REDUCIBILITY AND IRREDUCIBILITY OF THE GAUSS–MANIN SYSTEM ASSOCIATED WITH A SELBERG TYPE INTEGRAL

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Dedicated to professor Joji Kajiwara on his 60th birthdy

Consider the function with variables $z_1, \ldots, z_m$

$$(0.1) \qquad \int_{\Gamma} \prod_{1 \leq i \leq j \leq n} (t_i - t_j)^{\nu} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (t_i - z_j)^{\lambda_j} \, dt_1 dt_2 \ldots dt_n,$$

where ν and λ_j $(j = 1, \ldots, m)$ are complex numbers and Γ is a suitably chosen integral domain. In case $m = 2$, if we set $[z_1, z_2]^n$ as Γ, it is the Selberg integral [22]. Our function can be regarded as an extention of it; so we may call (0.1) a *Selberg type integral*. It is known that (0.1) satisfies a Gauss–Manin system, i.e. a system of rationally holonomic differential equations [3], [21].

The purpose of the present paper is to study the (ir)reducibility conditions of the Gauss–Manin system associated with the integral above. Here the (ir)reducibility of a system means the (ir)reducibility of its monodromy representation.

Throughout our arguments, we adopt a framework of the theory of twisted rational de Rham cohomology and twisted homology, which will be briefly reviewed in Section 1. Let the many-valued n-form $\Phi dt_1 \ldots dt_n$ be the integrand of (0.1). A basis of the twisted n-th de Rham cohomology group attached to Φ naturally induces a system of matrix valued differential equations of the first order, which is called the Gauss–Manin system. Our first result is its explicit expression (Proposition 2.1). Luckily enough, our system happens to have logarithmic poles with residues of constant matrices, which makes our discussion much simpler. A study about (ir)reducibility is made in Section 3. Our second result is the irreducibility condition when $\nu = 0$. In this case, it is shown that if λ_j $(1 \leq j \leq m) \notin \mathbf{Z}$ and $\sum_{j=1}^{m} \lambda_j \notin \mathbf{Z}$ then our system is irreducible (Theorem 3.1). This suggests that

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our Gauss-Manin system is *generically* irreducible, though we could not succeed in finding (ir)reducibility conditions in general setting. If a given system of differential equations has a subsystem, then the monodromy representation of the original one is reducible, of course. It seems an interesting problem to find a subsystem of our system. We find four reducible cases, when $m = 3$ (Theorem 3.3). We also prove that the Lie algebra generated by the residue matrices is isomorphic to the general linear Lie algebra $\mathfrak{gl}(n + 1 ; \mathbf{C})$ (Proposition 3.4). The author believes that this Lie algebra should shed light on the irreducibility conditions in the future. When m is arbitrary, we obtain a reducibility condition and a corresponding subsystem in Theorem 3.5: We find that if $0 \leq \ell \leq n - 1$ and $2\lambda_j + \nu\ell = 0$ ($j = 1, \ldots, m - 2$) then our system is reducible. By virtue of the formula in Theorem 3.5 in the case $\ell = 1$, we can derive a system of the differential equations of the second order which is satisfied by the integral (0.1) (Theorem 4.1). This system is related with the spherical functions of BC type defined in [11].

We finally give a comment that the integral (0.1) is used to give an integral representation of the solution of $\mathfrak{sl}(2)$ Knizhnik-Zamolodchikov equation in Conformal field theory. Its monodromy representation is studied in various aspects and related with our work. See [6], [8], [21], [26].

1. Twisted cohomology and homology

Let $z_1, \ldots, z_m$ be complex parameters. Let

$$(1.1) \qquad \Phi = \prod_{1 \leq i \leq j \leq n} (t_i - t_j)^{\nu} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (t_i - z_j)^{\lambda_j}$$

be a many valued holomorphic function with variables $t_1, \ldots, t_n$ on $X = \mathbf{C}^n \setminus D$, where

$$D := \bigcup_{1 \leq i < j \leq n} \{t_i - t_j = 0\} \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \{t_i - z_j = 0\},$$

and ν and λ_j ($1 \leq j \leq m$) complex numbers. The symmetric group S_n acts freely on X as permutations of the coordinates $t_1, \ldots, t_n$.

Let $\Omega^p(* D)$ be the space of rational p-forms on $\mathbf{C}^n$ that are holomorphic on X. Then the (single-valued holomorphic) 1-form $\omega := d\Phi / \Phi$ on X determines a rationally integrable connection (Gauss-Manin connection)

$$\nabla_\omega : \Omega^p(* D) \ni \varphi \mapsto \nabla_\omega \varphi = d\varphi + \omega \wedge \varphi \in \Omega^{p+1}(* D).$$

Then we have the following complex:

$$0 \to \Omega^0(*D) \xrightarrow{\nabla_\omega} \Omega^1(*D) \xrightarrow{\nabla_\omega} \cdots \xrightarrow{\nabla_\omega} \Omega^n(*D) \to 0.$$

Its p-th cohomology, denoted by $H^p(\Omega^{\cdot}(*D), \nabla_\omega)$, is called the p-th twisted rational de Rham cohomology associated with the connection form ω. The following fact is fundamental.

THEOREM 1.1 ([3], [9]).   (1) Suppose that

(Mon): non of the values of $i\left(\lambda_j + \dfrac{1}{2}(i-1)\nu\right)$ $(1 \le j \le m, 1 \le i \le n)$,

$\dfrac{1}{2} i(i+1)\nu$ $(1 \le i \le n)$, $(i-1-n)\sum\limits_{k=1}^{m}\lambda_k + \left\{\left(\dbinom{i-1}{2}\right) + \left(\dbinom{n}{2}\right)\right\}\nu$ $(1 \le i \le n)$

lies in $\mathbf{N} \setminus \{0\}$.

Then the symmetric part $H^n(\Omega^{\cdot}(*D), \nabla_\omega)^{S_n}$ of the cohomology $H^n(\Omega^{\cdot}(*D), \nabla_\omega)$ is generated by the symmetrization of the logarithmic forms

(1.2) $d\log(t_1 - z_{i_1}) \wedge d\log(t_2 - z_{i_2}) \wedge \cdots \wedge d\log(t_n - z_{i_n})$

for $1 \le i_1, \ldots, i_n \le m$.

(2) If we suppose $\prod\limits_{0 \le k \le n-1}\left(\lambda_m + \dfrac{\nu}{2}k\right) \neq 0$, in addition to the assumption (Mon), then a basis of $H^n(\Omega^{\cdot}(*D), \nabla_\omega)^{S_n}$ is given by the symmetrization of

(1.3) $d\log(t_1 - z_{i_1}) \wedge d\log(t_2 - z_{i_2}) \wedge \cdots \wedge d\log(t_n - z_{i_n})$

for $1 \le i_1, \ldots, i_n \le m-1$.

Then the rank of $H^n(\Omega^{\cdot}(*D), \nabla_\omega)^{S_n}$ is $\dbinom{n+m-2}{m-2}$.

This theorem was first proved by Aomoto [3]. His proof depends on the arguments for the generalized Pochhammer differential equations. On the other hand, Esnault–Schechtman–Viehweg gave another proof of (1) under more general setting (They treat the case of arbitrary hyperplane arrangements [9].). Their proof is given by the combination of the arguments for the degeneracy of the Hodge–Deligne spectral sequence and a result by Brieskorn [5].

Let S_ω be the local system on X defined by the monodromy of Φ and S_ω^* the dual local system of S_ω. Let $H_p(X, S_\omega^*)$ be the p-th homology group with coefficients in S_ω^*. Then we have the perfect pairing (see [7], [23])

$$H_p(X, S_\omega^*) \times H^p(\Omega^{\cdot}(*D), \nabla_\omega) \to \mathbf{C},$$
$$(\Gamma, \varphi) \mapsto \int_\Gamma \varphi \Phi dt_1 \ldots, dt_n,$$

which induces the pairing of the symmetric parts:

$$H_p(X,\, S_\omega^*)^{S_n} \times H^p(\Omega^\cdot(*D),\, \nabla_\omega)^{S_n} \to \mathbf{C}.$$

In this paper we consider only the $(S_n\text{-})$symmetric (co)homology, i.e., the (twisted) cycles Γ in the integral (0.1) are so chosen that

$$(1.4) \qquad \int_\Gamma \varphi(\Phi dt_1\ldots,dt_n)^\sigma = \int_\Gamma \varphi \Phi dt_1\ldots,dt_n$$

for all symmetric rational functions $\varphi \in \mathbf{C}(t_1,\ldots,t_n)$ and $\sigma \in S_n$.

We refer the reader to [1], [14], [15], [16], [21], [25] for related works with the twisted cohomology and homology.

By using this framework, in the next section, we derive a Gauss–Manin system associated with this Selberg type integral.

2. Gauss–Manin system

This section is devoted to deriving a system of matrix valued differential equations of the first order associated with the Selberg type integral

$$\int_\Gamma \prod_{1\le i\le j\le n} (t_i - t_j)^\nu \prod_{\substack{1\le i\le n \\ 1\le j\le m}} (t_i - z_j)^{\lambda_j}\, dt_1 dt_2\ldots dt_n$$

with variables $z_1,\ldots,z_m$. In what follows we suppose $m \ge 2$ and fix a cycle Γ satisfying (1.4).

Notice that for a rational function φ, we have

$$(2.1) \qquad \frac{\partial}{\partial z_s} \int_\Gamma \varphi \Phi\, dt_1 dt_2\ldots dt_n = \int_\Gamma (\nabla_{z_s}\varphi)\, \Phi\, dt_1 dt_2\ldots dt_n,$$

where

$$\nabla_{z_s} := \frac{\partial}{\partial z_s} + \frac{\partial \log \Phi}{\partial z_s}.$$

For brevity we put

$$(2.2) \qquad \varphi_{(a_1,\ldots,a_n)} := \prod_{1\le i\le n} (t_i - z_{a_i})^{-1}$$

for $1 \le a_i \le m$ $(1 \le i \le n)$ and also use the notation

$$\varphi_{(1^{k_1},2^{k_2},\ldots,m^{k_m})} := \varphi_{(\underbrace{1,\ldots,1}_{k_1},\underbrace{2,\ldots,2}_{k_2},\ldots,\underbrace{m,\ldots,m}_{k_m})}$$

for $k_1, \ldots, k_m$ such that $k_i \geq 0$ $(1 \leq i \leq m)$ and $k_1 + \cdots + k_m = n$. Moreover we write

$$\langle \varphi \rangle := \int_\Gamma \varphi \Phi \, dt_1 dt_2 \ldots dt_n$$

for all symmetric twisted cycles Γ, and $\varphi_1 \equiv \varphi_2$ for $\langle \varphi_1 \rangle = \langle \varphi_2 \rangle$.

Under these notations, we derive a system of differential equations of the first order (Gauss–Manin system) with respect to the basis

$$(2.3) \qquad \{\langle \varphi_{(1^{k_1}, 2^{k_2}, \ldots, (m-1)^{k_{m-1}})} \rangle \; ; k_i \geq 0 \; (1 \leq i \leq m-1), \sum_{1 \leq i \leq m-1} k_i = n\}$$

Note that the cardinality of the set (2.3) is $\begin{pmatrix} n+m-2 \\ m-2 \end{pmatrix}$.

PROPOSITION 2.1.   (i) *For* $r = 1, \ldots, m-1$, *we have*

$$(2.4) \qquad \frac{\partial}{\partial z_r} \langle \varphi_{(1^{k_1}, 2^{k_2}, \ldots, (m-1)^{k_{m-1}})} \rangle =$$

$$- \left(\lambda_r + \frac{\nu}{2} k_r \right) \sum_{\substack{1 \leq s \leq m-1 \\ s \neq r}} \frac{k_s}{z_r - z_s} \langle \varphi_{(\ldots, r^{k_r+1}, \ldots, s^{k_s-1}, \ldots)} \rangle$$

$$+ \left[\sum_{\substack{1 \leq s \leq m-1 \\ s \neq r}} \frac{k_r k_s \nu + \lambda_s k_r + \lambda_r k_s}{z_r - z_s} + \frac{k_r \{\lambda_r + \lambda_m + \frac{\nu}{2} (k_r - 1)\}}{z_r - z_m} \right]$$

$$\times \langle \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} \rangle$$

$$+ \sum_{\substack{1 \leq s \leq m-1 \\ s \neq r}} k_r \left(\lambda_s + \frac{\nu}{2} k_s \right) \left\{ \frac{1}{z_r - z_m} - \frac{1}{z_r - z_s} \right\}$$

$$\times \langle \varphi_{(\ldots, r^{k_r-1}, \ldots, s^{k_s+1}, \ldots)} \rangle,$$

and

$$(2.5) \qquad \frac{\partial}{\partial z_m} \langle \varphi_{(1^{k_1}, 2^{k_2}, \ldots, (m-1)^{k_{m-1}})} \rangle =$$

$$\sum_{1 \leq s \leq m-1} \frac{k_s \{\lambda_m + \lambda_s + \frac{\nu}{2} (k_s - 1)\}}{z_m - z_s} \langle \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} \rangle$$

$$+ \sum_{1 \leq s \leq m-1} \sum_{\substack{1 \leq r \leq m-1 \\ r \neq s}} \frac{k_s \left(\lambda_r + \frac{\nu}{2} k_r \right)}{z_m - z_s} \langle \varphi_{(\ldots, s^{k_s-1}, \ldots, r^{k_r+1}, \ldots)} \rangle.$$

Proof. First we derive the equality (2.7) used below. By the Stokes theorem, we have

$$(2.6) \qquad 0 \equiv \frac{\nu}{2} \sum_{1 \le j \le n-1} \varphi_{(a_1,\ldots,a_{n-1},a_j)} + \sum_{1 \le a \le m} \lambda_a \, \varphi_{(a_1,\ldots,a_{n-1},a)} \, ;$$

indeed, the right hand side of (2.6) is

$$\sum_{j(\neq n)} \frac{\nu}{2} \left[\frac{1}{t_n - t_j} \frac{1}{t_j - z_{a_i}} + \frac{1}{t_j - t_n} \frac{1}{t_n - z_{a_j}} \right] \sum_{\substack{1 \le i \le n-1 \\ i \neq j}} (t_i - z_{a_i})^{-1}$$

$$+ \prod_{1 \le i \le n-1} (t_i - z_{a_i})^{-1} \sum_{1 \le a \le m} \lambda_a (t_n - z_a)^{-1}$$

$$\equiv \left[\sum_{j(\neq n)} \frac{\nu}{t_n - t_j} + \sum_{1 \le a \le m} \frac{\lambda_a}{t_n - t_a} \right] \prod_{1 \le i \le n-1} (t_i - z_{a_i})^{-1}$$

$$= \nabla_{t_n} \prod_{1 \le i \le n-1} (t_i - z_{a_i})^{-1} \equiv 0.$$

Set

$$a_1 = \cdots = a_{k_1 - 1} = 1,$$
$$a_{k_1 + \cdots + k_{s-1}} = \cdots = a_{k_1 + \cdots + k_s - 1} = s \text{ for } 2 \le s \le m - 1$$

and $a_n = m$ in the relation (2.6). Then the equality

$$(2.7) \qquad 0 \equiv \left\{ \lambda_1 + \frac{\nu}{2} (k_1 - 1) \right\} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$+ \sum_{2 \le s \le m-1} \left(\lambda_s + \frac{\nu}{2} k_s \right) \varphi_{(1^{k_1-1}, \ldots, s^{k_s+1}, \ldots)}$$

$$+ \lambda_m \, \varphi_{(1^{k_1-1}, \ldots, (m-1)^{k_{m-1}}, m^1)}$$

holds.

By the help of the equality (2.7), we derive the equations (2.4) and (2.5).

Direct calculation shows

$$(2.8) \qquad \nabla_{z_1} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$= \left[\sum_{1 \le i \le k_1} \left\{ - \nabla_{t_i} + \sum_{j(\neq i)} \frac{\nu}{t_i - t_j} + \sum_{1 \le s \le m} \frac{\lambda_s}{t_i - z_s} \right\} + \frac{\partial \log \Phi}{\partial z_1} \right]$$

$$\times \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$\equiv \left[\sum_{1 \le i \le k_1} \left\{ \sum_{j(\neq i)} \frac{\nu}{t_i - t_j} + \sum_{2 \le s \le m} \frac{\lambda_s}{t_i - z_s} \right\} + \sum_{k_1 + 1 \le i \le n} \frac{-\lambda_1}{t_i - z_1} \right]$$

$$\times \; \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$\equiv \left[k_1 \left\{ \sum_{k_1+1 \leq j \leq n} \frac{\nu}{t_1 - t_j} + \sum_{2 \leq s \leq m} \frac{\lambda_s}{t_1 - z_s} \right\} - \sum_{k_1+1 \leq i \leq n} \frac{\lambda_1}{t_i - z_1} \right]$$

$$\times \; \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}.$$

By virtue of the equality

$$\frac{1}{t_1 - t_j} \frac{1}{t_1 - z_1} \frac{1}{t_j - z_s} + \frac{1}{t_j - t_1} \frac{1}{t_j - z_1} \frac{1}{t_1 - z_s}$$

$$= \frac{-1}{z_1 - z_s} \left[\frac{1}{t_1 - z_1} - \frac{1}{t_1 - z_s} \right] \left[\frac{1}{t_j - z_1} - \frac{1}{t_j - z_s} \right],$$

we have

$$(2.9) \qquad \sum_{k_1+1 \leq j \leq n} \frac{\nu}{t_1 - t_j} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$= - \sum_{2 \leq s \leq m-1} \frac{k_s}{2(z_1 - z_s)} \{ \varphi_{(1^{k_1+1}, \ldots, s^{k_s-1}, \ldots)} - 2\varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} + \varphi_{(1^{k_1-1}, \ldots, s^{k_s+1}, \ldots)} \}.$$

The equality

$$(2.10) \qquad \frac{1}{t_i - z_1} \frac{1}{t_i - z_s} = \frac{1}{z_1 - z_s} \left[\frac{1}{t_i - z_1} - \frac{1}{t_i - z_s} \right]$$

and (2.7) lead to

$$(2.11) \qquad \sum_{2 \leq s \leq m} \frac{\lambda_s}{t_1 - z_s} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$= \sum_{2 \leq s \leq m-1} \frac{\lambda_s}{z_1 - z_s} \{ \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} - \varphi_{(1^{k_1-1}, \ldots, s^{k_s+1}, \ldots)} \}$$

$$+ \frac{\lambda_m}{z_1 - z_m} \{ \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} - \varphi_{(1^{k_1-1}, \ldots, (m-1)^{k_{m-1}}, m^1)} \}$$

$$\equiv \sum_{2 \leq s \leq m-1} \frac{\lambda_s}{z_1 - z_s} \{ \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} - \varphi_{(1^{k_1-1}, \ldots, s^{k_s+1}, \ldots)} \}$$

$$+ \frac{\lambda_m + \lambda_1 + \dfrac{\nu}{2}(k_1 - 1)}{z_1 - z_m} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$+ \sum_{2 \leq s \leq m-1} \frac{\lambda_s + \dfrac{\nu}{2} k_s}{z_1 - z_m} \varphi_{(1^{k_1-1}, \ldots, s^{k_s+1}, \ldots)}.$$

Similarly, the equality (2.10) gives

(2.12)
$$\sum_{k_1+1 \leq i \leq n} \frac{1}{t_i - z_1} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$= \sum_{2 \leq s \leq m-1} \frac{k_s}{z_s - z_1} \{\varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} - \varphi_{(1^{k_1+1}, \ldots, s^{k_s-1}, \ldots)}\}.$$

Therefore we get the differential equation (2.4) with $r = 1$ by substituting (2.9), (2.11) and (2.12) into (2.8). The equations with respect to $\nabla_{z_r} (2 \leq r \leq m-1)$ are obtained by changing the suffices accordingly.

The equation for $\partial/\partial z_m$ is given by the same way as follows. First we note the equality

(2.13)
$$0 \equiv \left\{\lambda_s + \frac{\nu}{2}(k_s - 1)\right\} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$+ \sum_{\substack{1 \leq t \leq m-1 \\ t \neq s}} \left(\lambda_t + \frac{\nu}{2} k_t\right) \varphi_{(\ldots, s^{k_s-1}, \ldots, t^{k_t+1}, \ldots)}$$

$$+ \lambda_m \varphi_{(\ldots, s^{k_s-1}, \ldots, (m-1)^{k_{m-1}}, m^1)}$$

for $1 \leq s \leq m-1$, which is given by changing the suffices in (2.6) appropriately. By (2.10) and (2.13), we have

(2.14)
$$\nabla_{z_m} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$\equiv \sum_{1 \leq i \leq n} \frac{\lambda_m}{z_m - t_i} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}$$

$$= \sum_{1 \leq s \leq m-1} \frac{\lambda_m k_s}{z_m - z_s} \{\varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} - \varphi_{(\ldots, s^{k_s-1}, \ldots, m^1)}\}$$

$$\equiv \sum_{1 \leq s \leq m-1} \frac{k_s}{z_m - z_s} \left\{\left\{\lambda_m + \lambda_s + \frac{\nu}{2}(k_s - 1)\right\} \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})}\right.$$

$$\left. + \sum_{\substack{1 \leq t \leq m-1 \\ t \neq s}} \left(\lambda_t + \frac{\nu}{2} k_t\right) \varphi_{(\ldots s^{k_s-1}, \ldots, t^{k_t+1}, \ldots)}\right\},$$

which is (2.5).

Remark 2.2.   To derive the system (2.4) and (2.5), we need no conditions.

Remark 2.3.   Our system can be written as

(2.15)
$$d\Psi = \{\sum_{1 \leq i < j \leq m} A_{ij} d \log(z_i - z_j)\} \Psi.$$

Here A_{ij}, $1 \leq i < j \leq m$, are square matrices with size $\binom{n+m-2}{m-2}$ and with $\mathbf{C}$ coefficients, and $\Psi = {}^t(\langle \varphi_{(1^n)} \rangle, \ldots, \langle \varphi_{((m-1)^n)} \rangle)$. It is known ([18]) that (2.15) is integrable if and only if the following conditions are satisfied.

$$[A_{ij}, A_{ik} + A_{jk}] = [A_{ij} + A_{ik}, A_{jk}] = 0 \text{ for } i < j < k,$$

$$[A_{ij}, A_{k\ell}] = 0 \text{ for distinct } i, j, k, \ell.$$

These relations are called the *infinitesimal pure braid relations,* and also called the *classical Yang–Baxter relations,* which play an important role in Mathematical Physics. The linear independence of the basis

$$\{ \langle \varphi_{(1^{k_1},\, 2^{k_2}, \ldots,\, (m-1)^{k_{m-1}})} \rangle \; ; k_i \geq 0, \sum_{1 \leq i \leq m-1} k_i = n \}$$

guarantees the integrability of the system. Once we get the integrability under the conditions (Mon), it is guaranteed without any conditions, since the elements of A_{ij} are linear in λ_j $(j = 1, \ldots, m)$ and ν.

3. Reducibility and irreducibility

In this section, we study the (ir)reducibility of the Gauss-Manin system in Proposition 2.1.

THEOREM 3.1. *When $\nu = 0$, the Gauss-Manin system* (2.4-5) *is irreducible if $\lambda_i \notin \mathbf{Z}_{(1 \leq i \leq m)}$ and $\sum_{i=1}^{m} \lambda_i \notin \mathbf{Z}$.*

Proof. Notice that the integral (0.1) reduces to the product of integrals of Pochhammer type (i.e. case $n = 1$)

$$\int \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (t_i - z_j)^{\lambda_j} \, dt_1 dt_2 \ldots dt_n = \prod_{1 \leq i \leq n} \int \prod_{1 \leq j \leq m} (t_i - z_j)^{\lambda_j} \, dt_i.$$

Takano-Bannai [24] showed that when $n = 1$, if $\lambda_i \notin \mathbf{Z}$ and $\sum_{i=1}^{m} \lambda_i \notin \mathbf{Z}$ then the monodromy representation is irreducible. On the other hand T. Sasaki [20] showed that the Zariski closure of this monodromy group is the general linear group $GL(m-1; \mathbf{C})$. Hence there exists a basis B_1 of the homology $H_1(X, S_\omega^*)$ in the $n = 1$ case which gives a fundamental representation of $GL(m-1; \mathbf{C})$. On the other hand, each element in the homology $H_n(X, S_\omega^*)^{S_n}$ corresponds to the n-fold symmetric tensor of elements in B_1. Thus by Weyl's reciprocity, there exists a basis of $H_n(X, S_\omega^*)^{S_n}$ which gives an irreducible representation of $GL(m-1; \mathbf{C})$,

and consequently of the monodromy representation of (2.4-5).

Theorem 3.1 suggests us that under some *generic* condition for ν and λ_i Gauss-Manin system (2.4-5) is irreducibile, although we could not prove it generally.

We consider the case $m = 3$, for a while.

Thanks to Proposition 2.1, the Gauss-Manin system takes the form

$$(3.1) \qquad d\Psi = \{ \sum_{1 \leq i < j \leq 3} A_{ij} d \log(z_i - z_j) \} \Psi$$

with respect to the basis $\Psi = {}^t(\langle \varphi_{(1^n)} \rangle, \langle \varphi_{(1^{n-1},2^1)} \rangle, \ldots, \langle \varphi_{(2^n)} \rangle)$. Here the matrices $A_{ij} = (a_{st}^{(ij)})_{1 \leq s, t \leq n+1}$ do not depend on the z_j's and are given by

$$(3.2) \qquad \begin{cases} a_{i+1,i}^{(12)} = -i \left(\lambda_1 + \dfrac{\nu}{2}(n-i) \right), \\[2mm] a_{i,i+1}^{(12)} = -(n-i+1) \left(\lambda_2 + \dfrac{\nu}{2}(i-1) \right), \\[2mm] a_{i,i}^{(12)} = (i-1)((n-i+1)\nu + (n-i+1)\lambda_2 + (i-1)\lambda_1), \\[2mm] a_{ij}^{(12)} = 0 \quad \text{otherwise} \end{cases}$$

and

$$(3.3) \qquad \begin{cases} a_{i,i}^{(13)} = (n-i+1) \left(\lambda_1 + \lambda_3 + \dfrac{\nu}{2}(n-i) \right), \\[2mm] a_{i,i+1}^{(13)} = (n-i+1) \left(\lambda_2 + \dfrac{\nu}{2}(i-1) \right), \\[2mm] a_{ij}^{(13)} = 0 \quad \text{otherwise} \end{cases}$$

and

$$(3.4) \qquad \begin{cases} a_{i+1,i}^{(23)} = i \left(\lambda_1 + \dfrac{\nu}{2}(n-i) \right), \\[2mm] a_{i+1,i+1}^{(23)} = i \left(\lambda_2 + \lambda_3 + \dfrac{\nu}{2}(i-1) \right), \\[2mm] a_{ij}^{(23)} = 0 \quad \text{otherwise.} \end{cases}$$

These show the following: Fix an integer k such that $0 \leq k \leq n-1$. If $\lambda_1 + \dfrac{\nu}{2} k = 0$, we have $a_{n-k+1,\,n-k}^{(12)} = a_{n-k+1,\,n-k}^{(13)} = a_{n-k+1,\,n-k}^{(23)} = 0$. This yields a subsystem of the system (3.1) with a basis

$$\{ \varphi_{(1^{k_1}, 2^{k_2})} \, ; \, 0 \leq k_1 \leq k, \ k_1 + k_2 = n \}.$$

Similarly for each k such that $0 \leq k \leq n-1$, if $\lambda_2 + \dfrac{\nu}{2} k = 0$, we have a subsystem with a basis

$$\{ \varphi_{(1^{k_1}, 2^{k_2})} \, ; \, 0 \leq k_2 \leq k, \ k_1 + k_2 = n \}.$$

The following question naturally arises: Are there other subsystems of (3.1)? To answer this question, we transform A_{ij} by the triangular matrix P_{13}. Consider a matrix $P_{13} = (p_{ij}^{(13)})_{1 \le i,j \le n+1}$ defined by

(3.5)
$$\begin{cases} p_{1j}^{(13)} = 1 \text{ for } 1 \le i \le n + 1, \\ p_{ij}^{(13)} = \prod_{t=0}^{i-2} \dfrac{\rho_j^{(13)} - \rho_{t+1}^{(13)}}{\left\{ (n - t) \left(\lambda_2 + \dfrac{\nu}{2} t \right) \right\}} \quad \text{for} \quad 2 \le i \le j \le n + 1, \\ p_{ij}^{(13)} = 0, \quad \text{otherwise,} \end{cases}$$

where $\rho_i^{(13)} = (n - i + 1) \left(\lambda_1 + \lambda_3 + \dfrac{\nu}{2} (n - i) \right)$. Then we have the key lemma.

LEMMA 3.2. *Under the condition*

(3.6)
$$\prod_{t=0}^{n-1} \left(\lambda_2 + \frac{\nu}{2} t \right) \prod_{k=0}^{2n-2} (2\lambda_1 + 2\lambda_3 + k\nu) \ne 0,$$

we have

$$P_{13}^{-1} A_{13} P_{13} = \text{diag}(\rho_1^{(13)}, \rho_2^{(13)}, \dots, \rho_{n+1}^{(13)})$$

and

(3.7)
$$P_{13}^{-1} A_{23} P_{13} = B_{23} = (b_{ij}^{(23)})_{1 \le i,j \le n+1},$$

where

(3.8)
$$\begin{cases} b_{i,i+1}^{(23)} = - i \times \\ \dfrac{\left(\lambda_1 + \lambda_3 + \dfrac{\nu}{2} (n - i - 1) \right) \left(\lambda_3 + \dfrac{\nu}{2} (n - i) \right) \left(\lambda_1 + \lambda_2 + \lambda_3 + \dfrac{\nu}{2} (2n - i - 1) \right)}{(\lambda_1 + \lambda_3 + (n - i)\nu) \left(\lambda_1 + \lambda_3 + \dfrac{\nu}{2} (2n - 2i - 1) \right)}, \\ b_{i+1,i}^{(23)} = - (n - i + 1) \\ \qquad \times \dfrac{\left(\lambda_1 + \dfrac{\nu}{2} (n - i) \right) \left(\lambda_2 + \dfrac{\nu}{2} (i - 1) \right) \left(\lambda_1 + \lambda_3 + \dfrac{\nu}{2} (2n - i) \right)}{(\lambda_1 + \lambda_3 + (n - i)\nu) \left(\lambda_1 + \lambda_3 + \dfrac{\nu}{2} (2n - 2i + 1) \right)}, \\ b_{i,i}^{(23)} = - b_{i+1,i}^{(23)} - b_{i-1,i}^{(23)}, \\ b_{ij}^{(23)} = 0 \quad \textit{otherwise.} \end{cases}$$

Proof.   Since

$$\det P_{13} = \prod_{i=2}^{n+1} \prod_{t=0}^{i-2} \frac{\rho_j^{(13)} - \rho_{t+1}^{(13)}}{\left\{ (n-t)\left(\lambda_2 + \frac{\nu}{2} t\right)\right\}}$$

$$= \prod_{i=2}^{n+1} \prod_{t=0}^{i-2} \frac{(i-t-1)\{(i+t-2n)\nu - 2\lambda_1 - 2\lambda_3\}}{2\,(n-t)\left(\lambda_2 + \frac{\nu}{2} t\right)}$$

the condition $\det P_{13} \neq 0$ is equivalent to (3.6). Then it suffices to prove the equality $A_{23} P_{13} = P_{13} B_{23}$. The (i, j)-entry in this equality is

(3.9) $a_{i,i-1}^{(23)} p_{i-1,j}^{(13)} + a_{i,i}^{(23)} p_{ij}^{(13)}$

$$= p_{i,j-1}^{(13)} b_{j-1,j}^{(23)} + p_{i,j}^{(13)} b_{j,j}^{(23)} + p_{i,j+1}^{(13)} b_{j+1,j}^{(23)}.$$

Note that only the elements at $j = i-1, i, i+1, \ldots, n+1$ are not zero. At $j = i-1$, the above equality is obvious. Thus the only cases $j \geq i$ are crucial; so we suppose $j \geq i$.

First we get

$a_{i,i-1}^{(23)} p_{i-1,j}^{(13)} + a_{i,i}^{(23)} p_{ij}^{(13)}$

$$= (i-1) \prod_{t=0}^{i-2} \frac{\rho_j^{(13)} - \rho_{t+1}^{(13)}}{\left\{ (n-t)\left(\lambda_2 + \frac{\nu}{2} t\right)\right\}}$$

$$\times \left[\frac{(n-i+2)\left(\lambda_1 + \frac{\nu}{2}(n-i+1)\right)\left(\lambda_2 + \frac{\nu}{2}(i-2)\right)}{(i-j-1)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n+2-i-j)\right)} \right.$$

$$\left. + (\lambda_2 + \lambda_3 + \frac{\nu}{2}(i-2)) \right].$$

On the other hand, by noting

$$\prod_{t=0}^{i-2} (\rho_{j-1} - \rho_{t+1}) = \frac{(i-j)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-j+1)\right)}{(1-j)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-i-j+2)\right)} \prod_{t=0}^{i-2} (\rho_j - \rho_{t+1})$$

and

$$\prod_{t=0}^{i-2} (\rho_{j+1} - \rho_{t+1}) = \frac{-j\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-i-j+1)\right)}{(i-j-1)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-j)\right)} \prod_{t=0}^{i-2} (\rho_j - \rho_{t+1}),$$

we obtain

$$p_{i,j-1}^{(13)}\, b_{j-1,j}^{(23)} + p_{i,j}^{(13)}\, b_{j,j}^{(23)} + p_{i,j+1}^{(13)}\, b_{j+1,j}^{(23)}$$

$$= \prod_{t=0}^{i-2} \frac{\rho_j^{(13)} - \rho_{t+1}^{(13)}}{\left\{(n-t)\left(\lambda_2 + \frac{\nu}{2} t\right)\right\}} \left[\frac{(i-j)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-j+1)\right)}{(1-j)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-i-j+2)\right)} b_{j-1,j}^{(23)} \right.$$

$$+ (- b_{j-1,j}^{(23)} - b_{j+1,j}^{(23)}) + \frac{- j\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-i-j+1)\right)}{(i-j-1)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-j)\right)} b_{j+1,j}^{(23)} \left. \right]$$

$$= \prod_{t=0}^{i-2} \frac{\rho_j^{(13)} - \rho_{t+1}^{(13)}}{\left\{(n-t)\left(\lambda_2 + \frac{\nu}{2} t\right)\right\}} \left[\frac{(i-1)(\lambda_1 + \lambda_3 + \nu(n-j+1))}{(1-j)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-i-j+2)\right)} b_{j-1,j}^{(23)} \right.$$

$$- \frac{(i-1)(\lambda_1 + \lambda_3 + \nu(n-j))}{(i-j-1)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-j)\right)} b_{j+1,j}^{(23)} \left. \right]$$

$$= \prod_{t=0}^{i-2} \frac{\rho_j^{(13)} - \rho_{t+1}^{(13)}}{\left\{(n-t)\left(\lambda_2 + \frac{\nu}{2} t\right)\right\}} \frac{i-1}{\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-2j+1)\right)}$$

$$\times \left[\frac{\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(n-j)\right)\left(\lambda_3 + \frac{\nu}{2}(n-j+1)\right)\left(\lambda_1 + \lambda_2 + \lambda_3 + \frac{\nu}{2}(2n-j)\right)}{\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-i-j+2)} \right.$$

$$+ \frac{(n-j+1)\left(\lambda_1 + \frac{\nu}{2}(n-j)\right)\left(\lambda_2 + \frac{\nu}{2}(j-1)\right)}{i-j-1} \left. \right].$$

Therefore the equality (3.9) reduces to

$$\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-2j+1)\right)\left\{(n-i+2)\left(\lambda_1 + \frac{\nu}{2}(n-i+1)\right)\left(\lambda_2 + \frac{\nu}{2}(i-2)\right)\right.$$

$$+ (i-j-1)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-i-j+2)\right)\left(\lambda_2 + \lambda_3 + \frac{\nu}{2}(i-2)\right)\left. \right\}$$

$$= (i-j-1)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(n-j)\right)\left(\lambda_3 + \frac{\nu}{2}(n-j+1)\right)\left(\lambda_1 + \lambda_2 + \lambda_3 + \frac{\nu}{2}(2n-j)\right)$$

$$+ (n-j+1)\left(\lambda_1 + \lambda_3 + \frac{\nu}{2}(2n-i-j+2)\right)\left(\lambda_1 + \frac{\nu}{2}(n-j)\right)\left(\lambda_2 + \frac{\nu}{2}(j-1)\right),$$

which can be checked readily. This completes the proof.   □

Notice that $P_{13}^{-1}A_{13}P_{13}$ is diagonal and $P_{13}^{-1}A_{23}P_{13}$ is tri-diagonal.

THEOREM 3.3.   *If one of the followings holds, the Gauss-Manin connection* (3.1) *is reducible*:

(1) $\lambda_1 + \dfrac{\nu}{2}k = 0$ *for some k such that* $0 \le k \le n - 1$,

(2) $\lambda_2 + \dfrac{\nu}{2}k = 0$ *for some k such that* $0 \le k \le n - 1$,

(3) *under the condition* (3.6),

$$\lambda_3 + \frac{\nu}{2}k = 0 \ \textit{for some k such that} \ 0 \le k \le n - 1,$$

(4). *under the condition* (3.6),

$$\lambda_1 + \lambda_2 + \lambda_3 + \frac{\nu}{2}(2n - k - 2) = 0 \ \textit{for some k such that} \ 0 \le k \le n - 1.$$

Proof.   The assertions (1), (2) have already been proved. The assertions (3) and (4) are derived from Lemma 3.2, since one of the off-diagonal elements of $P_{13}^{-1}A_{23}P_{13}$ vanishes.   □

On the other hand, as a by-product of the argument above, we have

PROPOSITION 3.4.   *Under the condition* (3.6), *the Lie algebra* $\mathscr{L}ie(A_{12}, A_{13}, A_{23})$ *generated by three matrices* A_{12}, A_{13} *and* A_{23} *is isomorphic to the general linear Lie algebra* $\mathfrak{gl}(n + 1 ; \mathbf{C})$.

Proof.   It suffices to show that Lie algebra generated by two matrices $P_{13}^{-1}A_{13}P_{13}$ and $P_{13}^{-1}A_{23}P_{13}$ is isomorphic to $\mathfrak{gl}(n + 1 ; \mathbf{C})$ under the condition (3.11). We suppose (3.11) in what follows. By (1) of Lemma 3.2, $P_{13}^{-1}A_{13}P_{13}$ is a diagonal matrix with different entries each other, which will be written by D_{13}. Any elements of matrix $B_{23} = P_{13}^{-1}A_{23}P_{13}$ are not zero. Hence by considering the adjoint actions $[D_{13}[D_{13}[,,,[D_{13}, B_{23}]]]]$ and Vandermonde determinant, it is seen matrix units $E_{i,i+1}$ and $E_{i+1,i}$ for $1 \le i \le n$ and $E_{i,i}$ for $1 \le i \le n + 1$ are included into the Lie algebra generated by D_{13} and B_{23}. This completes the proof.   □

The author believes that the Lie algebra $\mathscr{L}ie(A_{12}, A_{13}, A_{23})$ generated by the residue matrices of the system (3.1) coincides with the Lie algebra Lie(PV) of the

Picard–Vessiot group PV of (3.1).

We refer the reader for related works [2], [10], [17], [18], [19].

As for general m, we obtain the following.

THEOREM 3.5. *Fix ℓ as $0 \leq \ell \leq n-1$ and $\lambda_i + \dfrac{\nu}{2}\ell = 0$ for $i = 1, \ldots, m-2$. Then the Gauss–Manin system (2.4-5) is reducible, indeed, we have a subsystem of it with a basis*

$$(3.10) \qquad \{\langle \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} \rangle \,;\, 0 \leq k_i \leq \ell \; (1 \leq i \leq m-2),\, 0 \leq k_{m-1} \}.$$

For $r = 1, \ldots, m-2\,;\, k_r = \ell$,

$$(3.11) \qquad \frac{\partial}{\partial z_r} \left\langle \varphi_{(1^{k_1}, \ldots, r^{\ell}, \ldots, (m-1)^{k_{m-1}})} \right\rangle$$

$$= \left[\sum_{\substack{1 \leq s \leq m-1 \\ s \neq r}} \frac{\ell\, k_s \nu + \lambda_s \ell + \lambda_r k_s}{z_r - z_s} + \frac{\ell\left\{ \lambda_r + \lambda_m + \dfrac{\nu}{2}(\ell - 1) \right\}}{z_r - z_m} \right]$$

$$\times \left\langle \varphi_{(1^{k_1}, \ldots, r^{\ell}, \ldots, (m-1)^{k_{m-1}})} \right\rangle$$

$$+ \sum_{\substack{1 \leq s \leq m-2 \\ s \neq r \\ k_s < \ell}} \ell\left(\lambda_s + \frac{\nu}{2} k_s \right) \left\{ \frac{1}{z_r - z_m} - \frac{1}{z_r - z_s} \right\} \left\langle \varphi_{(\ldots, r^{\ell-1}, \ldots, s^{k_s+1}, \ldots)} \right\rangle$$

$$+ \ell\left(\lambda_{m-1} + \frac{\nu}{2} k_{m-1} \right) \left\{ \frac{1}{z_r - z_m} - \frac{1}{z_r - z_{m-1}} \right\} \left\langle \varphi_{(\ldots, r^{\ell-1}, \ldots, (m-1)^{k_{m-1}+1})} \right\rangle,$$

for $r = 1, \ldots, m-2\,;\, k_r < \ell$,

$$(3.12) \qquad \frac{\partial}{\partial z_r} \left\langle \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} \right\rangle =$$

$$- \left(\lambda_r + \frac{\nu}{2} k_r \right) \sum_{\substack{1 \leq s \leq m-1 \\ s \neq r}} \frac{k_s}{z_r - z_s} \left\langle \varphi_{(\ldots, r^{k_r+1}, \ldots, s^{k_s-1}, \ldots)} \right\rangle$$

$$+ \left[\sum_{\substack{1 \leq s \leq m-1 \\ s \neq r}} \frac{k_r k_s \nu + \lambda_s k_r + \lambda_r k_s}{z_r - z_s} + \frac{k_r\left\{ \lambda_r + \lambda_m + \dfrac{\nu}{2}(k_r - 1) \right\}}{z_r - z_m} \right]$$

$$\times \left\langle \varphi_{(1^{k_1}, \ldots, (m-1)^{k_{m-1}})} \right\rangle$$

$$+ \sum_{\substack{1 \leq s \leq m-2 \\ s \neq r \\ k_s < \ell}} k_r\left(\lambda_s + \frac{\nu}{2} k_s \right) \left\{ \frac{1}{z_r - z_m} - \frac{1}{z_r - z_s} \right\} \left\langle \varphi_{(\ldots, r^{k_r-1}, \ldots, s^{k_s+1}, \ldots)} \right\rangle$$

$$+ k_r\left(\lambda_{m-1} + \frac{\nu}{2} k_{m-1} \right) \left\{ \frac{1}{z_r - z_m} - \frac{1}{z_r - z_{m-1}} \right\} \left\langle \varphi_{(\ldots, r^{k_r-1}, \ldots, (m-1)^{k_{m-1}+1})} \right\rangle,$$

for $r = m - 1$,

$$(3.13) \qquad \frac{\partial}{\partial z_{m-1}} \left\langle \varphi_{(1^{k_1},\ldots,(m-1)^{k_{m-1}})} \right\rangle =$$

$$- \left(\lambda_{m-1} + \frac{\nu}{2} k_{m-1} \right) \sum_{1 \leq s \leq m-2} \frac{k_s}{z_{m-1} - z_s} \left\langle \varphi_{(\ldots,s^{k_s-1},\ldots,(m-1)^{k_{m-1}+1})} \right\rangle$$

$$+ \left[\sum_{1 \leq s \leq m-2} \frac{k_{m-1}k_s\nu + \lambda_s k_{m-1} + \lambda_{m-1}k_s}{z_{m-1} - z_s} + \frac{k_{m-1}\left\{ \lambda_{m-1} + \lambda_m + \frac{\nu}{2}(k_{m-1} - 1) \right\}}{z_{m-1} - z_m} \right]$$

$$\times \left\langle \varphi_{(1^{k_1},\ldots,(m-1)^{k_{m-1}})} \right\rangle$$

$$+ \sum_{\substack{1 \leq s \leq m-2 \\ k_s < l}} k_{m-1} \left(\lambda_s + \frac{\nu}{2} k_s \right) \left\{ \frac{1}{z_{m-1} - z_m} - \frac{1}{z_{m-1} - z_s} \right\}$$

$$\times \left\langle \varphi_{(\ldots,s^{k_s+1},\ldots,(m-1)^{k_{m-1}+1})} \right\rangle$$

and for $r = m$,

$$(3.14) \qquad \frac{\partial}{\partial z_m} \left\langle \varphi_{(1^{k_1},\ldots,(m-1)^{k_{m-1}})} \right\rangle =$$

$$\sum_{1 \leq s \leq m-1} \frac{k_s \left\{ \lambda_s + \lambda_m + \frac{\nu}{2}(k_s - 1) \right\}}{z_m - z_s} \left\langle \varphi_{(1^{k_1},\ldots,(m-1)^{k_{m-1}})} \right\rangle$$

$$+ \sum_{1 \leq s \leq m-1} \sum_{\substack{1 \leq r \leq m-2 \\ r \neq s \\ k_r < \ell}} \frac{k_s \left\{ \lambda_r + \frac{\nu}{2}(k_r - 1) \right\}}{z_m - z_s} \left\langle \varphi_{(\ldots,s^{k_s-1},\ldots,r^{k_r+1},\ldots)} \right\rangle$$

$$+ \sum_{1 \leq s \leq m-2} \frac{k_s \left(\lambda_{m-1} + \frac{\nu}{2} k_{m-1} \right)}{z_m - z_s} \left\langle \varphi_{(\ldots,s^{k_s-1},\ldots,(m-1)^{k_{m-1}+1})} \right\rangle.$$

Proof. It is easily seen from the formulas in Proposition 2.1.

Remark 3.6. The dimension of our system is the cardinarity of the set

$$\{(k_1,\ldots,k_{m-1}) ; 0 \leq k_i \leq \ell \ (1 \leq i \leq m - 2), 0 \leq k_{m-1}, k_1 + \cdots k_{m-1} = n\}.$$

For instances, it is $(\ell + 1)^{m-2}$ if $n \geq (m-2)\ell$, and $\binom{n+m-2}{m-2}$ if $\ell \geq n \geq 1$.

Remark 3.7. In case $\ell = 1$, which will be discussed in the next section, the dimension of the system is estimated as

$$N(m, n) := 1 + \binom{m-2}{1} + \cdots + \binom{m-2}{n},$$

which is 2^{m-2} if $n \geq m - 2$.

4. Spherical functions of BC type

As an application of Theorem 3.5 in the case $l = 1$, we derive a system of differential equations of the second order which is satisfied by the integral (0.1), which is written by $\langle 1 \rangle$. It is known by Beerends-Opdam [4] and Kaneko [13] that this system characterizes some generalized hypergeometric series of matrix arguments under some conditions. Moreover, it was shown by Beerends-Opdam [4] that this generalized hypergeometric series is a special case of the hypergeometric function associated with the root system of BC_m type defined in [11], [12].

THEOREM 4.1. *For* $r = 1, \ldots, m - 2$, *put*

$$\begin{aligned}
(4.1) \quad \Delta_r = &(z_r - z_m)(z_r - z_{m-1})\partial_r^2 \\
&+ \left\{\left(\frac{\nu}{2} - \lambda_{m-1}\right)(z_r - z_m) - (\lambda_r + \lambda_m)(z_r - z_{m-1})\right\}\partial_r \\
&+ n\lambda_r\left\{1 + \lambda_m + \lambda_{m-1} + \frac{\nu}{2}(n - m + 1)\right\} \\
&+ \frac{\nu}{2}(z_r - z_{m-1})(z_r - z_m)\sum_{\substack{1 \leq s \leq m-2 \\ s \neq r}}\frac{1}{z_r - z_s}\partial_r \\
&+ \frac{\nu}{2}\sum_{\substack{1 \leq s \leq m-2 \\ s \neq r}}\frac{(z_s - z_{m-1})(z_m - z_s)}{z_r - z_s}\partial_s,
\end{aligned}$$

where $\partial_i = \partial/\partial z_i$ *and* $\lambda_r = -\nu/2$. *Then we have* $\Delta_r \langle 1 \rangle = 0$ *for* $r = 1, \ldots, m - 2$.

Proof. Fix r ($r = 1, \ldots, m - 2$). By (3.12), we have

$$(4.2) \quad \partial_r \langle \varphi_{((m-1)^n)} \rangle = \frac{n\lambda_r}{z_r - z_{m-1}}\{\langle \varphi_{((m-1)^n)} \rangle - \langle \varphi_{(r^1, (m-1)^{n-1})} \rangle\}.$$

Hence

$$(4.3) \quad \langle \varphi_{(r^1, (m-1)^{n-1})} \rangle = \langle \varphi_{((m-1)^n)} \rangle - \frac{z_r - z_{m-1}}{n\lambda_r}\partial_r \langle \varphi_{((m-1)^n)} \rangle.$$

On the other hand, by (3.11) we have

$$(4.4) \qquad \partial_r \langle \varphi_{(r^1, (m-1)^{n-1})} \rangle$$

$$= \left\{ \sum_{\substack{1 \le s \le m-2 \\ s \ne r}} \frac{\lambda_s}{z_r - z_s} + \frac{(n-1)(\nu + \lambda_r) + \lambda_{m-1}}{z_r - z_{m-1}} + \frac{\lambda_r + \lambda_m}{z_r - z_m} \right\} \langle \varphi_{(r^1, (m-1)^{n-1})} \rangle$$

$$+ \sum_{\substack{1 \le s \le m-2 \\ s \ne r}} \lambda_s \left\{ \frac{1}{z_r - z_m} - \frac{1}{z_r - z_s} \right\} \langle \varphi_{(s^1, (m-1)^{n-1})} \rangle$$

$$+ \left(\lambda_{m-1} + \frac{\nu}{2}(n-1) \right) \left\{ \frac{1}{z_r - z_m} - \frac{1}{z_r - z_{m-1}} \right\} \langle \varphi_{((m-1)^n)} \rangle .$$

By substituting (4.3) into both sides of (4.4), we finally obtain

$$(4.5)$$

$$\left\{ (z_r - z_m)(z_r - z_{m-1}) \partial_r^2 + \left[\left(1 + \frac{\nu}{2} - \lambda_{m-1} \right)(z_r - z_m) - (\lambda_r + \lambda_m)(z_r - z_{m-1}) \right] \partial_r \right.$$

$$+ n\lambda_r \left\{ \lambda_m + \lambda_{m-1} + \frac{\nu}{2}(n - m + 1) \right\} + \frac{\nu}{2}(z_r - z_{m-1})(z_r - z_m) \sum_{\substack{1 \le s \le m-2 \\ s \ne r}} \frac{1}{z_r - z_s} \partial_r$$

$$\left. + \frac{\nu}{2} \sum_{\substack{1 \le s \le m-2 \\ s \ne r}} \frac{(z_s - z_{m-1})(z_m - z_s)}{z_r - z_s} \partial_r \right\} \langle \varphi_{((m-1)^n)} \rangle = 0.$$

The equation (4.1) is given by substituting $\lambda_{m-1} + 1$ for λ_{m-1} in (4.5).

Remark 4.2. If we put $z_m = 0$ and $z_{m-1} = 1$ in (4.1), then the operators Δ_r $(1 \le r \le m-2)$ turn out to be

$$(4.6) \qquad \Delta_r = z_r(1 - z_r) \partial_r^2 + \left\{ \left(\frac{\nu}{2} - \lambda_m \right) - (\nu - \lambda_m - \lambda_{m-1}) z_r \right\} \partial_r$$

$$+ \frac{n\nu}{2} \left\{ 1 + \lambda_m + \lambda_{m-1} + \frac{\nu}{2}(n - m + 1) \right\}$$

$$+ \frac{\nu}{2} z_r(1 - z_r) \sum_{\substack{1 \le s \le m-2 \\ s \ne r}} \frac{1}{z_r - z_s} \partial_r$$

$$- \frac{\nu}{2} \sum_{\substack{1 \le s \le m-2 \\ s \ne r}} \frac{z_s(1 - z_s)}{z_r - z_s} \partial_s.$$

Remark 4.3. By taking various $\Gamma \in H_n(X, S_\omega^*)^{S_n}$, we can construct $N(m, n)$ solutions of (4.6) expressed by the integral (0.1) (see Remark 3.7). Notice

that

$$
\begin{cases}
N(m,\, n) = 2^{m-2}, \text{ when } n \geq m - 2, \\
N(m,\, n) < 2^{m-2}, \text{ when } n < m - 2.
\end{cases}
$$

In [13], Kaneko states without proof that the rank of the system (4.6) is 2^{m-2} (the author has been told by him that he has proved this fact). The author does not know if the rank of the system (4.6) equals to $N(m,\, n)$ when $n < m - 2$.

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