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ON FINITE p-GROUPS WITH ISOMORPHIC MAXIMAL SUBGROUPS

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Abstract

Finite p-groups with all of their maximal subgroups isomorphic are studied by means of the coclass. All such groups of coclass 1 and 2 are determined, while those of coclass 3 are shown to have order at most p^{13}. A general bound for the order is given as a function of p and the coclass only.

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The aim of the present paper is the investigation of finite p-groups with all maximal subgroups isomorphic, by means of their coclass. It will be shown that the order of such a p-group is bounded by a function of p and the coclass only. For the particular case of coclass at most 2, a complete characterization of the above groups is provided, while for coclass 3 we give a big improvement for the bound on the group order, at least compared to the one resulting from the general estimate. The methods used throughout are perfectly elementary and are mainly based on Lemma 1 below.

NOTATION. For a finite p-group X we denote by $1 = Z_0(X) < Z_1(X) < \cdots$ and by $X = \gamma_1(X) > \gamma_2(X) > \cdots$ the terms of the ascending and descending central series of X, respectively. (Higher) commutators of two subgroups A and B will be denoted by $[A, B; 1] = [A, B]$, $[A, B; 2] = [[A, B; 1], B]$, etc. Let $[A, B; 0]$ simply stand for A. The nilpotency class of X is abbreviated

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by $\mathrm{cl}(X)$; if X is of order p^n and of class k, the coclass of X is $\mathrm{cocl}(X) = n-k$. We will use the standard notation $\Phi(X)$, or merely Φ, for the Frattini subgroup of X. A group is said to be an (MI)-group if all of its maximal subgroups are isomorphic.

When determining the automorphism group of a finite p-group P, one can investigate the induced linear group A on the Frattini factor P/Φ as a first step. In doing so, information about the orbits of A on the set of all hyperplanes can be obtained by knowing the isomorphism classes of the maximal subgroups in P. (MI)-groups just provide the situation when that reduction fails.

In turn, if the automorphism group of P induces a transitive linear group on the hyperplanes (or, equivalently, on the lines) of P/Φ then P is (MI). In particular, the finitely generated (relatively) free groups in any variety defined by the identities $x^{p^m} = 1$, $\mathrm{cl}(X) \leq k$ are certainly (MI). There are nonfree groups with induced transitive linear group as the latter can be prescribed (compare with Bryant and Kovács [1]). In addition, for $p = 2$ some of the Suzuki 2-groups are also known to be (MI) (see Higman [4]). (MI)-groups with nontransitive induced linear groups appear in Theorem 2.

Since every finite p-group is a homomorphic image of some relatively free group, (MI)-groups of arbitrarily high class, coclass and solubility length exist. It can be seen for the same reason that the class (MI)-p is not closed under taking homomorphic images. Simple examples yield the same for subgroups and direct products. However, for all known (MI)-group P, $P/Z(P)$ is also (MI). Were that true for all (MI)-groups, the big bound $f(c)$ of the estimate in Theorem 1 could be replaced by $g(c) = 3c$. That is certainly fulfilled by any group with an induced transitive linear group.

LEMMA 1. *Let G be a finite p-group. If G is (MI) and $Z_n(G) \neq G$, then $Z_n(G) \leq \Phi(G)$.*

PROOF. Suppose the contrary and choose a subgroup R in G to be of minimal order with $G = Z_n(G) \cdot R$. Let $R \leq M$ and $|G : M| = p$. By assumption, we can find a subgroup R_1 in R such that $Z_n(G) \cap R \leq R_1$ and $|R : R_1| = p$. Then $Z_n(G) \cdot R_1$ is a maximal subgroup in G, hence it is isomorphic to M. Thus we can write $M = TK$ with $T \lhd M$, $K \leq M$, $[T, M; n] = 1$ and $|K| = |R_1|$. Clearly $Z_n(G) \cdot M = G$.

CLAIM. $[T, G; k] \leq Z_{n-k}(G) \cdot [T, M; k]$ for all k.

Our claim can be proved by induction on k; it holds trivially for $k = 0$. Assume that $[T, G; i] \leq Z_{n-i}(G) \cdot [T, M; i]$, then $[T, G; i+1] \leq [Z_{n-i}(G) \cdot [T, M; i], G] \leq Z_{n-i-1}(G) \cdot [[T, M; i], Z_n(G) \cdot M] \leq Z_{n-i-1}(G) \cdot \langle [T, M; i+1], [[T, M; i], Z_n(G)]^m : m \in M \rangle \leq Z_{n-i-1}(G) \cdot \langle [T, M; i+1], [\gamma_{i+1}(G), Z_n(G)] \rangle$;

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by the well-known assertion due to P. Hall, $[\gamma_{i+1}(G), Z_n(G)] \leq Z_{n-i-1}(G)$ (see [2] or [5, p. 265]), and therefore one can conclude that $[T, G; i+1] \leq Z_{n-i-1}(G) \cdot [T, M; i+1]$, as required. For $k = n$ we have $[T, G; n] \leq Z_0(G) \cdot [T, M; n] = 1$, that is $T \leq Z_n(G)$, thus $G = Z_n(G) \cdot M = Z_n(G) \cdot TK = Z_n(G) \cdot K$ with $|K| = |R_1| < |R|$, a contradiction.

We need some simple facts about (MI)-groups; their proofs are left to the reader.

LEMMA 2. *If G is a nonabelian finite p-group and it is (MI), then $|Z_2(G)| \geq p^3$.*

LEMMA 3. *Let G be a finite p-group. If G is (MI) and* $\mathrm{cocl}(G) = 1$ *(that is, G is of maximal class), then $p^2 \leq |G| \leq p^3$.*

COROLLARY 1. *G is (MI) of coclass* 1 *if and only if it is*

(i) *elementary Abelian of order p^2, or*
(ii) *nonabelian of order p^3 and of exponent p with $p > 2$, or*
(iii) *the quaternion group of order* 8.

LEMMA 4. *Assume that $|Z_i(G) : Z_{i-1}(G)| = p$ for all $t < i \leq s$ (for some $t < s$). If G is a p-group, $Z_t(G) \leq N \leq Z_s(G)$ and $N \lhd G$, then $N = Z_j(G)$ for some j.*

LEMMA 5. *If M_1 and M_2 are different maximal subgroups of a finite p-group G and $Z_s(M_1)$, $Z_s(M_2)$ are both contained in $Z_t(G)$ (for some s and t), then $Z_{s+1}(M_1) \cap Z_{s+1}(M_2) \leq Z_{t+1}(G)$.*

LEMMA 6. *Assume that the finite p-group G is (MI) and* $\mathrm{cocl}(G) = c$. *If M_1 and M_2 are different maximal subgroups in G with $Z_s(M_i) \leq Z_t(G)$ $(i = 1, 2)$, then $Z_{s+1}(M_i) \leq Z_{t+2c+1}(G)$ $(i = 1, 2)$. If $|G| > p^{2s+3c-2}$, then in addition $Z_{s+1}(M_i) \leq Z_{t+2c-1}(G)$.*

LEMMA 7. *There is no finite (MI)-group G with $Z(G) = Z(M)$ of order p for all maximal subgroups M.*

THEOREM 1. *Let G be a finite p-group of coclass c. If G is (MI), then $|G| \leq p^{f(c)}$, where $f(c) = (2c+1)^{c+1} + c + 2$.*

PROOF. Let M be any maximal subgroup of G. Denote $Z_j(G)$ simply by Z_j; then by Lemmas 1 and 6 we get from the trivial inclusion $Z_0(M) \leq Z_0$ that $Z_i \leq Z_i(M) \leq Z_{i(2c+1)}$ for all i. We proceed by induction on $|G| = p^n$.

CASE 1. There exists a minimal i in the set $\{1, (2c+1), (2c+1)^2, \dots, (2c+1)^{c-1}\}$ such that $|Z_{i(2c+1)}: Z_i| = p^{2ci}$.

We have $Z_i(M) = Z_{i+s}$ (for some $0 \leq s \leq 2ci$) by Lemma 4. (Thanks to the (MI)-property, s is independent of M.) Let $|Z_i| = p^{i+t}$ ($\geq p^{i+1}$ by Lemma 2), then $|Z_{i+s}| = p^{i+t+s}$, $\overline{G} = G/Z_{i+s}$ is of order $p^{n-i-t-s}$, it is (MI) and $\operatorname{cocl}(\overline{G}) = c - t < c$; therefore $|G| = p^{i+t+s} \cdot |\overline{G}| \leq p^{i+t+s+f(c-t)} \leq p^{i(2c+1)+t+f(c-t)} \leq p^{(2c+1)^c+t+f(c-t)} \leq p^{f(c)}$ by induction.

CASE 2. $|Z_{j(2c+1)} : Z_j| > p^{2cj}$ for all $j \in \{1, (2c+1), \dots, (2c+1)^{c-1}\}$.

Now $G/Z_{(2c+1)^c}$ is of maximal class, hence $T = Z_{(2c+1)^c}(M)$ is either independent of M, or $|G : T| \leq p$. In any case, $|G : T| \leq p^3$ by Lemma 3, thus $|G| \leq p^3 \cdot |T| \leq p^3 \cdot |Z_{(2c+1)^{c+1}}| = p^{3+(2c+1)^{c+1}+c-1} = p^{f(c)}$ since $T \leq Z_{(2c+1)^{c+1}}$.

For the case $c = 1$ the above formula gives $p^{f(1)} = p^{12}$ only (instead of the correct value of p^3). The situation becomes even worse for $c > 1$, when $f(c)$ will turn out to be far too big. To give some refinement for $c \leq 3$, we firstly deal with the case $c = 2$ by proving

THEOREM 2. *Assume that the finite p-group G is (MI) and that* $\operatorname{cocl}(G) = 2$; *then G is isomorphic to one of the groups listed below:*

(i) $$Z_{p^3};$$

(ii) $$Z_p \times Z_p \times Z_p;$$

(iii) $$\langle a, b : a^{p^2} = b^{p^2} = 1, b^{-1}ab = a^{1+p} \rangle;$$

(iv) $$\langle a, b : a^9 = b^9 = [a,b]^3 = [a,b,a,a] = [a,b,b,b] = 1, \; [a,b,a] = b^3, [a,b,b] = a^3 \rangle;$$

(v) $$\langle a, b : a^9 = b^9 = [a,b]^3 = [a,b,a,a] = [a,b,b,b] = 1, \; [a,b,a] = b^6, [a,b,b] = a^3 \rangle;$$

(vi) $$\langle a, b : a^{p^2} = b^{p^2} = [a,b]^p = [a,b,a,a] = [a,b,b,b] = 1, \; [a,b,a] = b^p, [a,b,b] = a^{pm} \rangle$$

(*for $p \geq 5$ and m the smallest quadratic nonresidue* mod p);

(vii) $$\langle a, b : a^{p^2} = b^{p^2} = [a,b]^p = [a,b,a,a] = [a,b,b,b] = 1, \; [a,b,a] = b^p, [a,b,b] = a^{pg} b^p \rangle$$

(*for $p \geq 5$, $1 \leq g \leq p-1$ and $4g+1$ any quadratic nonresidue* mod p, *which gives $(p-1)/2$ groups of this type*);

(viii) $$\langle a, b : a^p = b^p = [a,b]^p = [a,b,a]^p = [a,b,b]^p = [a,b,a,a] = [a,b,a,b] = [a,b,b,a] = [a,b,b,b] = 1 \rangle \quad (\textit{for } p \geq 5);$$

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(ix) $\langle a, b\colon a^{p^2} = b^{p^2} = [a,b]^p = [a,b,a,b] = [a,b,b,a] = 1,$
$[a,b,a] = a^p, [a,b,b] = b^p\rangle$ (*for* $p \geq 5$);

(x) $\langle a, b\colon a^p = b^p = [a,b]^p = [a,b,a]^p = [a,b,b]^p = [a,b,a,a]^p$
$= [a,b,a,b] = [a,b,b,a] = [a,b,a,a,a] = [a,b,a,a,b] = 1,$
$[a,b,b,b] = [a,b,a,a]^{-m}\rangle$

(*for* $p \geq 5$ *and* m *the smallest quadratic nonresidue* $\bmod p$);

(xi) $\langle a, b\colon a^9 = b^9 = [a,b]^3 = [a,b,a,a]^3 = [a,b,a,a,a] = [a,b,a,a,b] = 1,$
$[a,b,a] = b^3, [a,b,b] = a^3, [a,b,a,a] = [a,b,b,b]\rangle$;

(xii) $\langle a, b\colon a^9 = b^9 = [a,b]^3 = [a,b,a,a]^3 = [a,b,a,a,a] = [a,b,a,a,b] = 1,$
$[a,b,a]^2 \cdot [a,b,a,a] = b^3, [a,b,b] \cdot [a,b,a,a]^2 = a^3,$
$\langle [a,b,a,a] = [a,b,b,b]\rangle$.

REMARKS. (1) The groups in (iv), (v) are of order 3^5, those in (vi)–(ix) of order p^5, the two in (xi)–(xii) of order 3^6, while the groups in (x) have order p^6.

(2) All groups listed in (iv)–(vii) have their proper nonmaximal subgroups Abelian and have been characterized as (MI)-groups with this property in [3].

(3) The groups in (i), (ii) and (viii) are relatively free, and for the groups in (ix), (x) and (xii) the automorphism group also permutes the maximal subgroups transitively. In case of (iii), (iv), (v), (vi), (vii) and (xi) the automorphism group is nontransitive on the set of the maximal subgroups.

It is suitable to break the proof of Theorem 2 into several parts. However, most of the detailed calculation will be left to the reader.

LEMMA 8. *Assume that* $|G| = p^n$, $\mathrm{cocl}(G) = 2$ *and that* G *is* (MI). *If* $|G : Z_{n-3}(G)| = p^3$, *then* G *is elementary Abelian of order* p^3.

PROOF. Let M be any maximal subgroup of G, with $|Z(M)| = p^k$. Assume that $k > n-3$, so that M is Abelian. If G is nonabelian, then G is of class 2, that is $n = 4$, hence $p^2 = |G : Z(G)| = |G : Z_{n-3}(G)| = p^3$, a contradiction; thus G is Abelian and obviously it is elementary of order p^3. Suppose that $k \leq n-3$; $|Z_k(G)| = p^k$ by assumption, so since $Z(M) \leq Z_k(G)$, we get $Z(M) = Z_k(G)$. This means that $Z(M) = Z(G)$ is of order p, contradicting Lemma 7.

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LEMMA 9. *Assume that* $|G| = p^n$ *and* $\mathrm{cocl}(G) = 2$. *If* G *is* (MI) *and* $|G : Z_{n-3}(G)| = p^2$, *then* $4 \leq n \leq 7$. *For* $n = 4$, G *is isomorphic to the group listed in* (iii) *of Theorem* 2.

PROOF. Let M be any maximal subgroup of G, and denote the order of $Z(M)$ by p^k. Assume firstly that $k > n - 3$; then M is Abelian, whence the nilpotency class of G is at most 2. It follows that $n = 4$, and then G can be determined easily. If $k \leq n-3$ then $Z(M) \leq Z_k(G)$ and $|Z_k(G) : Z(M)| \leq p$. Let M_1 and M_2 be different maximal subgroups in G. As $Z(M_1) \cap Z(M_2) \leq Z_1(G)$, we have $|Z_1(G)| \geq p^{k-1}$, $|Z_k(G) : Z_1(G)| \leq p^2$, which yields $k \leq 3$. Suppose that $k = 3$, then $p = |Z_3(G) : Z_2(G)| = |Z_2(G) : Z_1(G)| = |Z_3(G) : Z(M)$, thus $|Z(M)| = p^3$ implies $|Z(G)| = p^2$. Therefore $Z(M)/Z(G) \leq Z(G/Z(G))$, that is $Z(M) \leq Z_2(G)$, so that $Z(M) = Z_2(G)$ as their orders are equal. Being valid for any M, this implies that $Z_2(G) = Z(M) = Z(G)$, a contradiction. If $k = 1$, we would get $|Z(M)| = |Z(G)| = p$, contradicting Lemma 7; thus $k = 2$. If $Z(M) = Z(G)$ for some—equivalently for all—M (in the light of Lemma 1), then $G/Z(G)$ is (MI) of coclass 1, hence $|G| = |Z(G)| \cdot |G/Z(G)| \leq p^2 \cdot p^3$ by Lemma 3. Assume finally that $Z(M) \neq Z(G)$. We have $|Z_2(G) : Z(M)| = p$, and thus $G/Z_2(G)$ is of maximal class. Suppose that $|G : Z_2(M)| \leq p$; then all maximal subgroups of $\overline{G} = G/Z_2(G)$ are Abelian, whence $|\overline{G}| \leq p^3$, so that $|G| \leq p^6$. If $|G : Z_2(M)| \geq p^2$, then $T = Z_2(M)$ is independent of the choice of M since $G/Z_2(G)$ is of maximal class, and therefore G/T is (MI). Moreover, G/T is either trivial or of coclass 1, thus $|G/T| \leq p^3$ by Lemma 3. On the other hand, $T \leq Z_3(G)$ by Lemma 5, so $|G| = |G/T| \cdot |T| \leq p^3 \cdot |Z_3(G)| = p^7$. (Let us mention that $Z_3(G) = G$ would give $p^2 = |G : Z_{n-3}(G)| \geq |G : Z_2(G)|$, that is $n \leq 5$ by assumption.

For the further treatment of the open cases we point out some facts which appeared in the above proof.

COROLLARY 2. *Suppose that* $|G| = p^n$, G *is* (MI), $\mathrm{cocl}(G) = 2$, $|G : Z_{n-3}(G)| = p^2$. *Then*

(a) *if* $|G : M| = p$, *then* $Z(M)$ *is elementary Abelian of order* p^2;
assume that $6 \leq n \leq 7$; *then*
(b) $|Z(G)| = p$, $|Z_2(G)| = p^3$, $\mathrm{cocl}(G/Z_2(G)) = 1$;
(c) $Z_2(G) = \langle Z(M) : |G : M| = p \rangle$ *is elementary Abelian*;
(d) $Z_3(G)$ *is Abelian*;
(e) *if* $n = 7$ *and* $|G : M| = p$ *then* $Z_2(M) = Z_3(G)$ *is elementary Abelian*;
(f) *if* $n = 6$ *and* $|G : M| = p$ *then either* $Z_2(M) = Z_2(G)$ *or* M *is of class* 2.

Proof. There is nothing to do in case (b). For (a), suppose that $Z(M)$ were cyclic; then G could not possess any noncyclic Abelian normal subgroup of order p^2. Therefore G would contain a cyclic maximal subgroup by [5, Satz 7.6, page 304], a contradiction. Being in $Z_2(G) \leq \Phi(G)$ the centres of the maximal subgroups of G centralize one another, and thus $Z_2(G) = \langle Z(M) : |G : M| = p\rangle$ is elementary Abelian.

$C_G(Z_2(G)) \geq \Phi(G) \geq Z_3(G)$, so $Z_3(G)$ is Abelian as $|Z_3(G) : Z_2(G)| = p$. If $n = 7$ and M is a maximal subgroup of G, then we must have equality in all estimates used in proving that $|G| \leq p^7$; in particular, $Z_2(M) = T = Z_3(G)$. Suppose that $Z_3(G)$ is of type (p^2, p, p); then $Z_1(G) = \langle x^p : x \in Z_2(M)\rangle$, so that $\overline{G} = G/Z_1(G)$ is also (MI). As $\operatorname{cocl}(\overline{G}) = 2$ and $|Z(\overline{G})| = |Z_2(G) : Z_1(G)| = p^2$, we have a contradiction to (b). Assume that $n = 6$ and $|G : M| = p$. If $|G : Z_2(M)| \neq p$ then $Z_2(M) \leq Z_3(G) = \Phi(G)$, as $G/Z_2(G)$ is a group of maximal class, and $Z_2(M) \geq Z_2(G)$ by Lemma 1. Suppose that $Z_2(M) = Z_3(G)$; then $|M : Z_2(M)| = p$, a contradiction; and thus $Z_2(M) = Z_2(G)$, by Lemma 4.

Lemma 10. *If G is a finite p-group, G is (MI) and* $\operatorname{cocl}(G) = 2$, *then* $|G| \leq p^6$.

Proof. Suppose the contrary. Then $|G| = p^7$ by Lemma 9.

Case (A). $Z_4(G)$ is Abelian.

As the centre of any maximal subgroup M in G is of index p^4 in M, $Z_4(G)$ is the unique Abelian subgroup in M of index p, Suppose that $Z_4(G)$ is of type (p^2, p, p, p); then $Z(G) = \langle x^p : x \in Z_4(G)\rangle$ and $\overline{G} = G/Z(G)$ is also (MI) of coclass 2; that is impossible by (b) of Corollary 2, as $|Z(\overline{G})| = |Z_2(G) : Z_1(G)| = p^2$, again by (b). Thus $Z_4 = Z_4(G)$ is elementary Abelian by (e) of Corollary 2. Let $x \in G \backslash Z_4$; then $M = Z_4 \cdot \langle x\rangle$ is a maximal subgroup of G. The group $M' = [Z_4, x]$ is of order $|Z_4 : C_{Z_4}(x)| = |Z_4 : Z(M)| = p^3$. Let $Z_4 = \langle a, b, c_1, c_2, z\rangle$, $Z_3(G) = Z_3 = \langle b, c_1, c_2, z\rangle$, $Z_2(G) = Z_2 = \langle c_1, c_2, z\rangle$, $Z_1(G) = Z_1 = \langle z\rangle$. We can choose x with $[a, x] \in Z_3 \backslash Z_2$, and we can assume

(1) $[a, x] = b$ and

(2) $Z(M) = \langle c_2, z\rangle$ (with $M = Z_4 \cdot \langle x\rangle$). By (e) of Corollary 2, we have $b \in Z_3 = Z_2(M)$, hence $[b, x] = c_2^\alpha z^\beta$ and $[c_1, x] = z^\gamma$. As $M' = \langle [a, x], [b, x], [c_1, x]\rangle$ is of order p^3, we have $\alpha\gamma \not\equiv 0 \pmod{p}$; therefore we can replace c_2 by $c_2^\alpha z^\beta$ and z by z^γ to assume that

(3) $[b, x] = c_2$,

(4) $[c_1, x] = z$.

Set $G = \langle x, y\rangle$; then $a^y = ab^{\alpha_1} c_1^{\beta_1} c_2^{\gamma_1} z^{\delta_1}$, $b^y = bc_1^{\beta_2} c_2^{\gamma_2} z^{\delta_2}$, $c_2^y = c_2 z^{\delta_3}$. As $c_2 \in Z(M) \backslash Z(G)$, $\delta_3 \not\equiv 0 \pmod{p}$. Now $a^{xy} = a^{yx}$ implies, by (1)–(4):

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(5) $\beta_2 \equiv 0 \pmod p$. Similarly, since $b^{xy} = b^{yx}$, we get

(6) $\beta_2 \equiv \delta_3 \pmod p$. Thus by (5) and (6), $\delta_3 \equiv 0 \pmod p$, a contradiction.

CASE (B). $Z_4(G)$ is nonabelian. As $G/Z_2(G)$ is of order p^4, it has at least one Abelian maximal subgroup, that is, we have $M_0' \leq Z_2(G)$ for some maximal subgroup M_0 in G. If $M_0' \neq Z_2(G)$, then $|M_0'| \leq p^2$, and thus $M' \leq Z_2(G)$ for every maximal subgroup M of G; in that case $\mathrm{cl}(G/Z_2(G)) \leq 2$, implying that $\mathrm{cl}(G) \leq 4$, a contradiction. So $M_0' = Z_2(G)$, hence $C_{M_0}(M_0') = C_{M_0}(Z_2(G)) = Z_4(G)$. As $Z_4(G)$ is nonabelian and $Z_2(G) = Z(Z_4(G))$ is of index p^2 in $Z_4(G)$, the derived group of $Z_4(G)$ is of order p, that is $Z_4(G)' = Z_1(G)$.

Suppose that, for all maximal subgroups M of G, $C_M(M') = Z_4(G)$; then $\overline{G} = G/Z_1(G) = G/Z_4(G)' = G/(C_M(M'))'$ is (MI) and of coclass 2 with $|Z(\overline{G}| = |Z_2(G) : Z_1(G)| = p^2$, contradicting (b) of Corollary 2. Thus there exists some maximal subgroup M_1 in G with $C_{M_1}(M_1') \neq C_{M_0}(M_0')$. As by (e) of Corollary 2 we have $Z_3(G) = Z_2(M_i) \leq C_{M_i}$ $(i = 1, 2)$, we get

$$G' \leq C_{M_0}(M_0') \cap C_{M_1}(M_1') = Z_3(G),$$

since $|G : C_{M_i}(M_i')| = p^2$, and this is the final contradiction.

LEMMA 11. *Assume that* $|G| = p^5$, $\mathrm{cocl}(G) = 2$. *If G is (MI), then $Z_2(G)$ is elementary Abelian of order p^3 and $Z_1(G)$ is of order p^2.*

PROOF. Clearly $Z_2(G) = \Phi(G)$ is noncyclic Abelian of order p^3 by Lemmas 1, 2 and (a) of Corollary 2. Suppose that $|Z_1(G)| = p$. For any maximal subgroups M_1, M_2 of G, $G/Z(M_1)$ is of order p^3, and hence $M_2' \leq Z(M_1)$. Taking $M_1 \neq M_2$ we get $M_2' \leq Z(M_1) \cap Z(M_2) = Z(G)$, that is $M' = Z(G)$ for all maximal subgroups M. The group $G/Z(G)$ is therefore (MI) and one step nonabelian of order p^4, so that $G/Z(G) = \langle \overline{x}, \overline{y} : \overline{x}^{p^2} = \overline{y}^{p^2} = 1, \overline{y}^{-1}\overline{x}\overline{y} = \overline{x}^{1+p} \rangle$. Obviously $G = \langle x, y \rangle$ $(\overline{x} = xZ(G), \overline{y} = yZ(G))$ is not metacyclic, hence $x^y = x^{1+p}c$ where $\langle c \rangle = Z(G)$. For any integer n, $(x^n)^y = x^{n(1+p)}c^n$; in particular, $[x^p, y] = x^{p^2}$. Suppose that $x^{p^2} \neq 1$; then $\langle x^{p^2} \rangle = Z(G)$, $\langle x \rangle \lhd G$, G is metacyclic, a contradiction; thus the order of x is p^2. For any integer n we have

$$x^{y^n} = x^{(1+p)^n} c^{((1+p)^n - 1)/p}.$$

In particular $[x, y^p] = 1$; that implies $x^p, y^p \in Z(G)$, whence $p = |Z(G)| \geq |\langle x^p, y^p \rangle| \geq |\langle \overline{x}^p, \overline{y}^p \rangle| = p^2$, a contradiction.

Suppose that $Z_2(G)$ is not elementary. Then by earlier remarks, it is Abelian of type (p^2, p). Since $|Z(G)| = p^2$, we have $Z(G) = Z(M)$ for every maximal subgroup M of G by Lemma 1 and (a) of Corollary 2; $G/Z(G)$ is therefore (MI) of order p^3. As $G/Z(G)$ is never quaternion, $p > 2$ and $G/Z(G)$ is of exponent p, that is $\mho(G) \leq Z(G)$. Suppose that $|\mho(G)| = p$.

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Then $\mho(G) = \mho(M)$ for all maximal subgroups M; hence $\overline{G} = G/\mho(G)$ is (MI) of order p^4 and of exponent p. The class of $\overline{G}$ must be 3 since $\overline{G}$ is 2-generated; but there is no (MI)-group of maximal class and of order p^4. Thus $\mho(G) = Z(G)$, and hence we can find an element g in G with $g^p \notin \mho(Z_2(G))$. Let $K = \langle g \rangle \cdot Z_2(G)$, so that K is a maximal subgroup in G and (K) is of order (at least) p^2. This implies that the order of $\Phi(K)$ is (at least) p^2; as $\Phi(K) = \mho(K) = \mho(G) = Z(G) = Z(K)$, all maximal subgroups of K are Abelian, hence G is isomorphic to one of the groups listed in (iv)–(vii), by [2]. On the other hand, all the above groups have their Frattini subgroup elementary Abelian, a contradiction.

Proposition 1. *Assume that* $|G| = p^5$ *and* $\mathrm{cocl}(G) = 2$. *If* G *is* (MI), *then* G *is isomorphic to one of the groups listed in* (iv)–(ix).

Proof. We have $G = \langle x, y \rangle$, $Z_2 := Z_2(G) = \langle a, z_1, z_2 \rangle = \Phi(G)$ is elementary Abelian, where $a = [x, y]$ and $Z_1 := \langle z_1, z_2 \rangle = Z(G) = Z(M)$ for all maximal subgroups M. By (MI), either $\Omega(G) = G$ or $\Omega(G) \leq \Phi(G)$; thus we can assume that the elements x and y have the same order p or p^2. Since the maximal subgroups of G are nonabelian, $Z_2 = C_G(Z_2) = C_G(a)$. Therefore $\langle [a, x], [a, y] \rangle = [a, G]$ is of order p^2, that is, one can assume that $a^x = az_1$, $a^y = az_2$. Suppose that $p = 2$, when $[x, y^2] = x^{-1}(xa)^y = z_2$, and consequently $y^2 \in Z_2 \backslash Z_1$. Thus we have $y^2 = ac$ (with $c \in Z_1$), and hence $z_2 = [x, y^2] = [x, a] = z_1$, a contradiction; so $p > 2$. For any integer n, $x^{y_n} = xa^n z_2^{\binom{n}{2}}$; this implies that $y^p \in Z(G)$, and similarly $x^p \in Z(G)$.

Case 1. $x^p = y^p = 1$.

The generators x, y obviously satisfy the relations of the presentation in (viii) of Theorem 2. That relations yield that G is (MI) if and only if $p \geq 5$.

Case 2. x and y are of order p^2.

Let M denote a maximal subgroup of G. If $\Phi(M)$ is of order p^2, then $\Phi(M) = Z(M)$, that is M is a one-step nonabelian group. That means that G is exactly one of the groups listed in (iv)–(vii), by [3]. Therefore we can assume $|\Phi(M)| = p$, equivalently $\mho(M) = M'$; in particular, $x^p = z_1^\alpha$, $y^p = z_2^\beta$. For $p = 3$ we get a contradiction; for $p \geq 5$ $\alpha = \beta$ can be assumed to be 1, that is G is the group in (ix), which in turn is (MI).

Lemma 12. *Assume that* G *is* (MI) *of order* p^6 *and that* $\mathrm{cocl}(G) = 2$; *then* $Z_3(G)$ *is elementary Abelian, and every maximal subgroup in* G *is of class* 3.

Proof. Suppose that all maximal subgroups of G are of class 2. Suppose that $G = \langle x, y \rangle$ and $[x, y] = a$. Since $Z_3(G)$ is Abelian by (d) of Corollary 2, $x^p \in Z(\langle Z_3(G), x \rangle)$. As $\langle Z_3(G), x \rangle$ is a maximal subgroup of G, x^p lies in

$Z_2(G)$ by (a); hence $\mho(G) \leq Z_2(G)$. It follows that $G/Z_2(G)$ is a nonabelian group of exponent p, so p must be odd.

Since every maximal subgroup of G is supposed to be of class 2, we have $[a,x] = c \in C_G(x)$, $[a,y] = d \in C_G(y)$, $c^y = cz^\alpha$, $d^x = dz^\beta$ (with $Z(G) = \langle z \rangle$). Since $Z_3(G) = \langle Z_2(G), a \rangle$, we have $Z_2(G) = Z(Z_3(G) \cdot \langle x \rangle) \cdot Z(Z_3(G) \cdot \langle y \rangle) = (Z_3(G) \cdot \langle x \rangle)' \cdot (Z_3(G) \cdot \langle y \rangle)' = \langle c, d, Z(G) \rangle$, and therefore $\alpha \neq 0 \neq \beta$. Set $K = Z_3 \cdot \langle xy \rangle$. Then equations $[a, xy] = a^{-1}(ac)^y = dcz^\alpha$ and $[a, yx] = a^{-1}(ad)^x = cdx^\beta$ imply that $\alpha = \beta$, as $Z_3(G)$ is Abelian. Since $K' \leq Z(K)$, we can deduce that $1 = [cd, xy] = z^{2\alpha}$. Thus $\alpha = 0$, a contradiction; it proves that the maximal subgroups of G are of nilpotency class 3, by (f).

Suppose that $Z_3(G)$ is (Abelian) of type (p^2, p, p); then $Z_3(G) = \langle g, b_1, b_2 \rangle$, $Z_2(G) = \langle b_1, b_2, g^p \rangle$, $Z_1(G) = \langle g^p \rangle$. Let $M_1 = \langle Z_3(G), x \rangle$ be a maximal subgroup of G with $Z(M_1) = \langle b_1, g^p \rangle$; then by choosing b_2 suitably we can assume that $[g, x] = b_2$, $[b_1, x] = 1$, $[b_2, x] = g^{\alpha p}$, $(\alpha \neq 0)$: Since the order of M_1' is p^2 we can find a maximal subgroup $M_2 = \langle Z_3(G), y \rangle$ such that $Z(M_2) = M_1' = \langle b_2, g^p \rangle$. Obviously $[g, y] = b_1^\gamma b_2^\delta \cdot g^{\varphi p}$, $[b_1, y] = g^{\beta p}$, $[b_2, y] = 1$ $(\gamma \neq 0)$; hence $g^{xy} = (gb_2)^y = gb_1^\gamma \cdot b_2^{1+\delta} \cdot g^{\varphi p}$, $g^{yx} = (gb_1^\gamma \cdot b_2^\delta \cdot g^{\varphi p})^x = (gb_2)b_1^\gamma \cdot (b_2 g^{\alpha p})^\delta \cdot g^{\varphi p} = gb_1^\gamma \cdot b_2^{1+\delta} \cdot g^{(\varphi + \alpha\delta)p}$. Since $g^{xy} = g^{yx}$, we have $\alpha\delta = 0$, and thus $\delta = 0$ (since $\alpha \neq 0$). Replacing b_1 by $b_1^\gamma \cdot g^{\varphi p}$ one can assume $[g, y] = b_1$, $[b_1, y] = g^{\beta p}$, $[b_2, y] = 1$ (with a new value for β).

We show that the quadratic character of α $(\bmod p)$ is an invariant of M_1. For, let $g' = g^u b_1^v b_2^t$, $x' = x^i s$ $(s \in Z_3(G))$, in which case

$$\begin{aligned} b_2' &:= [g', x'] = [g, x^i]^u \cdot g^{\alpha t i p} \\ &= (b_2^i g^{\binom{i}{2}\alpha p})^u g^{\alpha t i p} = b_2^{iu} g^{(u\binom{i}{2} + ti)\alpha p}, \\ [b_2', x'] &= g^{i^2 u \alpha p} = (g'^p)^{i^2 \alpha}. \end{aligned}$$

Now for $\{k, m\} \neq \{0, 0\}$, set $M_{km} = \langle Z_3(G), x^k y^m \rangle$; the invariant of M_{km} is $\alpha k^2 + \beta m^2$, as

$$\begin{aligned} &[b, x^k y^m] = b_2^k b_1^m \pmod{Z(G)}, \\ &[g, x^k y^m, x^k y^m] = [b_1^m b_2^k, x^k y^m] = g^{p(\alpha k^2 + \beta m^2)}. \end{aligned}$$

Now G can be (MI) only if $\alpha, \beta, \alpha k^2 + \beta m^2$ have the same quadratic character $(\bmod p)$ for all nonzero pairs $\{k, m\}$. However, this does not happen for any prime p, and we have a contradiction. Thus $Z_3(G)$ is elementary Abelian by (c) of Corollary 2.

Using the previous lemma we can finish the proof of Theorem 2 by a direct calculation to get

Proposition 2. *Assume that G is (MI) of order p^6 and* $\operatorname{cocl}(G) = 2$; *then G is isomorphic to one of the groups listed in* (x)–(xii).

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Before turning to the case of coclass 3, we prove

LEMMA 13. *If the finite p-group G is (MI) and has a maximal subgroup M with $Z(M) \nleq \Phi(G)$, then the nilpotency class of G is at most 2.*

PROOF. There exists a maximal subgroup M_1 in G with $Z(M) \nleq M_1$, that is, $G = Z(M) \cdot M_1$. Then $[Z(M_1), G] = Z(M_1), Z(M)] \leq Z(M_1) \cap Z(M) = Z(G)$, and thus $Z(M_1) \leq Z_2(G)$. Suppose that the class of G is greater than 2, then by Lemma 1 $Z(M_1) \leq Z_2(G) \leq \Phi(G)$. Thus $Z(M_1) \leq M \leq C_G(Z(M))$, hence $Z(M) \leq C_G(Z(M_1)) = M_1$, a contradiction.

Now our aim is to prove

THEOREM 3. *If G is a finite p-group, G is (MI) and the coclass of G is 3, then $|G| \leq p^{13}$.*

We will get Theorem 3 as a corollary to three propositions.

PROPOSITION 3. *Assume that G is (MI), $|G| = p^n$, $\mathrm{cl}(G) = n - 3$ and $|G : Z_{n-4}(G)| = p^4$; then either*

(i) *G is Abelian of order p^4, or*
(ii) *G is extraspecial of order p^5.*

PROOF. Let M be a maximal subgroup of G, and denote the order of $Z(M)$ by p^k.

CASE 1. $|G : Z(M)| = p$.

Either G is Abelian, giving (i) or G is one step nonabelian. In the latter case $2 = \mathrm{cl}(G) = n - 3$, so that $n = 5$, which yields that $p^2 = |G : Z(G)| = |G : Z_1(G)| = |G : Z_{n-4}(G)| = p^4$, a contradiction.

CASE 2. $|G : Z(M)| = p^3$.

Suppose that $Z(M) \leq \Phi(G) = \Phi$ for all maximal subgroups M. The equality $Z(M) = \Phi$ would imply that $Z(M) = Z(G)$, hence $|G : Z_1(G)| = p^3$, a contradiction; thus $|\Phi : Z(M)| = p$ and $Z(M) > Z(G)$. So, with different maximal subgroups M_1 and M_2, $|\Phi : Z_1(G)| = |\Phi : Z(M_1) \cap Z(M_2)| = p^2$ gives $|Z(M) : Z(G)| = p$, so that $|G : Z_1(G)| = p^4$, $n = 5$, $|Z_1(G)| = p$, $\mathrm{cl}(G) = 2$. Since G is 2-generator, say $G = \langle x, y\rangle$, and $|G'| = p$, $\langle x^p, y, Z(G)\rangle$ is an Abelian maximal subgroup of G, a contradiction; therefore $Z(M_0) \nleq \Phi$ for some maximal subgroup M_0, whence $\mathrm{cl}(G) = 2$ by Lemma 13. We have then $n = 5$, $|G'| = p$, $M' = G'$; hence G/G' is also (MI). If it were of type (p^2, p^2), then we would get the same contradiction as above. Thus G/G' is elementary Abelian, that is G is extraspecial.

CASE 3. $|G : Z(M)| \geq p^4$.

$|Z(M)| = p^k$ implies that $Z(M) \leq Z_k(G) \leq Z_{n-4}(G)$, so that $Z(M) = Z_i(G)$ for some i, by Lemma 4. As i is independent of M, that gives $Z(M) = Z(G)$, and thus $\overline{G} = G/Z(G)$ is also (MI) of coclass 3. By the induction hypothesis, either

(i) $\overline{G}$ is Abelian of order p^4, or
(ii) $\overline{G}$ is extraspecial of order p^5.

In case (i), $\mathrm{cl}(G) = 2$, $n = 5$, whence $|G : Z_1(G)| = p^4 = |G : Z(M)|$; that is $Z(G) = Z(M)$ is of order p, contradicting Corollary 2. In case (ii), $\mathrm{cl}(G) = 3$, $n = 6$, so $p^4 = |G : Z_2(G)|$. This implies that $Z(M) \leq Z_2(G)$, but then $Z(M) = Z_2(G)$, as $p^2 \leq |Z(M)| \leq |Z_2(G)| = p^2$. Since $Z(M_1) = Z(M_2)$ for any maximal subgroups M_1, M_2, we have $Z_2(G) = Z(M) = Z_1(G)$, a contradiction.

Lemma 14. *Suppose that* $|G| = p^n$, $\mathrm{cl}(G) = n - 3$ *and* $|Z_2(G)| = p^3 = |G : Z_{n-4}(G)|$. *Assume that* $Z_2(M) \geq Z_2(G) \geq Z(M)$ *and* $|Z_2(M)| \geq p^{n-3}$ *for all maximal subgroups* M *of* G; *then* $n \leq 8$.

Proof. $Z_2(M) \geq \gamma_3(G)$, as $\mathrm{cl}(G/Z_2(M)) \leq 2$. Since $\mathrm{cl}(G) = n - 3$, we have $\gamma_3(G) \leq Z_{n-5}(G) \leq Z_{n-4}(G)$, and hence there exists some i with $\gamma_3(G)Z_2(G) = Z_i(G)$, by Lemma 4. Certainly $i = n - 5$, and thus $Z_2(M) \geq \gamma_3(G)Z_2(G) = Z_{n-5}(G)$. Let $M_1 \neq M_2$ be maximal subgroups in G, so that, by Lemma 5, $Z_{n-5}(G) \leq Z_2(M_1) \cap Z_2(M_2) \leq Z_3(G)$, so $n - 5 \leq 3$.

Proposition 4. *If* G *is* (MI), $|G| = p^n$, $\mathrm{cl}(G) = n - 3$ *and* $|G : Z_{n-4}(G)| = p^3$, *then* $n \leq 8$.

Proof. Let M be any maximal subgroup of G and $|Z(M)| = p^k$.

Case 1. $|G : Z(M)| = p$.

G is either Abelian of order p^4 or one step nonabelian of order p^5. In the first case $p^3 = |G : Z_{n-4}(G)| = |G : Z_0(G)| = p^4$, a contradiction. If $n = 5$, then $p^3 = |G : Z_{n-4}(G)| = |G : Z_1(G)| = p^2$, also a contradiction.

Case 2. $|G : Z(M)| = p^3$.

If $Z(M) = \Phi(G) = \Phi$ for all M, then $Z(M) = \Phi = Z(G)$ is of index p^3 in G, and therefore $n = 5$. If $|\Phi : Z(M)| = p$ for all M, then with $M_1 \neq M_2$ (maximal subgroups in G), $|\Phi : Z_1(G)| \leq |\Phi : Z(M_1) \cap Z(M_2)| \leq p^2$, hence $|G : Z_1(G)| \leq p^4$ and $n - 4 \leq 2$. If $Z(M_0) \nleq \Phi$ for some M_0, then $n - 3 = \mathrm{cl}(G) \leq 2$, by Lemma 13.

Case 3. $|G : Z(M)| \geq p^4$.

$Z(M) \leq Z_k(G) \leq Z_{n-4}(G)$, so since $\mathrm{cocl}(G) = 3$ and $|G : Z_{n-4}(G)| = p^3$, we have $|Z_k(G) : Z(M)| = p$. Were $Z_k(G) = Z(M)$, one would get $k = 1$, contrary to Lemma 7; thus $|Z_k(G) : Z(M)| = p$. Suppose that

$Z(M) = Z_1(G)$; then $k = 2$, and $\overline{G} = G/Z_1(G)$ is (MI) with $\mathrm{cocl}(\overline{G}) = 2$ and $|\overline{G} : Z_{n-5}(\overline{G})| = p^3$. Now Lemma 8 says that $|\overline{G}| = p^3$, so that $|G| \leq p^5$. Assume from now on that $Z(M) > Z(G)$; for any $M_1 \neq M_2$ (maximal subgroups in G), $|Z_k(G) : Z_1(G)| = |Z_k(G) : Z(M_1) \cap Z(M_2)| = p^2$, so that $2 \leq k \leq 3$. The inequalities $Z_k(G) \geq Z(M) \geq Z_1(G)$ exclude the possibility of $k = 3$ by Lemma 4, and hence $k = 2$. Therefore $|Z_2(G) : Z(M)| = p = |Z(M) : Z_1(G)|$, and $|Z_1(G)| = p$. If $|Z_2(M)| \geq p^{n-3}$, then $|G| \leq p^8$ by Lemma 14. Thus $|Z_2(M)| \leq p^{n-4}$, whence $Z_2(G) \leq Z_2(M) \leq Z_{n-4}(G)$, so that $Z_2(M) = Z_i(G)$ for some $2 \leq i \leq n-4$ by Lemma 4. As $Z(M) \leq Z_2(G)$, Lemma 5 gives $i \leq 3$. On the other hand, $\overline{G} = G/Z_i(G)$ is (MI) of coclass 2 and $|\overline{G} : Z_{n-i-4}(\overline{G})| = p^3$, thus $|\overline{G}| = p^3$, consequently $|G| = p^3 \cdot |Z_i(G)| = p^{3+i+1} \leq p^7$.

Proposition 5. *If G is (MI), $|G| = p^n$, $\mathrm{cl}(G) = n-3$ and $|G : Z_{n-4}(G)| = p^2$, then $n \leq 13$.*

Proof. Let M be any maximal subgroup in G and $|Z(M)| = p^k$; we can assume that $Z_k(G) < G$. In the same way as earlier, we may assume that $|G : Z(M)| \geq p^3$, as well as $Z(M) \leq \Phi(G) = Z_{n-4}(G)$. We define

$$t(M) = \min\{i \colon Z(M) \leq Z_i(G)\}.$$

Assume that $t(M_0) = 1$ for some M_0. Then $t(M) = 1$ for any M by Lemma 1. Now $\overline{G} = G/Z(G)$ is (MI), $\mathrm{cocl}(\overline{G}) \leq 2$ (by Lemma 7), and thus $|\overline{G}| \leq p^6$ by Theorem 2 and Lemma 3; so $|G| \leq p^6|Z(G)| = p^{6+k} \leq p^9$. We can therefore assume that $t(M) > 1$ for every M; if M_1 and M_2 are different maximal subgroups in G, then

$$\text{(E)} \quad |Z_k(G) : Z_1(G)| = |Z_k(G) : Z(M_1) \cap Z(M_2)| \leq |Z_k(G) : Z(M)|^2 \leq p^4$$

(for any M), whence $2 \leq t(M) \leq k \leq 5$.

Suppose that $t(M_0) = 5$ for some M_0, then $k = 5$, so we have equalities in (E). For any M, $Z_1(G) \leq Z(M) \leq Z_5(G)$ implies that $Z(M) = Z_i(G)$ for some i, by Lemma 4; necessarily $i = 1$, a contradiction; thus we may assume that $2 \leq t(M) \leq 4$ for all M. Suppose that $t(M_0) = 4$ for some M_0. We have $|Z_4(G) : Z_1(G)| \leq p^4$ by (E); if equality holds here, then $k = 4$, and by the second part of (E), it follows that $|Z_4(G) : Z(M)| = p^2$, whence $|Z(M) : Z_1(G)| = p^2$. The latter implies that $Z(M)/Z_1(G) \leq Z_2(G/Z_1(G))$, that is $Z(M) \leq Z_3(G)$; particularly $Z(M_0) \leq Z_3(G)$, a contradiction. So $|Z_4(G) : Z_1(G)| = p^3$, hence $Z_4(G) \geq Z(M) \geq Z_1(G)$ leads to $Z(M) = Z_i(G)$ for some i by Lemma 4. Clearly $i = 1$, that is $t(M) = 1$, a contradiction; thus we may assume that $2 \leq t(M) \leq 3$. In particular, $Z(M) \leq Z_3(G)$ for all M.

Suppose that $Z_2(M_0) \nleq Z_{n-4}(G)$ for some M_0; then $G = Z_2(M_0) \cdot M_1$, with a suitable maximal subgroup $M_1 \neq M_0$. So $[Z_2(M_1), G] \leq Z(M_1) \cdot (Z_2(M_1) \cap Z_2(M_0)) \leq Z_4(G)$ by Lemma 5, thus $Z_2(M_1) \leq Z_5(G)$. We have $|Z_2(M)| = |Z_2(M_1)| \leq |Z_5(G)| \leq p^7$ (or $n \leq 8$) for all maximal subgroups M. As $Z_2(M_0) \leq Z_7(G) \nleq Z_{n-4}(G)$, we get that $n - 4 < 7$, $n \leq 10$.

Assume now that $Z_2(M) \leq Z_{n-4}(G)$ for all maximal subgroups M. Let s ($\leq n-4$) be chosen to be minimal, subject to $Z_2(M) \leq Z_s(G)$ for all M.

CLAIM. $|Z_s(G) : Z_2(M)| \leq p$ (for all M).

First, $|Z_2(G) : Z_1(G)| \geq p^2$. In fact, supposing that $|Z_2(G) : Z_1(G)| = p$, we can find $N_i \lhd G$ with $Z(M_i) \geq N_i \geq Z(G)$, $|N_i : Z_1(G)| = p$ (for all maximal subgroups M_i). Then $N_i/Z_1(G) \leq Z(G/Z_1(G))$, that is $N_i \leq Z_2(G)$; so $Z_2(G) = N_i \leq Z(M_i)$ for all i, resulting in $Z_2(G) \leq \bigcap_i Z(M_i) = Z_1(G)$, a contradiction. Therefore $\operatorname{cocl}(G/Z_2(G)) \leq 2$, which proves our claim.

For any $M_1 \neq M_2$ (maximal subgroups of G),

$$p^2 \geq |Z_s(G) : Z_2(M_1) \cap Z_2(M_2)| \geq |Z_s(G) : Z_4(G)|$$

(by Lemma 5); that shows $s \leq 6$. If $Z_s(G) = Z_2(M)$ (for all M) then $G/Z_s(G)$ is (MI) of coclass $c \leq 2$, so

$$|G| = |G/Z_s(G)| \cdot |Z_s(G)| \leq p^{3c} \cdot p^{s+(3-c)} \leq p^{13}$$

by Theorem 2 and Lemma 3. In the remaining case $|Z_s(G) : Z_2(M)| = p$ (for all M), thus $|Z_s(G) : Z_2(G)| = p^{s-2}$ cannot hold by Lemma 4. Hence $G/Z_s(G)$ is of maximal class (or it is trivial, yielding $|G| = |Z_s(G)| \leq p^{s+3} \leq p^9$). We have $Z_3(M) \geq Z_s(G)$ (for all M) since $|Z_s(G) : Z_2(M)| = p$; on the other hand, $Z_3(M) \leq Z_{s+5}(G) \leq Z_{11}(G)$ by Lemma 6, whence $|Z_3(M)| \leq |Z_{11}(G)|$, so that $|Z_3(M)| \leq p^{13}$. Now either $|Z_3(M)| = p^{n-1}$, whence $\operatorname{cl}(M) \leq 3$, $\operatorname{cl}(G) \leq 6$, $n \leq 9$, or $Z_3(M) = Z_i(G)$ for some $s \leq i \leq s+5$ by Lemma 4. As for that latter case, $G/Z_i(G)$ is (MI) of coclass ≤ 1, and we can use Lemma 5 to conclude that $|Z_i(G)| = |Z_3(M)| \leq |Z_{s+1}(G)| \leq |Z_7(G)|$, involving $|Z_i(G)| \leq p^9$. We can summarize our conclusions as follows:

$$|G| = |G/Z_i(G)| \cdot |Z_i(G)| \leq p^3 \cdot p^9 = p^{12}.$$

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