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ON THE PARTITION MONOID AND SOME RELATED SEMIGROUPS

D. G. FITZGERALD ✉ and KWOK WAI LAU

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Abstract

The partition monoid is a salient natural example of a *-regular semigroup. We find a Galois connection between elements of the partition monoid and binary relations, and use it to show that the partition monoid contains copies of the semigroup of transformations and the symmetric and dual-symmetric inverse semigroups on the underlying set. We characterize the divisibility preorders and the natural order on the (straight) partition monoid, using certain graphical structures associated with each element. This gives a simpler characterization of Green's relations. We also derive a new interpretation of the natural order on the transformation semigroup. The results are also used to describe the ideal lattices of the straight and twisted partition monoids and the Brauer monoid.

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1. Diagrams and products

Partition algebras, which are twisted semigroup algebras of the partition monoids, are important in the theory of group representations, combinatorics, and statistical mechanics, and have an extensive literature including significant studies in [7, 13, 14]. Generators and relations for partition monoids have been studied in [3], and their endomorphisms in [15]. Wilcox [20, Section 7] studied the structure of the partition monoid in an application of his quite general theorem about the cellularity of twisted semigroup algebras of regular semigroups. It is our intention to investigate the structure of partition monoids further. We use this first section to describe the elements of the partition monoid P_X and their multiplication, and to draw attention to some of the subsemigroups of P_X which are interesting in their own right.

Let X be a set. A *diagram over* X is an equivalence class of graphs on a vertex set $X \cup X'$ (consisting of two copies of X). Two such graphs are regarded as equivalent if they have the same connected components. We define P_X as the set of all diagrams over X. Also, if X is finite, say $X = \{1, 2, \ldots, n\}$, we conventionally write P_n in place of P_X, and similarly for other families of semigroups.

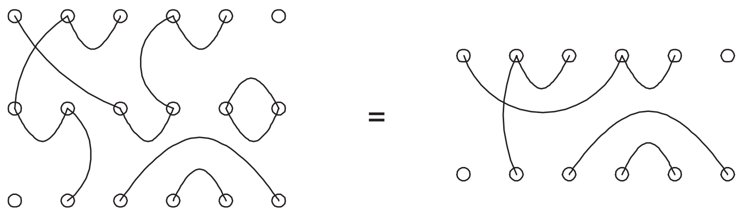

FIGURE 1. Multiplication of a and b in P_6 (left, expanded form; right, contracted form).

We need to have a standard representative of each equivalence class; for this purpose, we choose a graph with maximal edge set, so that each of its components is a complete graph. Proofs below will generally use standard representatives. When it comes to *drawing* the graphs, however, it is convenient to use a minimal number of edges, as in Figure 1. We draw the graph of $a \in P_X$ so that the undashed vertices (elements of X) are arranged in a horizontal row, and the corresponding dashed vertices (those in X') are directly below. Thus we will refer to the undashed elements as the *upper* vertices, and the dashed elements as the *lower* vertices.

To multiply two elements a, b of P_X, their diagrams are first drawn stacked vertically, with a above b. Then the lower vertices of a are identified with the upper vertices of b, in an 'interior' row called an *interface;* we call this diagram the *expanded* form of the product. Next, the connected components of the expanded form are constructed. Finally, we ignore the vertices in the interface, and any components using only these vertices. This results in another member of P_X, which defines the product ab; we call it the *contracted* form in contrast to the expanded form. Note too that a path in the expanded form becomes an edge in the standard representative of the contracted form. An example with $a, b \in P_6$ is seen in Figure 1. Here a has components

$$\{1, 3'\}, \{2, 3, 1'\}, \{4, 5, 4'\}, \{6\}, \{2'\}, \{5', 6'\}$$

and b has components

$$\{1, 2, 2'\}, \{3, 4\}, \{5, 6\}, \{1'\}, \{3', 6'\}, \{4', 5'\}.$$

We call P_X, with this multiplication, the *partition monoid* on the set X because the components of a diagram are the blocks of a partition of $X \cup X'$. An edge of a member a of P_X is called *transversal* if it is of the form $\{i, j'\}$ with $i \in X$ and $j' \in X'$. Likewise a component of a is *transversal* if its vertices include both upper and lower elements (and so it includes a transversal edge), and otherwise the component is *nontransversal*. For example, in Figure 1, the product ab has one transversal component, $\{2, 3, 2'\}$.

A Galois connection with binary relations.

DEFINITIONS. Let Rel_X denote the set of binary relations on X. We define a mapping $F : P_X \longrightarrow \mathrm{Rel}_X$ as follows. With $a \in P_X$ in standard form (union of complete

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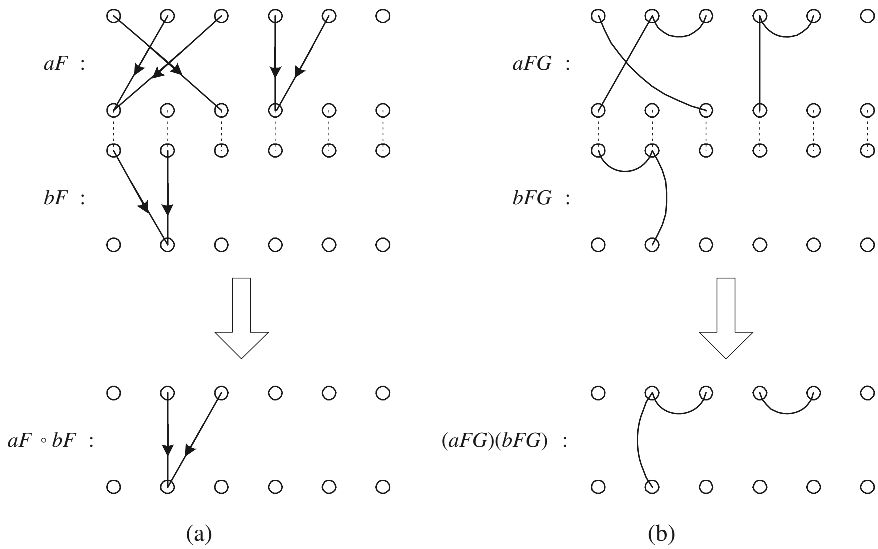

FIGURE 2. The action of the maps F and G : (a) F applied to diagrams a and b in Figure 1, and the composite relation $aF \circ bF$; (b) images under G of the relations in (a), and their product as diagrams.

subgraphs), put

$$aF = \{(i, j) \in X \times X : \{i, j'\} \in a \text{ for } i \in X \text{ and } j' \in X'\},$$

the relation on X induced by the transversal edges of a. In the reverse direction, we define $G : \mathrm{Rel}_X \longrightarrow P_X$ thus:

> for $\rho \in \mathrm{Rel}_X$, ρG is the graph on $X \cup X'$ with edge set being all (finite-length) paths produced from the edges $\{\{i, j'\} : (i, j) \in \rho\}$.

In consequence, ρG is an element of P_X, in fact in standard representative form. These definitions are illustrated in Figure 2.

We say a diagram is *earthed* if all its nontransversal components are singletons. We remind the reader that $\rho \in \mathrm{Rel}_X$ is *bifunctional* if $\rho \circ \rho^{-1} \circ \rho \subseteq \rho$, where $\circ$ means composition of binary relations and ρ^{-1} is the inverse relation of ρ.

We shall show that F, G constitute a Galois connection between Rel_X and P_X, in which the closed elements are the earthed diagrams and the bifunctional relations.

THEOREM 1.1.

(i) *F and G are monotone with respect to the usual inclusion orders; and*
(ii) *$GFG = G$ and $F = FGF$.*

For all $\rho \in \mathrm{Rel}_X$ and $a \in P_X$:

(iii) *$\rho G \subseteq a$ if and only if $\rho \subseteq aF$;*

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(iv) $aFG \subseteq a$, *with equality if and only if a is earthed; and*
(v) $\rho \subseteq \rho GF$, *with equality if and only if ρ is bifunctional.*

PROOF. (i) Suppose that $a, b \in P_X$ and $a \subseteq b$. If $(i, j) \in aF$, then $\{i, j'\}$ is an edge of a and hence of b. So $(i, j) \in bF$, and thus $aF \subseteq bF$. Suppose that $\rho, \sigma \in \mathrm{Rel}_X$ and $\rho \subseteq \sigma$. If $\{i, j\}$ is an edge in ρG, there is a path $(i, i_1, i_2, \ldots, i_m, j)$ with successive pairs $(i, i_1), (i_1, i_2), \ldots, (i_m, j)$ in ρ or ρ^{-1} and so in $\sigma \cup \sigma^{-1}$, whence i, j are in the same component of σG. So $\rho G \subseteq \sigma G$.

(ii) This will be proved last.

(iii) Suppose that $\rho G \subseteq a$ and $(i, j) \in \rho$. Then $\{i, j'\}$ is a transversal edge of ρG, hence of a. So $(i, j) \in aF$. Conversely, suppose that $\rho \subseteq aF$ and $\{i, j\}$ is an edge of ρG. Then there is a path $(i, i_1, i_2, \ldots, i_m, j)$ with successive pairs (i, i_1), $(i_1, i_2), \ldots, (i_m, j)$ in ρ or ρ^{-1}, hence in aF or $(aF)^{-1}$. Thus i, j are in the same component of a, so $\rho G \subseteq a$.

(iv) Taking $\rho = aF$ in (iii) implies that $aFG \subseteq a$. For any $\rho \in \mathrm{Rel}_X$, ρG is earthed by definition, so $a = aFG$ implies that a is earthed. Conversely, suppose that $a \in P_X$ is earthed and $\{i, j\}$ is an edge of a, with $i, j \in X \cup X'$. There is a path in a from i to j using only transversal edges. Therefore either $\{i, j\}$ is transversal, or there are transversal edges $\{i, k\}$ and $\{j, k\}$ in a. Then $(i, j) \in aF \cup (aF)^{-1}$ or $(i, k), (j, k) \in aF \cup (aF)^{-1}$. In either case, $\{i, j\}$ is an edge of aFG. Thus $a \subseteq aFG$. Together with $aFG \subseteq a$, this gives $a = aFG$.

(v) Taking $a = \rho G$ in (iii) implies that $\rho \subseteq \rho GF$. If $(i, k), (j, k), (j, l) \in aF$, then i, j, k', l' are in the same component of a, and so $\{i, l'\}$ is an edge of a. So $(i, l) \in aF$ and $aF = \rho GF$ is bifunctional. Hence $\rho = \rho GF$ implies that ρ is bifunctional. Conversely, suppose that ρ is bifunctional. If $(i, j) \in \rho GF$, then $\{i, j'\}$ is a transversal edge of ρG and so there is a path $(i, i_1, i_2, \ldots, i_{2m}, j)$ in ρG with $(i, i_1) \in \rho, (i_1, i_2) \in \rho^{-1}, \ldots, (i_{2m}, j) \in \rho$ (alternately in ρ and ρ^{-1}). Thus $(i, j) \in (\rho \circ \rho^{-1})^m \circ \rho$ and, by bifunctionality, $(i, j) \in \rho$. So $\rho = \rho GF$.

Finally, (ii) follows as usual: for any ρ and $\rho G = a$, (iv) gives $\rho GFG = \rho G$. The proof of (v) shows that aF is bifunctional for any a, and then we have $aF = aFGF$. □

It follows that the posets of earthed diagrams and bifunctional relations are isomorphic, under restrictions of the maps F and G. The natural multiplications on P_X and Rel_X are not respected by F and G in general, as may be seen in Figure 2, but there is a special case which is relevant to our concerns here.

PROPOSITION 1.2. *For all* $\rho, \sigma \in \mathrm{Rel}_X$, $(\rho \circ \sigma)G = (\rho G)(\sigma G)$ *if and only if* $(\rho G)(\sigma G)$ *is earthed.*

PROOF. As a preliminary, we shall show that in general $(\rho \circ \sigma)G \subseteq (\rho G)(\sigma G)$. Let $\{i, j\}$ be an edge of $(\rho \circ \sigma)G$, so there is a path $(i, i_1, i_2, \ldots, i_m, j)$ with successive pairs $(i, i_1), (i_1, i_2), \ldots, (i_m, j)$ in ρ, σ, ρ^{-1} or σ^{-1}, hence edges $\{i, i_1\}, \{i_1, i_2\}, \ldots$ in the same component of $(\rho G)(\sigma G)$. Thus $\{i, j\}$ is an edge of $(\rho G)(\sigma G)$.

So to the main proof. To prove the 'if' part, suppose that $(\rho G)(\sigma G)$ is earthed, and $\{i, j\}$ is an edge of $(\rho G)(\sigma G)$. Note that ρG and σG are earthed diagrams. Case (1):

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if $i \in X$ and $j \in X'$, there is a path $(i, i_1, i_2, \ldots, i_m, j)$ in $(\rho G)(\sigma G)$ such that

$$(i, i_1) \in \rho,\ (i_1, i_2) \in \sigma,\ (i_2, i_3) \in \sigma^{-1},\ (i_3, i_4) \in \rho^{-1}, \ldots, (i_m, j) \in \sigma.$$

Thus

$$(i, i_2) \in \rho \circ \sigma,\ (i_2, i_4) \in (\rho \circ \sigma)^{-1}, \ldots, (i_{m-1}, j) \in \rho \circ \sigma,$$

and so $\{i, j'\}$ is an edge of $(\rho \circ \sigma)G$. Case (2): if $i, j \in X$, there is $k \in X$ such that $\{i, k'\}$, $\{j, k'\}$ are transversal edges of $(\rho G)(\sigma G)$. The argument of case (1) shows that $\{i, k'\}$, $\{j, k'\}$ and so also $\{i, j\}$ are edges of $(\rho \circ \sigma)G$. The remaining cases ($i \in X'$ and $j \in X$; $i, j \in X'$) are similar.

Turning to the 'only if' part, if equality holds, then $(\rho G)(\sigma G)$ is earthed by Theorem 1.1(iv). □

Subsemigroups. We may use Proposition 1.2 to show that various kinds of relations form semigroups embeddable in P_X. Consider the restriction of G to the monoid of all functions of X to X, the full transformation monoid T_X. A diagram is an image under G of such a function if and only if it is earthed, every upper vertex is in a transversal component, and every component has a unique lower vertex. Similarly, the symmetric inverse monoid I_X of one-to-one relations on X is embedded by G in P_X; images are precisely the earthed diagrams in which transversal components have cardinality two. A bifunctional relation ρ which has both domain and range equal to X is called a *biequivalence* or *block bijection*; it represents a bijection between quotient sets of X. The block bijections, with an appropriate multiplication (not the composition $\circ$) form the dual symmetric inverse monoid I_X^*, studied in [4, 12]. It is easily verified that the multiplication in I_X^* is given by

$$\rho\sigma = ((\rho G)(\sigma G))F, \tag{1.1}$$

and so (using Theorem 1.1(iv)) G embeds I_X^* in P_X. The image of this embedding consists of the diagrams in which all components are transversal. East [3] has shown that, when X is finite, P_X is generated by the images of the symmetric group on X and the idempotents of both I_X and I_X^*; see also [15, Lemma 4.1].

The monoid PT_X of partial transformations of X is not embedded in P_X by G, by Proposition 1.2 (we are indebted to James East for pointing this out); PT_X embeds in T_Y, where $Y = X \sqcup \{0\}$, and hence embeds in P_Y. Alternative choices for multiplication of diagrams are canvassed in [9], and for multiplication of bifunctional relations in [19].

Here we shall concern ourselves with further submonoids of P_X. The *matching monoid* is the submonoid M_X of P_X consisting of matchings, that is to say, elements each of whose components has just two vertices—that is, an edge. This has also been called the *Brauer semigroup* in the literature, but we reserve that name for the twisted version in Section 4 below. For finite $|X| = n$, the *Jones monoid* J_n consists of the matchings which may be drawn in a planar manner in the region between the upper and lower rows. We began an investigation of J_n in [11], and here we extend those

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results to P_X. This also permits application of a lemma of Hall to deduce Green's relations and some order relations on M_X and J_n.

2. Patterns and an involution

Associated with each element a of P_X there are graphical structures we shall call *patterns*. These are the subgraphs of a induced on (respectively) the upper and lower vertex sets, and we give each a two-tone vertex colouring, so that the vertices of the transversal and nontransversal components are given different colours. To be specific, the subgraph induced on the upper vertices by the transversal (nontransversal) components of a will be denoted by $UT(a)$ $(UN(a))$, and we write $U(a) = UT(a) \cup UN(a)$ and consider $U(a)$ as a two-tone graph. Similarly the subgraph of a induced on the lower vertices by the transversal (nontransversal) components of a will be denoted by $LT(a)$ $(LN(a))$, and we write $L(a)$ for $LT(a) \cup LN(a)$. For example, in Figure 1, $UT(a)$ has components $\{1\}$, $\{2, 3\}$ and $\{4, 5\}$; $LN(a)$ has components $\{2'\}$ and $\{5', 6'\}$. Further, equality of $U(a)$ and $U(b)$ implies equality of their transversal components and also of their nontransversal components. We order graphs by inclusion of the vertex and edge sets:

$$(E, V) \subseteq (E_1, V_1) \quad \text{if and only if } E \subseteq E_1 \text{ and } V \subseteq V_1.$$

The cardinality of the set of transversal components of a pattern is its *rank*. We note that $U(a)$ and $L(a)$ have the same rank, and refer to this cardinal as the *rank* of a, denoted $\operatorname{rank}(a)$. The following lemma is then immediate from the definitions above; we use $r!$ to denote the cardinality of the set of permutations on a set of cardinality r (if r is infinite, $r! = 2^r$).

LEMMA 2.1. *Given two patterns on X, say Γ and Γ', of equal rank r, there exist $r!$ elements a in P_X such that $U(a) = \Gamma$ and $L(a) = \Gamma'$.*

In I_X^*, M_X and J_n there are further restrictions on the patterns which may arise as upper and lower patterns. If a is a block bijection, $UN(a)$ and $LN(a)$ are empty. In a matching a, every component of $UN(a)$ and $LN(a)$ has cardinality 2, and $UT(a)$ and $LT(a)$ are discrete graphs (no edges). Moreover, for finite $|X| = n$ and $a \in M_n$, $n - \operatorname{rank}(a)$ must be even. For J_n, in addition to the above, the nontransversal patterns must correspond to properly nested bracketings in which no transversal vertex occurs within an 'open' bracket. We refer to these patterns as *admissible* for each submonoid. For example, in Figure 1, $LN(b)$ has components $\{1'\}$, $\{4', 5'\}$ and $\{3', 6'\}$ and is inadmissible for both M_6 and J_6 because of the singleton; this pattern is shown in Figure 3(a). Figures 3(b), (c) show (upper) nontransversal patterns with edges $\{1, 2\}$, $\{3, 6\}$ and $\{3, 5\}$, $\{4, 6\}$ respectively, admissible for M_6 but not for J_6; and Figure 3(d) with edges $\{3, 6\}$, $\{4, 5\}$ is admissible for both.

LEMMA 2.2.

(a) *Given two patterns on X of equal rank r, say Γ and Γ', both admissible for M_X, there exist $r!$ elements a in M_X such that $U(a) = \Gamma$ and $L(a) = \Gamma'$.*

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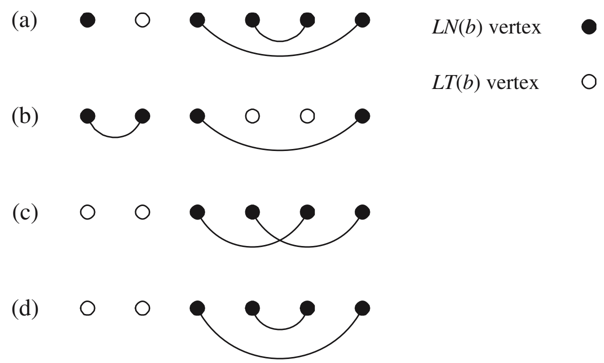

FIGURE 3. Examples illustrating patterns: (a) inadmissible for both M_6 and J_6; (b), (c), admissible for M_6 but not for J_6; (d), admissible for both.

(b) *Given two patterns on X of equal rank, say Γ and Γ', both admissible for J_n, there exists a unique element a in J_n such that $U(a) = \Gamma$ and $L(a) = \Gamma'$.*

Define, for $a \in P_X$, a diagram $a^* \in P_X$ obtained by 'turning a upside-down' or, more formally, exchanging dashed and undashed symbols. Together with the definition of multiplication, this gives the following lemma.

LEMMA 2.3. *For $a, b \in P_X$:*

(i) $U(a^*) = L(a)$;
(ii) $a^{**} = a$;
(iii) $(ab)^* = b^* a^*$;
(iv) $aa^*a = a$.

By the definitions of I_X, I_X^*, M_X, and J_n, each is closed under the unary operation $a \mapsto a^*$. So parts (ii) to (iv) assert that each of P_X, M_X, and J_n is a *regular $*$-semigroup* as introduced by Nordahl and Scheiblich [17]. Of course, for I_X and I_X^*, the operation $*$ is the inversion which makes them inverse semigroups.

3. Divisibility, Green's relations and the natural order

The relation $\leq_L$ on a semigroup S is defined by:

$$a \leq_L b \quad \text{if and only if } a = b \text{ or } a = xb$$

for some $x \in S$. It is a preorder induced by the inclusion relation on principal left ideals: $a \leq_L b \iff a \cup Sa \subseteq b \cup Sb$. When S is regular or a monoid, the definition simplifies to $a \leq_L b \iff a = xb$ for some $x \in S$. Dually, $a \leq_R b \iff a = b$ or $a = by$ for some $y \in S$.

LEMMA 3.1. *For $a, b \in P_X$, $a \leq_L b$ if and only if (i) every component of $LN(b)$ is a component of $LN(a)$, and (ii) every edge of $LT(b)$ is an edge of $L(a)$.*

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PROOF. If $\{i', j'\}$ is an edge of $LN(b)$, then it is an edge in $LN(xb)$. If $\{i', j'\}$ is an edge of $LN(xb)$ with $i' \in LN(b)$, then $j' \in LN(b)$. So if $a = xb$, (i) holds. Let $\{i', j'\}$ be an edge of $LT(b)$. Then there exists $k \in X$ such that $\{k, i'\}$ and $\{k, j'\}$ are edges of b. If there is a transversal edge $\{l, k'\}$ of x then $\{l, i'\}$ and $\{l, j'\}$ are edges of xb, and so $\{i', j'\}$ is an edge of $LT(xb)$. But if there is no such l, then $\{i', j'\}$ is an edge of $LN(xb)$. In either case, $\{i', j'\}$ is an edge of $L(xb)$. So if $a = xb$, (ii) holds.

Conversely, suppose that (i) and (ii) hold, and consider the projection b^*b. We intend to prove that $a = ab^*b$, by considering the different kinds of edges in turn. We have $U(b^*b) = L(b^*b) = L(b)$, and conditions (i) and (ii) imply that each component of $L(a)$ is a union of components of $L(b)$. Clearly $UN(a) \subseteq UN(ab^*b)$, and if $\{i, j\}$ is an edge of $UN(ab^*b)$ then either it is an edge of $UN(a)$, or there is a path $(i, j', \ldots, k)$ with

$$j' \in LT(a) \cap UN(b^*b) = LT(a) \cap LN(b) = \varnothing,$$

which is impossible. So $UN(a) = UN(ab^*b)$.

Next let $\{i, j'\}$ be a transversal edge of a. Then $j' \in LT(a) \subseteq LT(b) = LT(b^*b)$, and there is a path from i to j' in ab^*b, that is, $\{i, j'\}$ is a transversal edge of ab^*b. In the reverse direction, if $\{i, j'\}$ is a transversal edge of ab^*b, then there is a path from i to j' using transversal edges alternately from a and b^*b, and hence all in a. It follows $\{i, j'\}$ is an edge of a. So a and ab^*b have the same transversal edges.

Finally, if $\{i', j'\}$ is an edge of $LN(a)$ then either it is an edge of $LN(b) = LN(b^*b) \subseteq LN(ab^*b)$, or it joins two components of $LT(b) \cap LN(a)$, in which case there is a path from i' to j' in ab^*b. So $LN(a) \subseteq LN(ab^*b)$. On the other hand, if $\{i', j'\}$ is an edge of $LN(ab^*b)$, either it is in $LN(b^*b) = LN(b) \subseteq LN(a)$ or there is a path from i' to j' with edges alternately from $LT(b^*b) = LT(b)$ and $LN(a)$. All vertices in this path are in $LN(a) \cap LT(b)$ and so the path is in $LN(a)$. So $LN(a) = LN(ab^*b)$.

We have shown $a = ab^*b$ and hence that $a \leq_L b$. □

By the dual proof, or Lemma 3.1 applied to a^* and b^*, we have the following corollary.

COROLLARY 3.2. *For $a, b \in P_X$, $a \leq_R b$ if and only if (i) every component of $UN(b)$ is a component of $UN(a)$, and (ii) every edge of $UT(b)$ is an edge of $U(a)$.*

Green's relations. The equivalence relations of Green (see [5] or [8]) are important tools for describing and understanding the structure of a semigroup S. We remind the reader of their definitions. First, $a\mathcal{R}b$ if and only if a and b generate the same principal right ideal. For a, b elements of a monoid or a regular semigroup S, $a\mathcal{R}b$ if and only if $aS = bS$. Dually, $a\mathcal{L}b$ if and only if $Sa = Sb$. Note that $a\mathcal{L}b$ if and only if both $a \leq_L b$ and $b \leq_L a$, and so on. Further, $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, and $a\mathcal{J}b$ if and only if a and b generate the same two-sided ideal, which is to say that $SaS = SbS$ in a monoid or a regular semigroup. Wilcox [20, Section 7] described Green's relations on P_n; we use our divisibility results above to give an alternative description which seems a little more transparent.

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THEOREM 3.3. *For $a, b \in P_X$:*

(i) $a\mathcal{R}b$ *if and only if* $UT(a) = UT(b)$ *and* $UN(a) = UN(b)$*;*
(ii) $a\mathcal{L}b$ *if and only if* $LT(a) = LT(b)$ *and* $LN(a) = LN(b)$*;*
(iii) $a\mathcal{D}b$ *if and only if a and b have equal rank;*
(iv) $a \in P_X b P_X$ *if and only if* $\mathrm{rank}(a) \leq \mathrm{rank}(b)$*;*
(v) $\mathcal{D} = \mathcal{J}$*;*
(vi) *the ideals of P_X form a chain, and if X is finite, all ideals are principal.*

PROOF. (i) and (ii) follow directly from Lemma 3.1 and Corollary 3.2.

(iii) If there is $c \in P_X$ such that $a\mathcal{R}c$ and $c\mathcal{L}b$, then the ranks of a and b are equal to that of c. Conversely, given $a, b \in P_X$ of equal rank, there is by Lemma 2.1 an element $c \in P_X$ such that $U(c) = U(a)$ and $L(c) = L(b)$, whence $a\mathcal{R}c$ and $c\mathcal{L}b$. Thus $a\mathcal{D}b$.

(iv) Suppose that $a = xby$ for some $x, y \in P_X$. Then $a \leq_L by \leq_R b$, so $\mathrm{rank}(a) \leq \mathrm{rank}(by) \leq \mathrm{rank}(b)$. Conversely, suppose that $r = \mathrm{rank}(a) \leq \mathrm{rank}(b) = s$. There exist $I, J \subseteq X$ such that $I \subseteq J$, $|I| = r$, and $|J| = s$. Define elements e_I, e_J of P_X such that e_I has edge set $\{\{i, i'\} : i \in I\}$, and e_J has edge set $\{\{j, j'\} : j \in J\}$. Now $e_I e_J = e_I$, so $a\mathcal{D}e_I$ and $b\mathcal{D}e_J$ by (iii). Thus there are $x, y, z, t \in P_X$ such that $a = xe_I y = xe_I e_J y = xe_I zbty \in P_X b P_X$.

(v) In general, $\mathcal{D} \subseteq \mathcal{J}$. Suppose that $a\mathcal{J}b$. By (iv), $\mathrm{rank}(a) = \mathrm{rank}(b)$ and so $a\mathcal{D}b$ by (iii). Thus $\mathcal{J} \subseteq \mathcal{D}$ and equality holds.

(vi) The principal ideals of P_X form a chain by (iv), and any ideal is a union of the principal ideals it contains. If X is finite, an element of maximal rank generates the ideal. □

Now the submonoids T_X, I_X, I_X^*, M_X, and J_n are regular and so their divisibility preorders $\leq_L$ and $\leq_R$, and their $\mathcal{L}$ and $\mathcal{R}$ relations, are the restrictions of those on P_X, by a result of Hall in [6] (see also [8, Proposition 2.4.2]). Thus the well-known characterizations of Green's relations $\mathcal{L}$ and $\mathcal{R}$ on the first two monoids in the list above [8, 2.6 Exercise 16 and 5.11 Exercise 2] are corollaries of the theorem. For T_X, the UN and LN graphs are respectively empty and discrete, so part (i) of the theorem simplifies to equality of the UT graphs, which in this case can be recognized as kernels of mappings, and part (ii) reduces to equality of ranges. For I_X, all patterns are discrete, so the conditions reduce to equality of ranges and of domains. For I_X^*, the UN and LN graphs are empty and the conditions reduce to equality of set partitions [4, Theorem 2.2].

For M_X we have the following results, attributed to Mazorchuk in [10, Theorem 1] for the finite case.

COROLLARY 3.4. *Let $a, b \in M_X$. Then:*

(i) $a\mathcal{R}b$ *if and only if* $UN(a) = UN(b)$*;*
(ii) $a\mathcal{L}b$ *if and only if* $LN(a) = LN(b)$*; and*
(iii) $a \in M_X b M_X$ *if and only if* $\mathrm{rank}(a) \leq \mathrm{rank}(b)$.

PROOF. $UT(a)$ and $UT(b)$ consist of singleton components, so part (i) is immediate; likewise the LN graphs and part (ii). For part (iii), 'only if' is clear, so suppose that

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$\operatorname{rank}(a) = r \le \operatorname{rank}(b) = s$. There are $I, J \subseteq X$ such that $|I| = r$, $|J| = s$ and both $X \setminus I$ and $X \setminus J$ admit partition into two subsets of equal cardinality. Thus there are $Y_1, Y_2 \subseteq X \setminus I$ and a bijection $\phi : Y_1 \longrightarrow Y_2$, and similarly $Z_1, Z_2 \subseteq X \setminus J$ and a bijection $\psi : Z_1 \longrightarrow Z_2$. Let Γ_I be the rank-r pattern having singletons $\{i\}$ $(i \in I)$ for its transversal components, and edges $\{y, y\phi\}$ $(y \in Y_1)$ for its nontransversal components. It is admissible for M_X and so there is, by Lemma 2.2(a), $c \in M_X$ with $U(c) = U(a)$ and $L(c) = \Gamma_I$, and hence $a\mathcal{R}c$. Similarly, let Γ_J be the rank-s pattern having singletons $\{j\}$ $(j \in J)$ for its transversal components, and edges $\{z, z\psi\}$ $(z \in Z_1)$ for its nontransversal components. There is $d \in M_X$ with $U(d) = \Gamma_J$ and $L(d) = L(b)$, whence $d\mathcal{L}b$. Now let f_I have $U(f_I) = \Gamma_I = L(f_I)$; it follows that $f_I\mathcal{L}c\mathcal{R}a$, so $a\mathcal{D}f_I$. Similarly, $b\mathcal{D}f_J$. Now $f_I f_J f_I = f_I$ and it follows that there exist $x, y, z, t \in M_X$ such that

$$a = xf_I y = xf_I f_J f_I y = xf_I(zbt)f_I y \in M_X b M_X.$$

This concludes the proof. □

There is a completely analogous result for J_n, which we presented in [11, Theorem 3.5].

The natural order. The *natural* or *Mitsch order* [16] on a semigroup S is defined by:

$$a \le_M b \iff a = b \quad \text{or } a = xb = by = xa$$

for some $x, y \in S$. If it is necessary to specify the semigroup S involved, we write $\le_M^S$, and so on. When S is regular, $\le_M$ agrees with the more familiar natural order for a regular semigroup, in which $a \le_M b$ if and only if $a \le_L b$, $a \le_R b$ and $a = ab'a$ for some inverse b' of a. There are many equivalent formulations for regular semigroups—see [16, Lemma 1], which summarizes the work of multiple authors. From the formulation above, we have the following proposition.

PROPOSITION 3.5. *For $a, b \in P_X$, $a \le_M b$ if and only if every component of $UN(b)$ is a component of $UN(a)$, every component of $LN(b)$ is a component of $LN(a)$, every edge of $LT(b)$ is an edge of $L(a)$, every edge of $UT(b)$ is an edge of $U(a)$, and $a = ab^*a$.*

PROOF. It is enough to use Lemma 3.1 and Corollary 3.2 and choose b^* for the inverse. □

Clearly $\le_M$ is a refinement of the left and right divisibility orders treated earlier. There is a lemma of Hall's type for the natural order.

LEMMA 3.6. *If T is a regular subsemigroup of S and $a, b \in T$, then*

$$a \le_M^T b \iff a \le_M^S b.$$

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PROOF. If $a \leq_M^T b$ then of course $a \leq_M^S b$. Conversely, if $a = xb = by = xa$ for $x, y \in S$, then

$$\begin{aligned} a &= xb = xbb'b = by = bb'by \\ &= ab'b = bb'a, \end{aligned}$$

for any inverse b' of b, which may be chosen in T. Then $ab', b'a \in T$, and moreover $ab'a = xbb'by = xby = ay = a$. □

For P_X, the condition $a = ab^*a$ seems to be difficult to state simply in terms of patterns—loosely, the transversal edges of b have to stitch together components of $UN(a)$ and $LN(a)$ in just the right way. However, it simplifies for certain of the subsemigroups of P_X. By identifying T_X with its image $T_X G$ under the map G of Section 1, we have another quite transparent description of the natural order in T_X to add to the well-known ones mentioned and used in [16, Section 3]. We remind the reader that for $a \in T_X$, every component is transversal, so $UN(a)$ is empty, and $UT(a) = U(a)$; the blocks of $U(a)$ are the blocks of $\ker a$; every upper vertex i of a belongs to a component of a that in turn contains a unique lower vertex, which we denote as usual by ia; the components of $LT(a)$ are singletons of the range; and the components of $LN(a)$ are singletons of its complement.

PROPOSITION 3.7. *Let $a, b \in T_X$. Then the following are equivalent:*

(i) $a \leq_M b$;
(ii) $a = ab^*a$;
(iii) *every component of a contains a component of b, and $U(b) \subseteq U(a)$.*

PROOF. By Proposition 3.5, (i) implies (ii).

Suppose that (ii) holds. If $i' \in LT(a)$, then i' lies in a path of ab^*a at the ab^* interface. Thus $i \in UT(b^*) = LT(b)$. Hence $LT(a) \subseteq LT(b)$. Let $\{i, j\}$ be an edge of $U(b)$. There are two cases to consider.

First, if $ib \notin LT(a)$, then there are paths

$$(ib, i, ia) \quad \text{and} \quad (jb, j, ja)$$

in the expanded form of ab^*a. Since $ib = jb$, this gives an edge $\{ia, ja\}$ of $ab^*a = a$, and this is a contradiction unless $ia = ja$.

Second, if $ib \in LT(a)$, say $ib = ka$, then there is a path

$$(k, ka = ib, i, ia)$$

in $ab^*a = a$, and so $ka = ia$. Similarly, $ka = ja$.

In either case, $ia = ja$, that is, $\{i, j\}$ is an edge of a, and we have proved that $U(b) \subseteq U(a)$. Since, as seen above, $LT(a) \subseteq LT(b)$, every component of a contains a component of b as in the second case. So (ii) implies (iii).

Lastly, suppose that (iii) holds. Since every component of a contains a component of b, $a \subseteq ab^*a$. For the reverse inclusion, suppose that $\{i, j\}$ is an edge of ab^*a.

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Without loss of generality, we may take it to be transversal, with (say) $j = ka$. Then there is a path

$$(i, ia = kb, k, ka = j)$$

in the expanded form of ab^*a. Denote by A the component of a which contains i. There is, by hypothesis, a component of b contained in A, and since $kb = ia$, this must contain k. Hence $\{i, j\}$ is an edge of a. We have proved that $ab^*a = a$. The other conditions of Proposition 3.5 being satisfied, we thus have $a \leq_M^{P_X} b$, and by Lemma 3.6, $a \leq_M^{T_X} b$ follows. □

Simplification also occurs in the case of J_n. We need an intermediate result from [11, Lemma 3.8], which is correct as stated, but has a deficient proof. So we also repair the proof here. We require the following items from [11], given in our current notation.

Let $a, b \in J_n$. Then $U(a)$ and $UN(a)$ have the same edge sets, and likewise $L(a)$ and $LN(a)$. Denoting the number of edges in a graph Γ by $|\Gamma|$, one has $|L(a)| = |U(a)| = \frac{1}{2}(n - \text{rank}(a))$. Let $\omega(a, b)$ be the number of odd-length paths in the interface of the product ab; such a path has edges alternately from $L(a)$ and $U(b)$. Then by [11, Lemma 3.1(iv)],

$$2|U(ab)| = |L(a)| + |U(b)| + \omega(a, b). \tag{3.1}$$

LEMMA 3.8. *Let $a, b \in J_n$. Then:*

(i) *$a = aba$ if and only if $\omega(ab, a) = \omega(a, ba) = 0$; and*

(ii) *$a = aba$ and $b = bab$ if and only if $\omega(a, b) = \omega(b, a) = 0$.*

PROOF. (i) By equation (3.1) we have

$$\begin{aligned} 2|U(aba)| &= |L(a)| + |U(ba)| + \omega(a, ba) \\ &= |L(ab)| + |U(a)| + \omega(ab, a). \end{aligned}$$

If $a = aba$, then $ab\mathcal{R}a\mathcal{L}ba$ and so $\text{rank}(a) = \text{rank}(ab) = \text{rank}(ba)$, whence $\omega(a, ba) = 0 = \omega(ab, a)$. Conversely, if $\omega(a, ba) = 0$ then $U(aba) = U(a)$, and if $\omega(ab, a) = 0$ then $L(aba) = L(a)$, by [11, Theorem 3.1(i) and (ii)]. Then by Lemma 2.2(b), $aba = a$.

(ii) If $a = aba$ and $b = bab$, then $b\mathcal{D}a$, so $|U(b)| = |U(a)|$. Then $\omega(a, b) = 0$ by (3.1). Similarly, $\omega(b, a) = 0$. Conversely, if $\omega(a, b) = 0 = \omega(b, a)$, then by [11, Theorem 3.1(i) and (ii)],

$$\begin{aligned} U(ab) &= U(a) \quad \text{and} \quad L(ab) = L(b), \\ U(ba) &= U(b) \quad \text{and} \quad L(ba) = L(a). \end{aligned}$$

By definition, if $U(b) = U(c)$ then $\omega(a, b) = \omega(a, c)$. So $\omega(a, ba) = \omega(a, b) = 0$. Similarly, $\omega(ab, a) = 0$. By part (i), $aba = a$, and symmetrically, $b = bab$. □

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COROLLARY 3.9. *Let $a, b \in J_n$. Then $a \leq_M b$ if and only $U(b) \subseteq U(a)$, $L(b) \subseteq L(a)$, and $\omega(ab^*, a) = \omega(a, b^*a) = 0$.*

PROOF. This now follows from Corollary 3.6 and Proposition 3.7(i). □

4. Twisted monoids

In this final section, we shall treat only the case of finite X, $|X| = n$. Recall from Section 1 that the vertices in the interface were ignored in forming the product of two elements a, b of P_n. We can construct new semigroups from those above in the following manner. Let $\gamma(a, b)$ be the number of components (including singletons) in the expanded diagram for the product ab which have vertices only in the interface. We shall call these *interior cliques* or simply *cliques*. (In J_n they are called circles.) Figure 1 shows an example where $\gamma(a, b) = 1$. By definition, $\gamma(b^*, a^*) = \gamma(a, b)$ and $\gamma(1, a) = 0$ for all $a \in P_n$. Now define a product on the set $\mathbb{N} \times P_n$ by the rule

$$(k, a) \odot (l, b) = (k + \gamma(a, b) + l,\ ab),$$

and denote the resulting algebra $(\mathbb{N} \times P_n, \odot)$ by $\widehat{P_n}$.

LEMMA 4.1. *For all $a, b, c \in P_n$,*

$$\gamma(a, b) + \gamma(ab, c) = \gamma(a, bc) + \gamma(b, c); \tag{4.1}$$

consequently, $\widehat{P_n}$ is a monoid with identity $(0, 1)$.

PROOF. Consider the three-layer expanded diagram for the product abc in P_n. The cliques it contains are of three kinds: those on vertices in the upper interface, $\gamma(a, b)$ in number; those in the lower interface, of which there are likewise $\gamma(b, c)$; and those with vertices in both interfaces, of which there are (say) δ. Now we have

$$\gamma(ab, c) = \gamma(b, c) + \delta \quad \text{and} \quad \gamma(a, bc) = \delta + \gamma(a, b),$$

whence (4.1) follows. In turn (4.1) implies associativity of $\odot$, and from the definition,

$$(k, a) \odot (0, 1) = (k, a) = (0, 1) \odot (k, a).$$

This concludes the proof. □

This construction may be recognized as a twisting in the context of algebras; it is a special case of the *alteration* product discussed by Sweedler [18]. Now we introduce *augmented* diagrams, which are diagrams as described in Section 1 but with the possible addition of cliques. Placement and size of the cliques are irrelevant. Multiplication of augmented diagrams is similar to that for ordinary diagrams, except that the components within the interface are retained, each being depicted by a new clique in the augmented diagram for the product. Consider the map which associates to $(k, a) \in \widehat{P_n}$ the P_n-diagram a augmented by k cliques. It is clear this map gives a faithful representation of $\widehat{P_n}$ by augmented diagrams.

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If S is a subsemigroup of P_n then the subset $\mathbb{N}\times S$ is closed under the multiplication $\odot$, and so is a subsemigroup of $\widehat{P_n}$ which we denote by $\widehat{S}$. Consider the cases $S = J_n$, $M_n.\widehat{J_n}$ is the *Kauffman monoid* K_n investigated by Borisavljević *et al.* [1], and is generated by the generators of the Temperley–Lieb algebra TL_n. Likewise $\widehat{M_n}$ is the *Brauer monoid* B_n which has been well studied (beginning with [2]) because of the significance of the Brauer algebra which it generates. We conclude with some results on subsemigroups of $\widehat{P_n}$ of the form $\widehat{S}$ where S is a subsemigroup of P_n having the following property:

$$\begin{gathered}\text{for all } a, b \in S, \text{ there exist } a', b' \in S \text{ such that}\\ ab = ab' = a'b \text{ and } \gamma(a, b') = \gamma(a', b) = 0.\end{gathered} \tag{4.2}$$

LEMMA 4.2. *Let S be a subsemigroup of P_n with property (4.2), and $(k, a), (l, b) \in \widehat{S}$. Then in $\widehat{S}$, $(k, a)\mathcal{R}(l, b)$ if and only if $k = l$ and $a\mathcal{R}b$; and dually for $\mathcal{L}$.*

PROOF. Suppose that $(k, a)\mathcal{R}(l, b)$, so there are (m, x) and (n, y) such that $k \le l$, $ax = b$, $l \le k$ and $by = a$. Thus $k = l$ and $a\mathcal{R}b$. Conversely, if $a\mathcal{R}b$ then there are $x, y \in S^1$ such that $a = bx$ and $b = ay$, and by (4.2) we may assume that $\gamma(b, x) = 0 = \gamma(a, y)$. Then $(k, a) = (k, b)(0, x)$ and $(k, b) = (k, a)(0, y)$. □

When S is I_n, I_n^*, T_n, or any of their subsemigroups, $\gamma(a, b) = 0$ for $a, b \in S$, so (4.2) holds—indeed, $\widehat{S}$ is simply the direct product $\mathbb{N}\times S$. By [11, Lemma 4.1], J_n has property (4.2), and the same proof (without even the need for maintaining planarity) shows this is true also of M_n. Finally, P_n itself has property (4.2): each component entirely within the interface of the expanded form for the product ab may be joined by an edge to any component of $L(b)$, without changing the product, so we may construct b' by adjoining such edges to b. We have $ab = ab'$ and $\gamma(a, b') = 0$; and dually we construct a' with $ab = a'b$ and $\gamma(a', b) = 0$. Thus Lemma 4.2 applies not only to the Kauffman monoid (as shown in [11]) but also to the Brauer monoid B_n and the twisted form $\widehat{P_n}$ of the partition monoid. We conclude with a description of the poset of principal ideals for these monoids.

THEOREM 4.3. *Let $(k, a), (l, b) \in B_n$ [respectively, $\widehat{P_n}$]. Then:*

(i) *$(k, a)\mathcal{D}(l, b)$ if and only if $a\mathcal{D}b$ in $M_n[P_n]$ and $k = l$;*
(ii) *$\mathcal{D} = \mathcal{J}$ in $B_n[\widehat{P_n}]$;*
(iii) *the poset of principal ideals of $B_n[\widehat{P_n}]$ is the product of a chain isomorphic to $\mathbb{N}$ (with the order $0 > 1 > \cdots$) with a chain of length n; and*
(iv) *all ideals of $B_n[\widehat{P_n}]$ are finitely generated.*

PROOF. (i) $(k, a)\mathcal{D}(l, b)$ in $B_n[\widehat{P_n}]$ implies that there is $(m, c) \in B_n[\widehat{P_n}]$ such that $(k, a)\mathcal{L}(m, c)\mathcal{R}(l, b)$, when by Lemma 4.2, $k = m = l$ and $a\mathcal{L}c\mathcal{R}b$ in $M_n[P_n]$. Conversely, if $a\mathcal{L}c\mathcal{R}b$ in $M_n[P_n]$, then $(k, a)\mathcal{L}(k, c)\mathcal{R}(k, b)$ in $B_n[\widehat{P_n}]$.

(ii) $\mathcal{D} \subseteq \mathcal{J}$ in general, and if $(k, a)\mathcal{J}(l, b)$, then again $k = l$ and $a\mathcal{J}b$, that is, $a\mathcal{D}b$ by Theorem 3.3(v); so by part (i), $(k, a)\mathcal{D}(l, b)$.

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(iii) We use the bijection (well defined by parts (i) and (ii)) which associates to the $\mathcal{J}$-class $J(k, a)$ the pair $(k, J(a))$, where $J(a)$ is the $\mathcal{J}$-class of $a \in M_n[P_n]$. The result for $\widehat{P}_n$ follows from Theorem 3.3(iv), and for B_n from Corollary 3.4(iii).

(iv) Let I be an ideal of $B_n[\widehat{P}_n]$; it is a union of a set $\mathcal{X}$ of principal ideals. Consider the maximal elements of $\mathcal{X}$. If there are more than n of them, at least two must be comparable, by part (iii). □

5. Conclusion

The semigroup structure of P_n determines the ring structure of the partition algebra, so a thorough investigation of the former should be useful for a deeper understanding of the latter. For instance, Wilcox [20] provides a semigroup-based proof that the partition, Temperley–Lieb and Brauer algebras are cellular. We have given a description of the ideal structure of the partition monoid, but this is not enough. The relationship between congruences on a semigroup and ideals of its semigroup ring suggests that further work should be directed towards a determination of all congruences on P_n.

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D. G. FITZGERALD, School of Mathematics and Physics, University of Tasmania, Private Bag 37, Hobart, TAS 7001, Australia
e-mail: D.FitzGerald@utas.edu.au

KWOK WAI LAU, CSIRO Mathematics, Informatics and Statistics, Private Bag 5, Wembley, WA 6913, Australia
e-mail: Rex.Lau@csiro.au

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