ we have,

$$\langle \bar{u}u \rangle_{\Delta^{++}} = \frac{2}{3}, \qquad \langle \bar{d}d \rangle_{\Delta^{++}} = \frac{1}{6} \quad , \quad \langle \bar{s}s \rangle_{\Delta^{++}} = \frac{1}{6}. \tag{13.67}$$

In the constituent quark picture Δ^{++} contains just three u quarks. Both the d-quark and the s-quark content of the Δ^{++} come only from virtual quark pairs. Therefore in the $SU(3)$-symmetric case $\langle \bar{s}s \rangle_{\Delta^{++}} = \langle \bar{d}d \rangle_{\Delta^{++}}$, and $\langle \bar{s}s \rangle_{\Delta^+} = \langle \bar{s}s \rangle_{\Delta^{++}}$, as expected.

From eqn. (13.67) one can also read the results for $\Omega^- \sim z_3^3$, by replacing $u \leftrightarrow s$. In the general case of N_F flavors and N_C colors, one obtains,

$$\langle (\bar{q}q)_{\mathrm{sea}} \rangle_B = \frac{1}{N_\mathrm{C} + N_\mathrm{F}}, \tag{13.68}$$

where $(\bar{q}q)_{\mathrm{sea}}$ refers to the non-valence quarks in the baryon B. Moreover, one can also compute flavor content of valence quarks. Consider a baryon B containing k quarks of flavor v. The v-flavor content of such a baryon is,

$$\langle \bar{v}v \rangle_B = \frac{k+1}{N_\mathrm{C} + N_\mathrm{F}}. \tag{13.69}$$

This implies an "equipartition" for valence and sea, each with a content of $1/(N_C + N_F)$. It also follows that the total sea content of N_F flavors is,

$$\sum_{q=1}^{N_F} \langle (\bar{q}q)_{\text{sea}} \rangle_B = \frac{N_F}{N_C + N_F}, \tag{13.70}$$

which goes to zero for fixed N_F and $N_C \to \infty$, as expected.

It is interesting to compare these results with the Skyrme model in 3+1 dimensions. For the proton,

$$\langle \bar{u}u \rangle_p^{3+1} = \frac{2}{5}, \qquad \langle \bar{d}d \rangle_p^{3+1} = \frac{11}{30}, \qquad \langle \bar{s}s \rangle_p^{3+1} = \frac{7}{30}, \tag{13.71}$$

and for the Δ,

$$\langle \bar{s}s \rangle_\Delta^{3+1} = \frac{7}{24}, \qquad \langle \bar{s}s \rangle_{\Omega^-}^{3+1} = \frac{5}{12}. \tag{13.72}$$

The qualitative picture is similar, although the $\bar{s}s$ content in the non-strange baryons is lower in $1 + 1$ dimensions. One may speculate that in $1 + 1$ dimensions the effects of loops are smaller than in $3 + 1$ dimensions, since the theory is super-renormalizable and there are only longitudinal gluons. In the $SU_F(3)$-symmetric limit the strange quark content of baryons with zero net strangeness is significant, albeit smaller than that of either of the other two flavors. The situation obviously is reversed for Ω^-.

In the real world the current mass of the strange quark is much larger than the current masses of u and d quarks. It is natural to expect that this will have the effect of decreasing the strange quark content from its value in the $SU_F(3)$ symmetry limit. We do not know the exact extent of this effect, but it is likely that the strange content decreases by a factor which is less than two. This estimate is based on both explicit model calculations and what we know from PCAC, namely that the analogous quark bilinear expectation values in the vacuum are not dramatically different from their $SU(3)$ symmetric values,

$$0.5 \leq \frac{\langle \bar{s}s \rangle_0}{\langle \bar{u}u \rangle_0} \leq 1. \tag{13.73}$$

13.6 Multibaryons

Let us now explore the possibility of having multi-baryons states.[6] The procedure follows similar lines to that of the baryonic spectrum, namely, we look for classical solution of the equation of motions with baryon number kN_C, and then we semiclassically quantize this. The ansatz for the classical solution of the

[6] Multibaryonic states were studied in [102] and [103].

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low-lying k-baryon state is taken now to be,

$$g_0(k) = \begin{pmatrix} \overbrace{1 \quad\quad}^{(N_F - k)} & & & \\ & \ddots & & \\ & & \overbrace{\exp[-i(\frac{4\pi}{N_C})^{\frac{1}{2}} \varphi_c] \quad\quad\quad}^{k} & \\ & & & \ddots \end{pmatrix}. \quad (13.74)$$

For the semi-classical quantization we generalize the parametrization given in (13.16) to,

$$A = \begin{pmatrix} A_{ij} & z_{i\alpha} \end{pmatrix}, \quad (13.75)$$

where i represents the rows $(1, \ldots, N_F)$ and α the columns $(N_F - k + 1, \ldots, N_F)$. The effective action in its covariant form (13.33) becomes,

$$S[z_\alpha] = \frac{1}{2M} \int \mathrm{d}t \mathrm{Tr}(Dz)^\dagger Dz + iN_C \int \mathrm{d}t \mathrm{Tr} \dot{z}^\dagger z, \quad (13.76)$$

where now, instead of (13.34),

$$(Dz)_{i\alpha} = \dot{z}_{i\alpha} + z_{i\beta}(\dot{z}^\star_{j\beta} z_{j\alpha}) eDz. \quad (13.77)$$

Using the same steps as those which led to (13.54) one finds now the Hamiltonian,

$$H = 2M\left[C_2(N_F) - \frac{N_C^2}{2N_F} k(N_F - k)\right] + kE_c, \quad (13.78)$$

with E_c the classical contribution for one baryon, the first term in (13.60).

13.7 States, wave functions and binding energies

It was shown in [102] that the allowed k-baryon states contain (kN_C) boxes in the Young tableaux representation of the flavour group $SU(N_F)$. Let us recall that this result followed from the constraint implied by the local invariance,

$$z_{i\alpha} \rightarrow \mathrm{e}^{i\delta(t)} z_{i\alpha}. \quad (13.79)$$

Performing a variation corresponding to this invariance we find that the action S changes by

$$\Delta S = (kN_C) \int \dot{\delta}\, \mathrm{d}t. \quad (13.80)$$

This means that the N_z number is equal to (kN_C). Thus for any wave function, written as a polynomial in z and $z^\star$, the number of zs minus the number of $z^\star$s must equal (kN_C). Note that for $k = 1$ the transformation (13.79) represents also the N_F^{th} flavor number. Thus (13.80) entails that the representation contains a

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state with $N_{\rm C}$ boxes of the $N_{\rm F}$ flavor, and therefore must be the totally symmetric representation.

Now, the effective action (13.76) is invariant under a larger group of local transformations. In fact, we have extra (k^2-1) generators, which correspond to $SU(k)$ under which (13.76) is locally invariant. This can be exhibited by defining "local gauge potentials",

$$\tilde{A}_{\beta\alpha}(t) \;=\; -(z^\dagger \dot{z})_{\beta\alpha}, \tag{13.81}$$

so that,

$$Dz \;=\; \dot{z} \,+\, z\,\tilde{A}. \tag{13.82}$$

Under the local gauge transformation corresponding to $\Lambda(t)$, $\tilde{A}$ transforms as,

$$\tilde{A}(t) \;\rightarrow\; \mathrm{e}^{i\Lambda}\,\tilde{A}\mathrm{e}^{-i\Lambda} \,+\, (\partial_t \mathrm{e}^{i\Lambda})\,\mathrm{e}^{-i\Lambda}, \tag{13.83}$$

which implies,

$$(Dz)_{i\alpha} \;\rightarrow\; (Dz)_{i\beta}(\mathrm{e}^{-i\Lambda})_{\beta\alpha}, \tag{13.84}$$

and so $\Delta S = 0$. If we perform the $U(1)$ transformation (13.79) we obtain a contribution (13.80) from the Wess–Zumino term, which implies $N_z = (kN_{\rm C})$. But due to the larger local symmetry we have more restrictions; they imply that the allowed states have to be singlets under the above mentioned $SU(k)$ symmetry. This is analogous to the confinement property of QCD, which tells that, due to the non-abelian gauge invariance, the physical states have to be color singlets. Here we have an analogous singlet structure of the $SU(k)$ in the flavor space. Taking a wave function that has zs only (analogous to quarks only for QCD), it must be of the form,

$$\psi_k(z) \;=\; \prod_{i=1}^{N_{\rm C}} \Big(\epsilon_{\alpha_1 \ldots \alpha_k}\, z_{i_1\alpha_1} \ldots z_{i_k\alpha_k} \Big), \tag{13.85}$$

for a given set of $1 \le i_1, ..i_k \le N_{\rm F}$.

The most general state will then be of the form,

$$\tilde{\psi}(z, z^\star) \;=\; \psi_k(z)[\prod_{\{i,j\}} (z^\star_{i\alpha} z_{j\alpha})^{n_{ij}}], \tag{13.86}$$

and the products are over given sets of indices.

Using the explicit formula from [103], we obtain the mass of the state represented by (13.85),

$$E[\psi_k] = Mk(N_{\rm F} - k)N_{\rm C} \,+\, kE_c. \tag{13.87}$$

To obtain binding energies, consider our k-baryon as built from constituents k_r, such that $k = \sum_r k_r$. Then,

$$\begin{aligned} B[k|k_r] &= -(MN_C)\left[k^2 - \textstyle\sum_i k_i^2\right] \\ &= -(2MN_C)\textstyle\sum_{r>s} k_r k_s. \end{aligned} \tag{13.88}$$

When all $k_r = 1$, the sum gives us $\frac{1}{2}k(k-1)$, i.e. the number of one-baryon pairs in the k-baryon state. Note that the binding energy is always negative, thus the k-baryon is stable. The maximal binding corresponds to the case when all $k_r = 1$.

Note also that in the $N_C \to \infty$ limit, the binding tends to a finite value, since then,

$$\lim_{N_C \to \infty} (2MN_C) = (Cm e_c)^{\frac{1}{2}} \left(\frac{2N_F}{\pi}\right)^{\frac{1}{4}} \pi^{\frac{3}{2}}. \tag{13.89}$$

Let us take as an example an analog of a deutron, namely a di-baryon $k = 2$. Then for $N_C = 3$, $N_F = 2$ we find that its representation is a flavor singlet (this is the limiting case of $k = N_F$). The ratio of the binding to twice the baryon mass is given by,

$$\epsilon_2 = \frac{1}{1 + \frac{24}{\pi^2}} = 0.29. \tag{13.90}$$

For $k = 2$, $N_C = 3$ and $N_F = 3$ we find that the di-baryon is represented by, $\overline{10}$ and the ratio is given by,

$$\epsilon_3 = \frac{1}{2 + \frac{24}{\pi^2}} = 0.23. \tag{13.91}$$

For general N_F we obtain,

$$\epsilon_F = \frac{1}{(N_F - 1) + \frac{24}{\pi^2}} = \frac{1}{N_F + 1.43}. \tag{13.92}$$

Finally, let us make the following comment. The ratio of the quantum fluctuations term to the classical term, in the expression for the mass, eqn. (13.87), is given by,

$$\frac{(\text{Quantum corrections})}{(\text{Classical term})} = \left(\frac{\pi^2}{8}\right) \frac{N_F - k}{N_C}. \tag{13.93}$$

Thus, we do not expect our approximations to hold in the region $N_F \geq (N_C + 1)$. We expect it to start for $N_C \geq N_F$, and to be good in the region $N_C \gg N_F$.

13.8 Meson-baryon scattering

So far we have analyzed, using semiclassical quantization of the bosonized theory in the strong coupling limit, the spectrum of the baryons and their flavor content. Applying the same technique one can also study the scattering processes of mesons from baryons. The idea is to introduce perturbations around the classical soliton solutions and to compute the forward phase shifts. We start with the computation in the strong coupling limit [98] and then we discuss the general case of any coupling [87].

Our starting point is the soliton solution that describes the static classical baryon $g_c(x) = \exp[-i\Phi_c(x)]$ where,

$$\Phi_c(x) = \begin{pmatrix} \phi_c(x) & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 0 \end{pmatrix}, \tag{13.94}$$

and,

$$\phi_c(x) = 4\mathrm{arctg}\left(\mathrm{e}^{\mu x}\right), \quad \mu = m\sqrt{\frac{8\pi}{N_\mathrm{C}}}. \tag{13.95}$$

Note that we have shifted the non-trivial phase factor to the upper left-hand corner, whereas in (13.77) it was put in the lower right-hand one.

We introduce a fluctuation around it of the form,

$$g = \exp\left\{-i\left[\Phi_c(x) + \delta\phi(x,t)\right]\right\} \tag{13.96}$$

$$g \approx \mathrm{e}^{-i\Phi_c(x)} - i\int_0^1 d\tau\, \mathrm{e}^{-i\tau\Phi_c(x)}\delta\phi(x,t)\mathrm{e}^{-i(1-\tau)\Phi_c(x)}. \tag{13.97}$$

Actually, to avoid integrals as in eqn. (13.97), which yield rather complicated expressions for fluctuations, we will adopt a different expansion, namely,

$$\begin{aligned} g &= \mathrm{e}^{-i\Phi_c(x)}\,\mathrm{e}^{-i\tilde{\delta}\phi(x,t)}, \\ &\approx \mathrm{e}^{-i\Phi_c(x)} - i\mathrm{e}^{-i\Phi_c(x)}\tilde{\delta}\phi(x,t), \end{aligned} \tag{13.98}$$

where we have denoted by $\tilde{\delta}\phi$ the new variation, different from the $\delta\phi$ of eqn. (13.97), but still a fluctuation about the classical solution. Now,

$$\begin{aligned} &\frac{N_c}{4\pi}\,\partial_+\left[\mathrm{e}^{-i\Phi_c(x)}\left(\partial_-\tilde{\delta}\phi(x,t)\right)\mathrm{e}^{i\Phi_c(x)}\right] \\ &\quad + m^2\left[\mathrm{e}^{-i\Phi_c(x)}\tilde{\delta}\phi(x,t) + \tilde{\delta}\phi(x,t)\mathrm{e}^{i\Phi_c(x)}\right] = 0. \end{aligned} \tag{13.99}$$

Obviously the two expressions coincide in the abelian case. In fact, the relation between $\delta\phi$ and $\tilde{\delta}\phi$ is

$$\tilde{\delta}\phi(x,t) = \int_0^1 \mathrm{d}\tau\, \mathrm{e}^{i\tau\Phi_c(x)}\,\delta\phi(x,t)\,\mathrm{e}^{-i\tau\Phi_c(x)}. \tag{13.100}$$

Physical quantities should obviously come out to be the same for both types of fluctuation.

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13.8.1 Abelian case

We start with the abelian fluctuation $\delta\phi$ that commutes with Φ_c. Denote this case by $\delta\phi_{\rm ab}$, where the subscript "ab" stands for "abelian."

Then the fluctuation reads

$$\delta g = -i\delta\phi_{\rm ab}(x)\mathrm{e}^{-i\phi_c(x)}, \tag{13.101}$$

where,

$$\Box\,\delta\phi_{\rm ab} + \mu^2(\cos\phi_c)\delta\phi_{\rm ab} = 0, \tag{13.102}$$

and,

$$\cos\phi_c = \left[1 - \frac{2}{\cosh^2\mu x}\right]. \tag{13.103}$$

This equation of motion can be derived from the following effective action,

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\left(\partial_\mu\delta\phi_{\rm ab}\right)^2 - \frac{1}{2}V(x)\left(\delta\phi_{\rm ab}\right)^2 \tag{13.104}$$

$$V(x) = \mu^2\cos\phi_c(x) = \mu^2\left[1 - \frac{2}{\cosh^2\mu x}\right]. \tag{13.105}$$

For a solution with an harmonic time dependence of the form,

$$\delta\phi_{\rm ab}(x,t) = \mathrm{e}^{-i\omega t}\chi_{\rm ab}(x), \tag{13.106}$$

the spatial part has to solve,

$$-\omega^2\chi_{\rm ab} - \chi_{\rm ab}'' + V(x)\chi_{\rm ab} = 0. \tag{13.107}$$

Note that asymptotically the potential approaches $x \to \pm\infty$, the potential $\to \mu^2$, and so asymptotically,

$$\chi_{\rm ab}''(\pm\infty) + \omega^2\,\chi_{\rm ab}(\pm\infty) = \mu^2\,\chi_{\rm ab}(\pm\infty). \tag{13.108}$$

For asymptotic behavior of the form,

$$\chi_{\rm ab}(x) \underset{|x|\to\infty}{\longrightarrow} \mathrm{e}^{ikx}, \tag{13.109}$$

with,

$$\omega^2 = k^2 + \mu^2, \tag{13.110}$$

the two asymptotic solutions are,

$$\chi_{\rm ab}(x) \sim A(\omega)\sin kx + B(\omega)\cos kx, \tag{13.111}$$

and the S-matrix is,

$$S_{\rm forward} = \frac{1}{2}(B - iA) \tag{13.112}$$

$$S_{\rm backward} = \frac{1}{2}(B + iA), \tag{13.113}$$

for an incoming wave e^{ikx} from $x = -\infty$.

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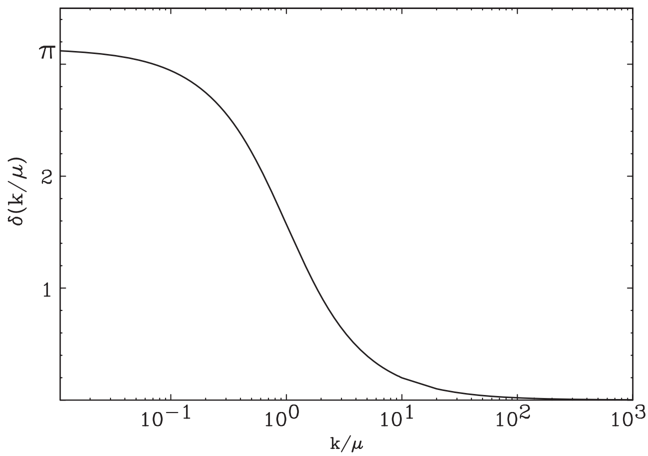

Fig. 13.1. The phase shift $\delta = 2\mathrm{ctg}^{-1}(k/\mu)$, eqn. (13.117), as a function of the normalized momentum k/μ, for the potential (13.105), governing the small fluctuations around the soliton in the abelian case. The phase shift is smooth and monotonically decreasing with momentum, indicating that no resonance is present. Note logarithmic momentum scale.

We can now proceed to derive the scattering matrix, using the standard procedure. The solution for $x \to \infty$ contains only the transmitted wave, $\psi(x \to \infty) \sim \mathrm{e}^{ikx}$.[7]

It turns out that for the particular potential (13.105) there is no reflection at all, i.e. the wave function for $x \to -\infty$ contains only the incoming wave,

$$\psi(x \to -\infty) \sim \mathrm{e}^{ikx-\delta} = \left(-\frac{1+ik/\mu}{1-ik/\mu}\right) \mathrm{e}^{ikx}. \tag{13.114}$$

Thus,

$$\frac{1}{T} = -\frac{1+ik/\mu}{1-ik/\mu} \tag{13.115}$$

$$T = -\frac{1-ik/\mu}{1+ik/\mu} = \mathrm{e}^{i\delta} \tag{13.116}$$

$$\mathrm{ctg}\,\frac{1}{2}\delta = \frac{k}{\mu}. \tag{13.117}$$

As shown in Fig. 13.1, δ varies smoothly and decreases monotonically from $\delta = \pi$ at $k = 0$ to $\delta = 0$ at $k = \infty$, indicating that there is no resonance.

The no-reflection potential we found is a special case of a well-known class of reflectionless in quantum mechanics.

[7] We take the convention where the scattering phase is taken to be zero at $x \to \infty$ and is therefore extracted from the wave function at $x \to -\infty$.

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13.8.2 The non-abelian case

We got a no-reflection potential in the previous section, in the case of one flavor. We want to examine now the non-abelian case.

Following eqn. (13.99), we get,

$$\Box\,\tilde{\delta}\phi - i\left(\partial_+\Phi_c\right)\left(\partial_-\tilde{\delta}\phi\right) + i\left(\partial_-\tilde{\delta}\phi\right)\left(\partial_+\Phi_c\right) + \frac{1}{2}\mu^2\left[\tilde{\delta}\phi \mathrm{e}^{-i\Phi_c(x)} + \mathrm{e}^{i\Phi_c(x)}\tilde{\delta}\phi\right] = 0. \tag{13.118}$$

The equation for $\tilde{\delta}\phi_{ij}$ with $i, j \neq 1$ is as for the free case,

$$\Box\,\tilde{\delta}\phi_{ij} + \mu^2\tilde{\delta}\phi_{ij} = 0\,, \qquad i \text{ and } j \neq 1, \tag{13.119}$$

whereas the $i = 1, j = 1$ matrix element is as in the abelian case,

$$\Box\,\tilde{\delta}\phi_{11} + \mu^2\left(\cos\phi_c(x)\right)\tilde{\delta}\phi_{11} = 0. \tag{13.120}$$

with no reflection and no resonance.

So in order to proceed beyond these results, we need to consider $\tilde{\delta}\phi_{1j}$, $j \neq 1$, or $\tilde{\delta}\phi_{i1}$, $i \neq 1$. As $\tilde{\delta}\phi$ is Hermitian, it is sufficient to discuss one of the above.

Thus we take,

$$\tilde{\delta}\phi_{1j} = \mathrm{e}^{-i\omega t}u_j(x) \qquad j \neq 1, \tag{13.121}$$

resulting in,

$$u_j''(x) - i\phi_c'(x)u_j'(x) + \left[\omega^2 + \omega\phi_c'(x) - \tfrac{1}{2}\mu^2\left(1 + \mathrm{e}^{i\phi_c(x)}\right)\right]u_j(x) = 0. \tag{13.122}$$

Defining,

$$u_j \equiv \mathrm{e}^{\frac{i}{2}\phi_c}\,v_j\,, \tag{13.123}$$

we find,

$$v_j'' + \left[\omega^2 + \omega\phi_c' - \tfrac{1}{2}\mu^2\left(1 + \cos\phi_c\right) + \tfrac{1}{4}\left(\phi_c'\right)^2\right]v_j = 0. \tag{13.124}$$

Using,

$$\tfrac{1}{2}\left(\phi_c'\right)^2 = \mu^2\left(1 - \cos\phi_c\right), \tag{13.125}$$

we get,

$$v_j'' + \left[\omega^2 + \omega\phi_c' - \mu^2\cos\phi_c\right]v_j = 0. \tag{13.126}$$

This can be rewritten as,

$$-v_j'' - \omega^2 v_j + V(x)v_j = 0, \tag{13.127}$$

where,

$$\begin{aligned} V(x) &= -\omega\phi_c' + \mu^2\cos\phi_c = \\ &= \mu^2 - 2\mu^2\left[\frac{(\omega/\mu)}{\cosh\mu x} + \frac{1}{\cosh^2\mu x}\right], \end{aligned} \tag{13.128}$$

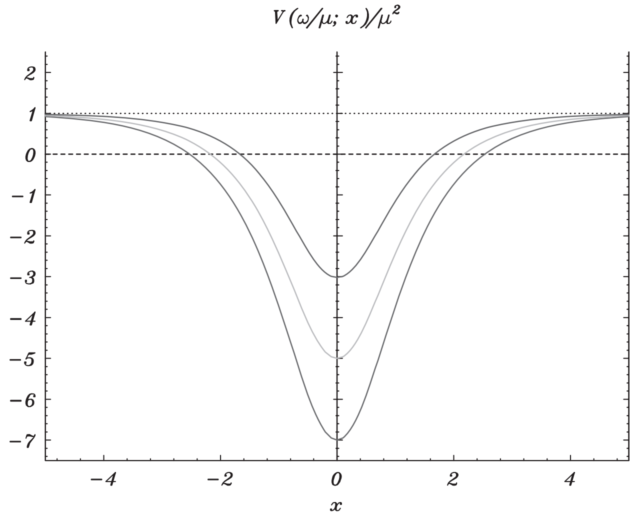

Fig. 13.2. The normalized potential $V(\omega/\mu; x)/\mu^2$ of eqn. (13.128), for $\omega/\mu =$ 1.01 (upper), 2 (middle) and 3 (lower).

with $\omega = \sqrt{k^2 + \mu^2}$, as before. Note that the potential depends on the momentum of the incoming particle, as shown in Fig. 13.2.

Next we proceed to solve numerically for the reflection and transmission coefficient. It turns out that for numerical solution of the scattering problem it is more convenient to take the coefficient of the outgoing wave at $x \sim +\infty$ to be 1, instead of the T prefactor, and integrate eqn. (13.127) backward, reading off the T and R amplitudes from the solution at $x \sim -\infty$.

We thus use,

$$\begin{aligned} v_j(x) &= \mathrm{e}^{ikx}, & x &\to +\infty \\ v_j(x) &= \tfrac{1}{T}\,\mathrm{e}^{ikx} + \tfrac{R}{T}\,\mathrm{e}^{-ikx}, & x &\to -\infty. \end{aligned} \tag{13.129}$$

Since the potential is symmetric, the symmetric and anti-symmetric scattering amplitudes don't mix, yielding two independent phase shifts δ_S and δ_A, respectively. This leads to,

$$\begin{aligned} T &= \tfrac{1}{2}\left(\mathrm{e}^{i\delta_S} + \mathrm{e}^{i\delta_A}\right) \\ R &= \tfrac{1}{2}\left(\mathrm{e}^{i\delta_S} - \mathrm{e}^{i\delta_A}\right). \end{aligned} \tag{13.130}$$

Defining,

$$\delta_\pm = \frac{1}{2}\left(\delta_S \pm \delta_A\right), \tag{13.131}$$

we find that,

$$\begin{aligned} T &= \mathrm{e}^{i\delta_+} \cos\delta_- \\ R &= i\mathrm{e}^{i\delta_+} \sin\delta_-. \end{aligned} \tag{13.132}$$

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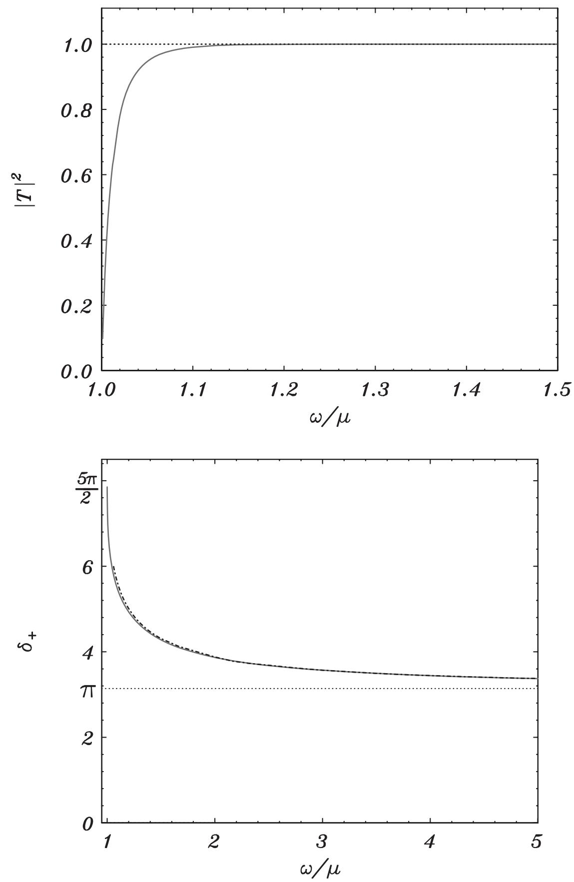

Fig. 13.3. Scattering by the potential eqn. (13.128) as a function of the normalized energy ω/μ. Upper plot: transmission probability $|T|^2$; lower plot: phase of T, δ_+ (continuous line). Also shown is the approximate result for δ_+ from WKB (dot-dashed line).

Note that R/T is purely imaginary. The transmission and reflections probabilities are,

$$\begin{aligned} |T|^2 &= \cos^2 \delta_- \\ |R|^2 &= \sin^2 \delta_- . \end{aligned} \tag{13.133}$$

The numerical results for the transmission probability $|T|^2$ and for the phase of T, δ_+ are presented in Fig. 13.3. For comparison and as an extra check we also plot the WKB result for δ_+. Note that no resonance appears.

Note that the asymptotic value of the phase shift is π. This can also be obtained from a WKB calculation, which becomes exact at infinite energies.

13.8.3 Extension to arbitrary coupling

To analyze the system at any gauge coupling we go back to bosonized action prior to the implementation of the strong coupling limit, which we now rewrite in the form,

$$S_{\rm eff}\,[u] = S_0[u] + \frac{e_c^2 N_f}{8\pi^2}\int {\rm d}^2x\,{\rm Tr}\,\left[\partial_-^{-1}\left(u\,\partial_- u^\dagger\right)_c\right]^2 + m'^2 N_{\tilde{m}}\int {\rm d}^2x\,{\rm Tr}\,\left(u+u^\dagger\right). \tag{13.134}$$

The strong coupling limit eliminates the second term of (13.134), for arbitrary coupling,

$$\frac{e_c^2 N_f}{8\pi^2}\int {\rm d}^2x{\rm Tr}\left[\partial_-^{-1}\left(u\,\partial_- u^\dagger\right)_c\right]^2, \tag{13.135}$$

where $\left(u\,\partial_- u^\dagger\right)_c$ is the color part of $M \equiv u\partial_- u^\dagger$, to be computed as,

$$M_c = {\rm Tr}_f M - 1/N_c {\rm Tr}_{f\,\&\,c} M. \tag{13.136}$$

As already mentioned, this term represents the interactions, as it arises from integrating out the gauge potentials. However, we will see that, for the physical situation we discuss, this term does not contribute to meson-baryon scattering for any coupling. As a result, the latter is described by the effective action $\tilde{S}_{\rm eff}\,[u]$, whereas in the strong coupling limit it is described by $\tilde{S}_{\rm eff}\,[g]$.

In a similar manner to (13.95) we take u to be of the form,

$$u = \exp(-i\Phi_c)\exp(-i\delta\Phi), \tag{13.137}$$

corresponding to a classical soliton Φ_c, and a small fluctuation $\delta\Phi$ around it, representing the meson. The resulting action is then expanded to second order in $\delta\Phi$, yielding a linear equation of motion for $\delta\Phi$ in the soliton background. The latter serves as an external potential in which the meson is propagating.

We start by evaluating,

$$\begin{aligned} M &\equiv u\partial_- u^\dagger = \\ &= \exp(-i\Phi_c)\,\partial_-(\exp i\Phi_c) + \exp(-i\Phi_c)\exp(-i\delta\Phi)\,[\partial_-\exp(i\delta\Phi)]\exp(i\Phi_c), \end{aligned} \tag{13.138}$$

and obtain the equations of motion for the meson field by varying with respect to $\delta\Phi$. The variation of (13.135) with respect to $\delta\Phi$ is proportional to,

$$\frac{\delta M_c}{\delta(\delta\Phi)}\,\partial^{-2} M_c. \tag{13.139}$$

To compute its variation with respect to $\delta\Phi$, we need only the second term M_2 of M, as the first term M_1 is independent of $\delta\Phi$.

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We take for the soliton a diagonal ansatz (13.94) now in the form of a u matrix rather then a g one,

$$\begin{aligned}[\exp(-i\Phi_c)]_{aa'jj'} = \delta_{aa'}\delta_{jj'} \exp\left(-i\sqrt{4\pi}\chi_{\alpha j}\right) : \\ a = 1, \ldots, N_c, \\ j = 1, \ldots, N_f,\end{aligned} \tag{13.140}$$

so that,

$$\begin{aligned}&\{\exp(-i\Phi_c)[\exp(-i\delta\Phi)\,\partial_-\exp(i\delta\Phi)]\exp(i\Phi_c)\}_{aj,a'j'} \\ &= \exp(-i\sqrt{4\pi}\chi_{aj})\,[\exp(-i\delta\Phi)\,\partial_-\exp(i\delta\Phi)]_{aj,a'j'}\,\exp(i\sqrt{4\pi}\chi_{a'j'}).\end{aligned} \tag{13.141}$$

The part of M that contributes to the effective action is its color projection (13.136). We note that $\mathrm{Tr}_{f\&c}M_2 = 0$, and thus,

$$[(M_2)_c]_{a,a'} = \sum_j \exp(-i\sqrt{4\pi}\chi_{aj})[\exp(-i\delta\Phi)\,\partial_-\exp(i\delta\Phi)]_{aj,a'j}\exp(i\sqrt{4\pi}\chi_{a'j}). \tag{13.142}$$

The mesons $\delta\Phi$ have to be diagonal in color, so,

$$[(M_2)_c]_{a,a'} = \sum_j [\,\exp(-i\delta\Phi)\,\partial_-\exp(i\delta\Phi)\,]_{aj,aj}\,\delta_{a,a'}. \tag{13.143}$$

We recall that the flavor structure of the mesons is independent of their color indices, and restrict our attention to mesons that have no $U(1)$ flavor part. In this way, we may be sure that classical solutions lead to stable particles, since their non-vanishing flavor quantum numbers put them in a different sector from the vacuum. We then have,

$$\sum_j [\,\exp(-i\delta\Phi)\,\partial_-\exp(i\delta\Phi)\,]_{\alpha j,\alpha j} = 0, \tag{13.144}$$

as shown earlier, and the effective meson-baryon action is,

$$\tilde{S}_{m\text{-}b}[\delta\Phi] = S_0[u] + m^2 N_m \int \mathrm{d}^2x \left(\,\mathrm{Tr}\ u + \ \mathrm{Tr}\ u^\dagger\right), \tag{13.145}$$

with u depending on $\delta\Phi$ for fixed Φ_c as in (13.137).

Next we would like to evaluate the potential. The equation of motion for $\delta\Phi$ is obtained from (13.145), by first varying with respect to u and then varying u with respect to $\delta\Phi$. To first order in $\delta\Phi$, we find,

$$\delta u = -i[\exp(-i\Phi_c)]\delta\Phi. \tag{13.146}$$

The resulting equation of motion is then,

$$\frac{1}{4\pi}\partial_+\left[(\partial_- u)\,u^\dagger\right] + \left(um^2 - m^2u^\dagger\right) = 0, \tag{13.147}$$

where m is the diagonal mass matrix: $m = \delta_{ij}m_j$ with (possibly different) entries m_j corresponding to flavors j. We note that there is the possibility of an overall

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scale ambiguity in m, since, when the masses are different, there is a question of which normal-ordering scale to use. The resulting equation of motion for $\delta\Phi$ is,

$$\Box\,\delta\Phi - i\,(\partial_+\Phi_c)\,(\partial_-\delta\Phi) + i\,(\partial_-\delta\Phi)\,(\partial_+\Phi_c) + \frac{1}{2}\left[\delta\Phi\mu^2\exp(-i\Phi_c) + \exp(i\Phi_c)\mu^2\delta\Phi\right] = 0, \tag{13.148}$$

where $\mu \equiv m\sqrt{8\pi}$.

As discussed before, both Φ_c and $\delta\Phi$ are diagonal in color. Moreover, Φ_c is diagonal in flavor too. So, taking the $aajj'$ matrix element of the equation of motion (13.148), we find,

$$\Box\,\delta\Phi_{ajj'} - i(\partial_+\Phi_c)_{aj}(\partial_-\delta\Phi)_{ajj'} + i\,(\partial_-\delta\Phi)_{ajj'}(\partial_+\Phi_c)_{aj'} + \frac{1}{2}\{\delta\Phi_{ajj'}\mu^2{}_{j'}[\exp(-i\Phi_c)]_{aj'} + [\exp(i\Phi_c)]_{aj}\mu^2{}_j\delta\Phi_{ajj'}\} = 0. \tag{13.149}$$

Examining the classical solutions for the quark solitons inside the baryons, we see that, for a given color index a, there is only one flavor for which Φ_c is non-zero. We can now distinguish three cases:

- The first is when an index a and indices j and j' are chosen in such a way that both $(\Phi_c)_{aj}$ and $(\Phi_c)_{aj'}$ are zero. In such a case,

$$\Box\,\delta\Phi_{ajj'} + \frac{1}{2}[\mu^2{}_j + \mu^2{}_{j'}]\delta\Phi_{ajj'} = 0, \tag{13.150}$$
$$\text{where } (\Phi_c)_{aj} = 0 \text{ and } (\Phi_c)_{\alpha j'} = 0.$$

  Thus $\delta\Phi_{ajj'}$ is a free field with squared mass given by the average of m_j^2 and $m_{j'}^2$ in this case, which we do not discuss further.
- The second case is that of $j = j'$, with a such that $(\Phi_c)_{aj}$ is a quark soliton inside the baryon. In this case,

$$\Box\,\delta\Phi_{ajj} + \mu^2{}_j\cos[(\Phi_c)_{aj}]\delta\Phi_{ajj} = 0. \tag{13.151}$$

- The third case is when j is different from j', now with one of the Φ_c being a soliton and the other vanishing. Taking $(\Phi_c)_{aj}$ to be the soliton, we obtain,

$$\Box\,\delta\Phi_{ajj'} - i(\partial_+\Phi_c)_{aj}(\partial_-\delta\Phi)_{ajj'} + \frac{1}{2}\{\mu^2{}_{j'} + \mu^2{}_j[\exp(i\Phi_c)]_{aj}\}\delta\Phi_{ajj'} = 0, \tag{13.152}$$

  where $\quad j' \neq j \text{ and } (\Phi_c)_{\alpha j'} = 0.$

Next we want to proceed and evaluate the meson-baryon scattering. For that purpose we need to analyze the equations that determine the static solution $(\Phi_c)_{aj}$. First one defines,

$$(\Phi_c)_{aj} = \sqrt{4\pi}(\chi_c)_{aj}, \tag{13.153}$$

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where the $(\chi_c)_{aj}$ are canonical fields, whose equations of motion are,

$$\chi''_{\alpha j} - 4\alpha_c \left(\sum_l \chi_\alpha - \frac{1}{N_c} \sum_{\beta l} \chi_{\beta l} \right) - 2\sqrt{4\pi} m_j^2 \sin \sqrt{4\pi} \chi_{\alpha j} = 0.$$

Note the extra factor 2 in front of the mass term, as compared with eqn. (22) of [86], due to an error in this reference.

Choosing the boundary conditions $\chi_{aj}(-\infty) = 0$, we get as constraints for $\chi_{aj}(+\infty)$, denoted hereafter simply by χ_{aj},

$$\frac{1}{\sqrt{\pi}} \chi_{\alpha j} = n_{\alpha j} \quad \text{integers,} \tag{13.154}$$

and

$$\sum_l n_\alpha = n \quad \text{independent of } a. \tag{13.155}$$

The baryon number[8] associated with any given flavor l is given by,

$$B_l = \sum_a n_\alpha.$$

Combining the last two equations, we find,

$$B = \sum_l B_l = n N_c,$$

for the total baryon number.

We now continue in a similar manner to the discussion in the strong coupling limit, starting with the first non-trivial case (13.151) identified above. As the soliton solutions are such that there is a unique correspondence between the color index a and the flavour index j, we suppress a in what follows. Putting,

$$\delta\Phi_{jj} = \mathrm{e}^{-i\omega_j t} u_j(x), \tag{13.156}$$

with,

$$u_j(x) \underset{x\to\infty}{\longrightarrow} \mathrm{e}^{ikx}, \tag{13.157}$$

we find,

$$\omega_j^2 = k^2 + \mu_j^2, \tag{13.158}$$

and the equation for $u_j(x)$ is,

$$u''_j(x) + {\omega_j}^2 u_j - {\mu^2}_j [\cos{(\Phi_c)_j}] u_j = 0. \tag{13.159}$$

We define the potential V_j for this scattering process via,

$$u''_j(x) + {\omega_j}^2 u_j - V_j u_j = 0, \tag{13.160}$$

[8] In our normalization, a single quark carries one unit of baryon number.

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and find,

$$V_j = \mu^2{}_j[\cos{(\Phi_c)}_j]. \tag{13.161}$$

In our normalization the outgoing wave has coefficient 1, which is more convenient for numerical calculations, and the wave for $x \to -\infty$ is now,

$$u_j(x) = \frac{1}{T_j}\,\mathrm{e}^{ikx} + \frac{R_j}{T_j}\,\mathrm{e}^{-ikx}, \qquad x \to -\infty, \tag{13.162}$$

in this case.

In the second non-trivial case (13.152), we put,

$$\delta\Phi_{jj'} = \mathrm{e}^{-i\omega_{jj'}t}u_{jj'}(x), \tag{13.163}$$

so that,

$$\begin{aligned} &u''_{jj'}(x) - i(\Phi_c)'_j(x)u'_{jj'}(x) + \{\omega^2_{jj'} + \omega_{jj'}(\Phi_c)'_j(x) \\ &\qquad -\frac{1}{2}\{\mu^2{}_{j'} + \mu^2{}_j[\exp(i\Phi_c)]_j\}\}u_{jj'} = 0. \end{aligned} \tag{13.164}$$

To eliminate the first derivative term in u, we substitute,

$$u_{jj'} = \left[\exp\left(\frac{i}{2}\Phi_c\right)\right]_j\, v_{jj'}. \tag{13.165}$$

This results in,

$$\begin{aligned} &v''_{jj'}(x) + \{\omega^2_{jj'} + \omega_{jj'}(\Phi_c)'_j(x) - \mu^2{}_j[\cos(\Phi_c)]_j\}v_{jj'} + \frac{1}{2}(\mu^2{}_j - \mu^2{}_{j'})v_{jj'} \\ &\qquad + \left\{\frac{1}{4}[(\Phi_c)'_j(x)]^2 - \frac{1}{2}\mu^2{}_j(1 - [\cos(\Phi_c)]_j)\right\}v_{jj'} \\ &\qquad + \frac{i}{2}\left\{(\Phi_c)_j{}''(x) - \mu^2{}_j[\sin(\Phi_c)_j]\right\}v_{jj'} = 0. \end{aligned} \tag{13.166}$$

We note that the last three lines vanish when all the quark masses are equal, as then the soliton is a sine-Gordon one. Thus, the scattering would then only be elastic.

The potential of the scattering is defined here via,

$$v''_{jj'}(x) + \omega^2_{jj'}v_{jj'} - V_{jj'}v_{jj'} = 0, \tag{13.167}$$

so that,

$$\begin{aligned} V_{jj'} = &-\omega_{jj'}(\Phi_c)'_j(x) + \mu^2{}_j[\cos(\Phi_c)]_j \\ &-\frac{1}{2}(\mu^2{}_j - \mu^2{}_{j'}) \\ &-\left\{\frac{1}{4}[(\Phi_c)'_j(x)]^2 - \frac{1}{2}\mu^2{}_j(1 - [\cos(\Phi_c)]_j)\right\} \\ &-\frac{i}{2}\{(\Phi_c)_j{}''(x) - \mu^2{}_j[\sin(\Phi_c)_j]\}. \end{aligned} \tag{13.168}$$

Taking again,

$$v_{jj'}(x) \underset{x\to\infty}{\longrightarrow} \mathrm{e}^{ikx}, \tag{13.169}$$

we get,

$$\omega_{jj'} = \frac{1}{2}({\mu^2}_j + {\mu^2}_{j'}), \tag{13.170}$$

and the wave for $x \to -\infty$ is,

$$v_{jj'}(x) = \frac{1}{T_{jj'}}\,\mathrm{e}^{ikx} + \frac{R_{jj'}}{T_{jj'}}\,\mathrm{e}^{-ikx}, \qquad x \to -\infty, \tag{13.171}$$

in this case.

To summarize we have shown that meson-baryon scattering in QCD_2 in the large-N_c limit is non-trivial for non-zero quark masses, and is described by two distinct effective potentials when the quark masses are unequal. These effective potentials are not of the sine-Gordon type found in previous cases, and we expect the scattering amplitudes also to be non-trivial. Their calculation will require numerical analysis.