COMPOSITIO MATHEMATICA

Quotient closed subcategories of quiver representations

Steffen Oppermann, Idun Reiten and Hugh Thomas

Compositio Math. **151** (2015), 568–602.

doi:10.1112/S0010437X14007623

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY 150 YEARS

Compositio Math. 151 (2015) 568–602
doi:10.1112/S0010437X14007623

Quotient closed subcategories of quiver representations

Steffen Oppermann, Idun Reiten and Hugh Thomas

Abstract

Let Q be a finite quiver without oriented cycles, and let k be an algebraically closed field. The main result in this paper is that there is a natural bijection between the elements in the associated Weyl group W_Q and the cofinite additive quotient closed subcategories of the category of finite dimensional right modules over kQ. We prove this correspondence by linking these subcategories to certain ideals in the preprojective algebra associated to Q, which are also indexed by elements of W_Q.

1. Introduction

Let Q be a finite quiver without oriented cycles. Let k be an algebraically closed field. The main result in this paper is that there is a natural bijection between the elements in the associated Weyl group W_Q and the cofinite additive quotient closed subcategories of the category $\operatorname{mod} kQ$ of finite dimensional right modules over the path algebra kQ. Here we call a subcategory $\mathcal{A}$ in $\operatorname{mod} kQ$ cofinite if there are only a finite number of indecomposable modules of $\operatorname{mod} kQ$ which are not in $\mathcal{A}$. From now on, when we refer to subcategories, we mean full, additive subcategories.

The natural bijection is given via the following map. Let $\mathcal{A}$ be a cofinite quotient closed subcategory of $\operatorname{mod} kQ$. We label the vertices of Q by $1, \ldots, n$ so that if $P_1, \ldots, P_n$ are the corresponding projective modules, then $\operatorname{Hom}(P_i, P_j) = 0$ for $i > j$. List the indecomposable modules not in $\mathcal{A}$, starting with the projective ones, with indices in increasing order, then similarly for $\tau^{-1}P_1, \ldots, \tau^{-1}P_n$, etc., where τ denotes the AR-translation. The sequence of modules gives rise to a word w by replacing $\tau^{-i}P_j$ by the simple reflection s_j of W_Q. For example, if Q is the quiver $1 \longrightarrow 2 \longrightarrow 3$, and $\tau^{-1}P_1, \tau^{-2}P_1$ are the indecomposable modules of a quotient closed subcategory of $\operatorname{mod} kQ$, then the missing indecomposables in the required order are $\{P_1, P_2, P_3, \tau^{-1}P_2\}$. The associated word is therefore $w = s_1 s_2 s_3 s_2$. Conversely, starting with an element w of length t, we describe explicitly how to find the t indecomposable kQ-modules which are not in the corresponding quotient closed subcategory.

Our method for proving this correspondence is to work with the preprojective algebra Π associated to Q. For each element w in W_Q, there is an associated ideal I_w in Π (see [IR08, BIRS09]), such that Π/I_w is a finite dimensional algebra. We associate to I_w the subcategory $\mathscr{C}(I_w) = \operatorname{add}((I_w)_{kQ}) \cap \operatorname{mod} kQ$. This is a subcategory of $\mathcal{P}$, the preprojective kQ-modules. We show that the additive category generated by $\mathscr{C}(I_w)$ together with the regular and preinjective kQ-modules is quotient closed and coincides with the subcategory corresponding as above to w in W_Q; we also show that any cofinite quotient closed subcategory of $\operatorname{mod} kQ$ is of this form. In our proofs, we have to distinguish between the Dynkin and non-Dynkin cases, with the

Received 14 December 2012, accepted in final form 29 April 2014, published online 15 October 2014.
2010 Mathematics Subject Classification 16G20, 16G70, 05E10 (primary).
Keywords: quotient closed subcategories, Weyl groups, preprojective algebras, sorting order.

Dynkin case being the more complicated one. To get the flavour of our results, the reader might prefer on a first reading to skip §§ 5 and 6, which deal with the Dynkin case.

For the most part, in this paper, we work over an algebraically closed ground field. This is necessary because of our reliance on [BIRS09], which makes this assumption. However, using the technology of *Frobenius maps*, we show that our main result extends to arbitrary finite dimensional hereditary algebras over finite fields.

Another interesting subcategory of mod kQ associated to an element w in W_Q is $\mathscr{C}(\Pi/I_w)$. When Q is Dynkin, we use our main theorem to show that the map from w to $\mathscr{C}(\Pi/I_w)$ is a bijection from W_Q to the subclosed subcategories of mod kQ. In the general case, we conjecture that there is a correspondence between the elements of W_Q and a certain specific subclass of the subclosed subcategories containing finitely many indecomposables.

The correspondence $w \longrightarrow \mathscr{C}(\Pi/I_w)$ was already investigated in a special case in [AIRT12]. It was shown that this gives a bijection between a special class of elements of W_Q called c-sortable [Rea07a] and torsion-free subcategories of mod kQ with a finite number of indecomposable objects. (Such a bijection had previously been constructed in [IT09, Theorem 4.3].)

By analogy, it would be interesting to describe the elements w of W_Q such that $\mathscr{C}(I_w)$ is a torsion class. Also, given such an element w, one might wish to determine the element of W_Q corresponding to the associated torsion-free class. We solve these problems for finite type, and we state a conjecture for the general case.

Quotient closed subcategories have not been extensively studied previously, though torsion classes are an important and well-studied special case. The dual concept, that of subclosed subcategories, arises in recent work of Ringel [Rin12] and of Krause and Prest [KP13]. In particular, Ringel has dealt with subclosed subcategories of infinite type, and has shown that any infinite subclosed subcategory of finite dimensional modules over a finite dimensional algebra contains a minimal infinite such subcategory. His work was motivated by previous work on the Gabriel–Roiter measure.

The inclusion order on cofinite quotient closed subcategories transfers over to give a partial order on the Weyl group W. This partial order was first studied by Armstrong [Arm09], under the name of 'sorting order'. Part of our motivation for this paper was to provide a representation-theoretic interpretation for this family of partial orders.

One of the referees raised a similar question: how could one transfer the group structure on W to the collection of cofinite quotient closed subcategories? This is an interesting question, which we are not able to answer.

The paper is organized as follows. In § 2 we give some essential background material, and we state our main theorem giving a bijection between elements of W_Q and cofinite, quotient closed subcategories of mod kQ. In § 3 we establish preliminary results on subfunctors of Ext^1_Π and on interpretations of reflection functors, which are important for the proof of the main result. We show in § 4 that I_w is determined by $\mathscr{C}(I_w)$ in the non-Dynkin case, and in § 5 that Π/I_w is determined by $\mathscr{C}(\Pi/I_w)$ in the Dynkin case. In § 6 we give some more results when Q is a Dynkin quiver, including the relationship between subcategories of the form $\mathscr{C}(\Pi/I_w)$ and those of the form $\mathscr{C}(I_w)$. The proof of the main theorem is given in § 7. In § 8 we extend our main theorem to give a bijection between arbitrary quotient closed subcategories of $\mathcal{P}$ and a suitable class of possibly infinite words. In § 9 we extend our main theorem to cover cofinite quotient closed subcategories of representations of hereditary algebras which are finite dimensional over a finite field. In § 10 we deal with the categories $\mathscr{C}(\Pi/I_w)$, and show that they are exactly the subclosed subcategories in the Dynkin case. We also state a conjecture for the non-Dynkin case. In § 11 we investigate when the categories $\mathscr{C}(I_w)$ are torsion classes, and how to describe the associated

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

torsion-free classes in that case. We give a complete answer in the Dynkin case, and state a conjecture in the general case. In § 12 we show how our main theorem can be used to recover the characterization by Postnikov [Pos06] in terms of Ⅎ-diagrams of the leftmost reduced subwords (equivalently, positive distinguished subexpressions) in the type A Grassmannian permutations. We discuss the connections to the work of Armstrong, mentioned above, in § 13.

2. Statement of main results

In this section we state our main results, and give relevant background material and an example for illustration.

Let Q be a quiver without oriented cycles and with vertices $1, \ldots, n$, and let k be an algebraically closed field. Denote by kQ the associated path algebra. The Weyl group $W = W_Q$ associated to Q has a distinguished set of generators $s_1, \ldots, s_n$, with relations $s_i^2 = e$ (the identity element), $s_i s_j = s_j s_i$ if there is no arrow between i and j, and $s_i s_j s_i = s_j s_i s_j$ if there is exactly one arrow between i and j. For an element w in W, an expression $\underline{w} = s_{i_1} \ldots s_{i_t}$ (called a *word*) is said to be *reduced* if t is as small as possible. In this case, $t = \ell(w)$ is the *length* of w.

Our main result is the following theorem.

Theorem 2.1. *There is a natural bijection between the elements in the Weyl group W_Q and the cofinite (additive) quotient closed subcategories of the category* $\operatorname{mod} kQ$ *of finitely generated kQ-modules.*

The following observation shows that we can equally well consider the cofinite quotient closed subcategories of the category $\mathcal{P}$ of preprojective kQ-modules.

Proposition 2.2. *Any cofinite quotient closed subcategory of* $\operatorname{mod} kQ$ *contains all the non-preprojective indecomposable kQ-modules. Further, any cofinite quotient closed subcategory of $\mathcal{P}$ can be extended to a cofinite quotient closed subcategory of* $\operatorname{mod} kQ$ *by taking the additive subcategory generated by it together with all the non-preprojective indecomposable kQ-modules.*

Proof. For Q Dynkin, $\mathcal{P} = \operatorname{mod} kQ$, so there is nothing to prove. Assume that Q is not Dynkin, and let $\mathcal{B}$ be an additive cofinite quotient closed subcategory of $\operatorname{mod} kQ$. Since $\mathcal{B}$ is cofinite, $\tau^{-i} kQ$ is in $\mathcal{B}$ for i sufficiently large. Since $\mathcal{B}$ is quotient closed and τ^{-1} preserves epimorphisms, it follows that $\mathcal{B}$ contains the regular and preinjective indecomposables of $\operatorname{mod} kQ$. This proves the first point.

Now suppose that $\mathcal{A}$ is a cofinite, quotient closed subcategory of $\mathcal{P}$. Let $\overline{\mathcal{A}}$ be the additive subcategory of $\operatorname{mod} kQ$ generated by $\mathcal{A}$ together with all the non-preprojective indecomposable objects of $\operatorname{mod} kQ$. Clearly, $\overline{\mathcal{A}}$ is cofinite in $\operatorname{mod} kQ$, and it is quotient closed because there are no non-zero maps from a regular or preinjective module to an object of $\mathcal{P}$. □

We introduce some more terminology in order to state the main theorem more explicitly. Let $\mathcal{A}$ be a cofinite, quotient closed subcategory of $\mathcal{P}$, and let $\mathcal{X}$ be the finite set of indecomposable preprojective modules not in $\mathcal{A}$. Let $P_1, \ldots, P_n$ be an ordering of the indecomposable projective kQ-modules, compatible with the orientation of Q, that is, such that if $i < j$ then $\operatorname{Hom}(P_j, P_i) = 0$. Consider the ordering $P_1, \ldots, P_n, \tau^{-1}P_1, \ldots, \tau^{-1}P_n, \tau^{-2}P_1, \ldots$ of the indecomposable preprojective kQ-modules, dropping any $\tau^{-i}P_j$ which are zero.

From this, we get an induced ordering on $\mathcal{X}$. We replace each module in $\mathcal{X}$ of the form $\tau^{-i}P_j$ for some i, by s_j, thereby obtaining a word $\underline{w}$ associated to the subcategory $\mathcal{A}$.

Conversely, start with an element $w \in W_Q$. Consider the infinite word $\underline{c}^\infty = \underline{c}\,\underline{c}\,\underline{c} \ldots$, where $\underline{c} = s_1 \ldots s_n$ is what is called a *Coxeter element*. We match the reflections in $\underline{c}^\infty$ with the indecomposable objects in $\mathcal{P}$, so that the first s_i corresponds to P_i, the second to $\tau^{-1}P_i$, and so on. Now, among all the reduced expressions $s_{i_1}\dots s_{i_t}$ in $\underline{c}^\infty$ for w, we choose the leftmost one, in the sense that s_{i_1} is as far to the left as possible in $\underline{c}^\infty$, and, among such expressions, s_{i_2} is as far to the left as possible (but to the right of s_{i_1}), and so on for each s_{i_j}. In this way, we determine a unique subword $\underline{w}$ of $\underline{c}^\infty$. Consider the associated set $\mathcal{X}$ of indecomposable preprojective modules corresponding to this subword, as discussed above. Then we associate to w the additive subcategory $\mathcal{A}$ of $\mathcal{P}$, whose indecomposable objects are the indecomposable objects of $\mathcal{P}$ which do not lie in $\mathcal{X}$. We can now state the following more explicit version of our main theorem.

THEOREM 2.3. *There is a bijective correspondence between elements $w \in W_Q$ and cofinite quotient closed subcategories of $\mathcal{P}$, which can be described as follows:*

(a) *the correspondence $w \longrightarrow \mathcal{A}$ is given by removing from $\mathcal{P}$ the indecomposable modules corresponding to the leftmost word $\underline{w}$ for w in $\underline{c}^\infty$;*
(b) *the correspondence $\mathcal{A} \longrightarrow w$ is given by taking the finite set $\mathcal{X}$ of indecomposable preprojective modules not in $\mathcal{A}$, and associating to it a word as described above.*

In order to prove these results, we work with the preprojective algebra $\Pi = \Pi_Q$ associated to kQ. For each arrow a in kQ, add an arrow a^* in the opposite direction to get a new quiver $\overline{Q}$. Then, by definition,

$$\Pi = k\overline{Q} \Big/ \sum_a (aa^* - a^*a).$$

We write $\operatorname{mod}\Pi$ for the category of finitely generated right Π-modules, and $\operatorname{Mod}\Pi$ for the category of all right Π-modules.

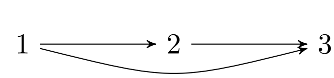
Let e_i be the idempotent corresponding to the vertex i. Then consider the ideal $I_i = \Pi(1-e_i)\Pi$ in Π. When $\underline{w} = s_{i_1}\dots s_{i_t}$ is a reduced expression for $w \in W_Q$, then I_w is (well-)defined by $I_w = I_{i_t}\dots I_{i_1}$ [BIRS09]. (Note that the product of ideals is taken in the opposite order to the product of reflections in $\underline{w}$. This follows the convention of [AIRT12].) Any Π-module, like I_w, is a kQ-module by restriction.

Consider the subcategory $\mathscr{C}(I_w)$ of $\operatorname{mod}\Pi$, whose indecomposable modules are those which appear as indecomposable summands of I_w as a kQ-module. We then have the following theorem.

THEOREM 2.4. *The cofinite quotient closed subcategory of $\mathcal{P}$ which corresponds to $w \in W_Q$ under the bijection of Theorem 2.3 can also be described as $\mathscr{C}(I_w)$.*

We now illustrate the above results.

Example 2.5. Let Q be the following quiver.



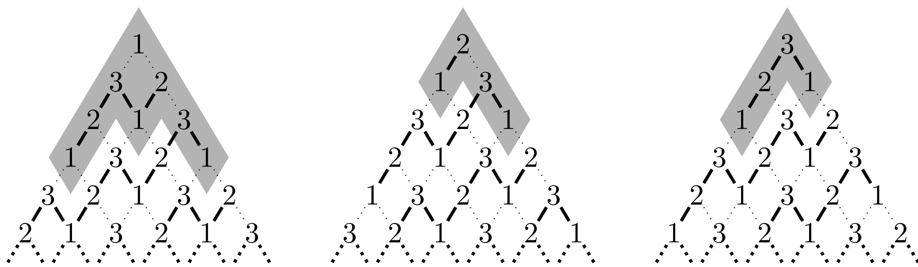
Then the indecomposable projective kQ-modules are as follows.

$$P_1 = 1, \quad P_2 = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \quad P_3 = \begin{smallmatrix} & & 3 & \\ & 2 & & 1 \\ 1 & & & \end{smallmatrix}$$

Let $w = s_1s_2s_3s_2s_1$. We have $I_w = I_1I_2I_3I_2I_1$ and $I_w = \widetilde{P}_1I_w \oplus \widetilde{P}_2I_w \oplus \widetilde{P}_3I_w$, where $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ are the indecomposable projective Π-modules. The modules $\widetilde{P}_i$, together with their submodules

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

$\widetilde{P}_i I_w$, are illustrated below. The regions in grey indicate the parts that do not appear in I_w. Solid lines indicate what remains connected upon restriction to kQ.



Then we compute that $(\widetilde{P}_1 I_w)_{kQ} = \tau^{-1}P_3 \oplus \big(\bigoplus_{i=3}^{\infty} \tau^{-i} P_1\big)$, $(\widetilde{P}_2 I_w)_{kQ} = \tau^{-1}P_1 \oplus \big(\bigoplus_{i=2}^{\infty} \tau^{-i} P_2\big)$, and $(\widetilde{P}_3 I_w)_{kQ} = \bigoplus_{i=1}^{\infty} \tau^{-i} P_3$. We see that the indecomposable kQ-modules not in $\mathscr{C}(I_w)$ are $P_1, P_2, P_3, \tau^{-1}P_2, \tau^{-2}P_1$.

We also illustrate how to see this by using our direct description of the missing set of $\ell(w) = 5$ indecomposable kQ-modules. Consider the infinite word $\underline{c}^{\infty} = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_3 \ldots$. We indicate the leftmost subword for the element w by underlining the corresponding s_i: $\underline{s_1}\underline{s_2}\underline{s_3}s_1\underline{s_2}s_3\underline{s_1}\ldots$. Hence we obtain the associated set of indecomposable modules P_1, P_2, P_3, $\tau^{-1}P_2, \tau^{-2}P_1$.

3. Results on preprojective algebras

In this section we give two results, which will be useful later, on the relationship between the path algebra kQ and the associated preprojective algebra Π. The first gives a long exact sequence involving the subfunctor of the ordinary $\operatorname{Ext}^1_{\Pi}$ functor, given by short exact sequences of Π-modules which split as kQ-modules. The second one gives a comparison between the functor $- \otimes_{\Pi} I_i$ and APR-tilting for kQ at the vertex i. Similar statements appear as [AIRT12, Corollary 2.11], [BK12, Proposition 22].

Relative homological algebra was investigated by Auslander and Solberg in [AS93]. They consider certain subfunctors of $\operatorname{Ext}^1_{\Pi}$ given by a choice of short exact sequences. In our context, we will be interested in those short exact sequences of Π-modules which split upon restriction to kQ. We write $\underline{\operatorname{Ext}}^1_{\Pi}$ for the subfunctor of $\operatorname{Ext}^1_{\Pi}$ given by these short exact sequences.

In the following lemma, we use a description of preprojective algebras which first appeared in [BGL87, Proposition 3.1],

$$\Pi = \mathrm{T}_{kQ}\Omega \quad \text{with } \Omega = \operatorname{Ext}^1_{kQ}(D(kQ), kQ).$$

Here T_{kQ} denotes the tensor algebra over kQ, that is

$$\mathrm{T}_{kQ}\Omega = kQ \oplus \Omega \oplus (\Omega \otimes_{kQ} \Omega) \oplus \Omega^{\otimes 3} \oplus \cdots .$$

In this description, a Π-module is given by a kQ-module M and a multiplication rule $\varphi_M : M \otimes_{kQ} \Omega \longrightarrow M$.

One may note that for finite dimensional M we have $M \otimes_{kQ} \Omega = \tau^{-1}M$, so in this case the above description coincides with Ringel's [Rin98].

Lemma 3.1. *For two Π-modules (A, φ_A) and (B, φ_B) we have an exact sequence*

$$0 \longrightarrow \operatorname{Hom}_{\Pi}(A,B) \xrightarrow{f} \operatorname{Hom}_{kQ}(A,B) \xrightarrow{g} \operatorname{Hom}_{kQ}(A \otimes_{kQ} \Omega, B) \xrightarrow{h} \underline{\operatorname{Ext}}^1_{\Pi}(A,B) \longrightarrow 0,$$

with maps given by

$f\colon$ *restriction functor*

$g\colon g(\alpha) = \varphi_B \circ (\alpha \otimes 1_\Omega) - \alpha \circ \varphi_A$

$h\colon h(\beta)$ *is given by the kQ-split short exact sequence*

$$0 \longrightarrow (B, \varphi_B) \longrightarrow \left(A \oplus B, \begin{pmatrix} \varphi_A & \\ \beta & \varphi_B \end{pmatrix}\right) \longrightarrow (A, \varphi_A) \longrightarrow 0.$$

Proof. Injectivity of f and surjectivity of h are clear.

It follows from the definition of g that $\operatorname{Ker} g = \operatorname{Im} f$.

We determine $\operatorname{Ker} h$: $\beta \in \operatorname{Ker} h$ if and only if the following diagram can be completed commutatively,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (B,\varphi_B) & \longrightarrow & \left(A \oplus B, \begin{pmatrix} \varphi_A & \\ & \varphi_B \end{pmatrix}\right) & \longrightarrow & (A,\varphi_A) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \Psi} & & \| & & \\ 0 & \longrightarrow & (B,\varphi_B) & \longrightarrow & \left(A \oplus B, \begin{pmatrix} \varphi_A & \\ \beta & \varphi_B \end{pmatrix}\right) & \longrightarrow & (A,\varphi_A) & \longrightarrow & 0 \end{array}$$

i.e. there is some $\Psi \in \operatorname{End}_{kQ}(A \oplus B)$ such that:

– $\Psi \circ \left(\begin{smallmatrix} \varphi_A & \\ & \varphi_B \end{smallmatrix}\right) = \left(\begin{smallmatrix} \varphi_A & \\ \beta & \varphi_B \end{smallmatrix}\right) \circ (\Psi \otimes 1_\Omega)$ (Ψ is a morphism of Π-modules);

– $\Psi \circ \left(\begin{smallmatrix} 0 \\ 1_B \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1_B \end{smallmatrix}\right)$ (the left square commutes); and

– $(1_A\ 0) \circ \Psi = (1_A\ 0)$ (the right square commutes).

Writing $\Psi = \left(\begin{smallmatrix} \Psi_{AA} & \Psi_{BA} \\ \Psi_{AB} & \Psi_{BB} \end{smallmatrix}\right)$ the latter two points amount to $\Psi_{AA} = 1_A$, $\Psi_{BB} = 1_B$, and $\Psi_{BA} = 0$. Thus the first one becomes

$$\begin{pmatrix} 1_A & \\ \Psi_{AB} & 1_B \end{pmatrix} \begin{pmatrix} \varphi_A & \\ & \varphi_B \end{pmatrix} = \begin{pmatrix} \varphi_A & \\ \beta & \varphi_B \end{pmatrix} \begin{pmatrix} 1_A \otimes 1_\Omega & \\ \Psi_{AB} \otimes 1_\Omega & 1_B \otimes 1_\Omega \end{pmatrix},$$

that is

$$\Psi_{AB} \circ \varphi_A = \beta + \varphi_B \circ (\Psi_{AB} \otimes 1).$$

Hence we have $\beta = g(-\Psi_{AB}) \in \operatorname{Im} g$.

The same calculation read backwards shows that $h \circ g = 0$. □

Proposition 3.2. *Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of* Π*-modules which splits upon restriction to* kQ*. Then for any* $X \in \operatorname{mod} \Pi$ *there are induced exact sequences*

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_\Pi(X, A) \longrightarrow \operatorname{Hom}_\Pi(X, B) \longrightarrow \operatorname{Hom}_\Pi(X, C) \\ \longrightarrow \underline{\operatorname{Ext}}^1_\Pi(X, A) \longrightarrow \underline{\operatorname{Ext}}^1_\Pi(X, B) \longrightarrow \underline{\operatorname{Ext}}^1_\Pi(X, C) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_\Pi(C, X) \longrightarrow \operatorname{Hom}_\Pi(B, X) \longrightarrow \operatorname{Hom}_\Pi(A, X) \\ \longrightarrow \underline{\operatorname{Ext}}^1_\Pi(C, X) \longrightarrow \underline{\operatorname{Ext}}^1_\Pi(B, X) \longrightarrow \underline{\operatorname{Ext}}^1_\Pi(A, X) \longrightarrow 0. \end{aligned}$$

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

Proof. Note that since $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is split exact over kQ the sequences

$$0 \longrightarrow \operatorname{Hom}_{kQ}(C, X) \longrightarrow \operatorname{Hom}_{kQ}(B, X) \longrightarrow \operatorname{Hom}_{kQ}(A, X) \longrightarrow 0 \text{ and}$$
$$0 \longrightarrow \operatorname{Hom}_{kQ}(C \otimes_{kQ} \Omega, X) \longrightarrow \operatorname{Hom}_{kQ}(B \otimes_{kQ} \Omega, X) \longrightarrow \operatorname{Hom}_{kQ}(A \otimes_{kQ} \Omega, X) \longrightarrow 0$$

are exact. Hence the proposition follows from Lemma 3.1 and the snake lemma. □

Now we investigate the interaction of tensoring with an ideal I_i and restricting to kQ. It turns out that on the level of kQ-modules, tensoring with I_i corresponds to applying an APR-tilt. A similar observation had already been made in [AIRT12]. We start by recalling the notion of APR-tilting [APR79], which is a module-theoretic interpretation of the reflections of [BGP73].

Let Q be a (connected, acyclic, finite) quiver, and i be a source of Q, so the corresponding indecomposable projective kQ-module P_i is simple.

The kQ-module

$$T = \tau^{-1} P_i \oplus kQ/P_i$$

is an APR-tilting module. We set Q' to be the Gabriel quiver of $\operatorname{End}_{kQ}(T)$, so that $kQ' = \operatorname{End}_{kQ}(T)$. Then we have the mutually inverse equivalences

$$\mathbf{R}\operatorname{Hom}_{kQ}(T, -) : \mathrm{D^b}(\operatorname{mod} kQ) \longleftrightarrow \mathrm{D^b}(\operatorname{mod} kQ') : - \otimes^L_{kQ'} T. \tag{3.1}$$

Recall from [BGL87, Rin98] that we have

$$\Pi = \bigoplus_{n \geqslant 0} \operatorname{Hom}_{\mathrm{D^b}(\operatorname{mod} kQ)}(kQ, \tau^{-n} kQ). \tag{3.2}$$

Since T is obtained from kQ by replacing one summand by its (inverse) AR-translation we also have

$$\Pi = \bigoplus_{n \geqslant 0} \operatorname{Hom}_{\mathrm{D^b}(\operatorname{mod} kQ)}(T, \tau^{-n} T) = \bigoplus_{n \geqslant 0} \operatorname{Hom}_{\mathrm{D^b}(\operatorname{mod} kQ')}(kQ', \tau^{-n} kQ'). \tag{3.3}$$

Lemma 3.3. *Via the identifications above we have isomorphisms of* Π–Π*-bimodules*

$$\Pi \cong \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathrm{D^b}(\operatorname{mod} kQ)}(kQ, \tau^{-n} T),$$
$$S_i \cong \bigoplus_{n < 0} \operatorname{Hom}_{\mathrm{D^b}(\operatorname{mod} kQ)}(kQ, \tau^{-n} T),$$
$$I_i \cong \bigoplus_{n \geqslant 0} \operatorname{Hom}_{\mathrm{D^b}(\operatorname{mod} kQ)}(kQ, \tau^{-n} T),$$

where in all cases the term on the right gets its right Π*-module structure via (3.2) and its left* Π*-module structure via (3.3).*

Proof. The first claim is seen similarly to the identification in (3.3).

For the second claim note that

$$\bigoplus_{n < 0} \operatorname{Hom}_{\mathrm{D^b}(\operatorname{mod} kQ)}(kQ, \tau^{-n} T) = \operatorname{Hom}_{\mathrm{D^b}(\operatorname{mod} kQ)}(P_i, \tau(\tau^{-1} P_i)),$$

so this module is isomorphic to S_i on both sides.

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

The final claim follows by looking at the short exact sequence

$$0 \longrightarrow \bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(kQ, \tau^{-n}T) \longrightarrow \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(kQ, \tau^{-n}T) \longrightarrow \bigoplus_{n<0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(kQ, \tau^{-n}T) \longrightarrow 0$$

of Π–Π-bimodules. □

Theorem 3.4. *Let Q be a quiver with a source i, and let T be the associated APR-tilting module as above. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Mod}\,\Pi & \xrightarrow{-\otimes_\Pi I_i} & \mathrm{Mod}\,\Pi \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{Mod}\,kQ' & \xrightarrow{-\otimes_{kQ'} T} & \mathrm{Mod}\,kQ \end{array}$$

Proof. By Lemma 3.3 the commutativity of the diagram in the theorem is equivalent to commutativity of the following diagram.

$$\begin{array}{ccc} \mathrm{Mod}\left[\bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(T, \tau^{-n}T)\right] & \xrightarrow{-\bigotimes\left[\bigoplus_{n\geqslant 0}\mathrm{Hom}_{\mathcal{P}}(kQ,\tau^{-n}T)\right]} & \mathrm{Mod}\left[\bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(kQ, \tau^{-n}kQ)\right] \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{Mod}\,\mathrm{End}_{kQ}(T) & \xrightarrow{-\otimes T} & \mathrm{Mod}\,kQ \end{array}$$

Here the restriction functors are given by restriction along the natural inclusions

$$\mathrm{End}_{kQ}(T) \hookrightarrow \bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(T, \tau^{-n}T) \text{ and}$$

$$kQ = \mathrm{End}_{kQ}(kQ) \hookrightarrow \bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(kQ, \tau^{-n}kQ), \text{ respectively.}$$

Equivalently we may regard the restriction functors as being given by (derived) tensoring with $\bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(T, \tau^{-n}T)$ and $\bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(kQ, \tau^{-n}kQ)$, respectively.

Thus the commutativity of the diagram is equivalent to the Π–kQ-bimodules

$$\bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(T, \tau^{-n}T) \otimes_{kQ'} T \quad \text{and} \quad \bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(kQ, \tau^{-n}T)$$

being isomorphic.

Clearly we have a morphism from the first to the second bimodule, which is given by evaluating. To see that this morphism is bijective it suffices to check that evaluation

$$\mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(T, X) \otimes_{kQ'} T \longrightarrow \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\, kQ)}(kQ, X)$$

is bijective for any indecomposable $X \in \mathrm{add} \bigoplus_{n\geqslant 0} \tau^{-n}T$. If X is concentrated in degree 0 then this follows from the mutually inverse equivalences in (3.1). If X is concentrated in degree -1 or lower the right hand side vanishes, so we need to show that the term on the left also vanishes. We have $\mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\,kQ)}(T, X) = 0$ unless $X = S_i[1]$, so this is the only case to consider. But

$$\mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\,kQ)}(T, S_i[1]) = S_i, \quad \text{and} \quad S_i \otimes_{kQ'} T = \mathrm{H}^0(S_i \otimes^{\mathbf{L}}_{kQ'} T) = \mathrm{H}^0(S_i[1]) = 0,$$

again by the mutually inverse equivalences of (3.1). □

4. Describing I_w for Q non-Dynkin

Note that kQ is a subalgebra of Π in a natural way. To a Π-module X, we associate the subcategory

$$\mathscr{C}(X) = \mathrm{add}\, X_{kQ} \cap \mathrm{mod}\, kQ.$$

For Q a non-Dynkin quiver, we show that for any element w in the Weyl group W, the Π-module I_w is uniquely determined by the category $\mathscr{C}(I_w)$, and even by $\mathrm{Fac}\,\mathscr{C}(I_w)$. We use this to show, in the non-Dynkin case, that if $\mathscr{C}(I_v) \subseteq \mathscr{C}(I_w)$, then $I_v \subseteq I_w$. This is crucial for the proof of the main theorem. The Dynkin case of this result will be treated in § 6.

For $\mathcal{A}$ any (additive) subcategory of $\mathrm{mod}\, kQ$, we write $\mathrm{Fac}\,\mathcal{A}$ for the category consisting of the quotients of objects in $\mathcal{A}$. We write $\mathrm{filt}(\mathcal{A})$ for the minimal subcategory of finite length Π-modules containing the objects from $\mathcal{A}$ and closed under kQ-split extensions. We make the straightforward observation that $\mathrm{filt}(\mathscr{C}(I_w)) \subseteq \mathrm{filt}(\mathscr{C}(I_v))$ if and only if $\mathscr{C}(I_w) \subseteq \mathscr{C}(I_v)$, since the kQ-modules in $\mathrm{filt}(\mathcal{A})$ are exactly the objects of $\mathcal{A}$.

We set

$${}^{\perp}I_w = \{M \in \mathrm{mod}\,\Pi \mid \underline{\mathrm{Ext}}^1_{\Pi}(M, I_w) = 0\}.$$

Throughout this section, let Q be a connected quiver which is not Dynkin. Then we know that the ideals I_w are tilting Π-modules [BIRS09], and that the bounded derived category of finite length Π-modules is 2-Calabi–Yau [GLS07].

Proposition 4.1. *For any $w \in W$ we have $\mathscr{C}(I_w) \subseteq {}^{\perp}I_w$.*

Proof. We consider $F \in \mathrm{Fac}\, I_w$ finite dimensional. Then we have an epimorphism $I_w^n \twoheadrightarrow F$ for some n. Applying $\mathrm{Hom}_{\Pi}(I_w, -)$ and using that the projective dimension of I_w is at most 1, we obtain an epimorphism $\mathrm{Ext}^1_{\Pi}(I_w, I_w^n) \twoheadrightarrow \mathrm{Ext}^1_{\Pi}(I_w, F)$. The first space is zero since I_w is tilting, so $\mathrm{Ext}^1_{\Pi}(I_w, F) = 0$ as well.

Next we look at $\mathrm{Ext}^1_{\Pi}(F, I_w)$. The short exact sequence $0 \to I_w \to \Pi \to \Pi/I_w \to 0$, together with the fact that $\mathrm{Ext}^i_{\Pi}(F, \Pi) = 0$ for $i = 0, 1$, gives $\mathrm{Ext}^1_{\Pi}(F, I_w) \cong \mathrm{Hom}_{\Pi}(F, \Pi/I_w)$. Thus we have

$$\begin{aligned} \mathrm{Ext}^1_{\Pi}(F, I_w) &\cong \mathrm{Hom}_{\Pi}(F, \Pi/I_w) \\ &\cong D\,\mathrm{Ext}^2_{\Pi}(\Pi/I_w, F) \quad \text{by the 2-CY property} \\ &\cong D\,\mathrm{Ext}^1_{\Pi}(I_w, F) \quad \text{since } I_w = \Omega(\Pi/I_w) \\ &= 0. \end{aligned}$$

Now let $X \in \mathscr{C}(I_w)$. That is, X is a kQ-split subquotient of (a finite sum of copies of) I_w. That is, X is a kQ-split submodule of some kQ-split quotient F of I_w^n. By the discussion above we know that $\underline{\mathrm{Ext}}^1_{\Pi}(F, I_w) = 0$. Now, using the right exactness of $\underline{\mathrm{Ext}}^1_{\Pi}$, we obtain $\underline{\mathrm{Ext}}^1_{\Pi}(X, I_w) = 0$. □

Lemma 4.2. *For any $w \in W$, the category ${}^{\perp}I_w$ is closed under taking factors modulo finite dimensional modules.*

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

Proof. Let $0 \longrightarrow A \longrightarrow X \longrightarrow F \longrightarrow 0$ be a short exact sequence of Π-modules, such that A is finite dimensional and $X \in {}^{\perp}I_w$. Then $\mathrm{Hom}_{\Pi}(A, I_w) = 0$, and so we have a monomorphism $\mathrm{Ext}^1_{\Pi}(F, I_w) \rightarrowtail \mathrm{Ext}^1_{\Pi}(X, I_w)$. Note that the pullback of a kQ-split short exact sequence is kQ-split, so we obtain the following commutative square.

$$\begin{array}{ccc} \underline{\mathrm{Ext}}^1_{\Pi}(F, I_w) & \hookrightarrow & \mathrm{Ext}^1_{\Pi}(F, I_w) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Ext}}^1_{\Pi}(X, I_w) & \hookrightarrow & \mathrm{Ext}^1_{\Pi}(X, I_w) \end{array}$$

Since the right map is injective, so is the left one. Thus, since $X \in {}^{\perp}I_w$, also $F \in {}^{\perp}I_w$. □

COROLLARY 4.3. *For any $w \in W$ we have $\mathrm{Fac}\,\mathscr{C}(I_w) \subseteq {}^{\perp}I_w$.*

COROLLARY 4.4. *For any $w \in W$ we have $\mathrm{filt}(\mathrm{Fac}\,\mathscr{C}(I_w)) \subseteq {}^{\perp}I_w$.*

We say that a short exact sequence of Π-modules

$$0 \longrightarrow Z \longrightarrow A \longrightarrow M \longrightarrow 0,$$

with M in some subcategory $\mathcal{X}$ of $\mathrm{mod}\,\Pi$, is a universal extension of Z by $\mathcal{X}$, if the induced map $M \longrightarrow Z[1]$ is a right $\mathcal{X}$-approximation. The extension is called *minimal* if the map is right minimal. It follows directly that a minimal universal extension is unique up to isomorphism. We then have the following consequence of Corollaries 4.3 and 4.4.

COROLLARY 4.5. *For any $w \in W$, and n such that $I_{c^n} \subseteq I_w$, the sequence*

$$0 \longrightarrow I_{c^n} \longrightarrow I_w \longrightarrow M \longrightarrow 0 \tag{4.1}$$

is a minimal universal kQ-split extension of I_{c^n} by objects in any of $\mathrm{filt}(\mathscr{C}(I_w))$, $\mathrm{filt}(\mathrm{Fac}\,\mathscr{C}(I_w))$, or ${}^{\perp}I_w$.

Proof. Since the sequence $0 \longrightarrow I_{c^n} \longrightarrow \Pi \longrightarrow \Pi/I_{c^n} \longrightarrow 0$ is clearly kQ-split, so is the sequence $0 \longrightarrow I_{c^n} \longrightarrow I_w \longrightarrow M \longrightarrow 0$. Since $M \in \mathrm{filt}(\mathscr{C}(I_w)) \subseteq \mathrm{filt}(\mathrm{Fac}\,\mathscr{C}(I_w)) \subseteq {}^{\perp}I_w$ by Corollary 4.4, the sequence in (4.1) is a kQ-split extension of I_{c^n} by objects in any of $\mathrm{filt}(\mathscr{C}(I_w))$, $\mathrm{filt}(\mathrm{Fac}\,\mathscr{C}(I_w))$, or ${}^{\perp}I_w$. Moreover, since by definition $\underline{\mathrm{Ext}}^1_{\Pi}({}^{\perp}I_w, I_w) = 0$, the extension is universal.

To see that it is minimal, consider the associated map $M \longrightarrow I_{c^n}[1]$. Since I_w has no non-zero finite dimensional summand, there is also no non-zero summand of M which splits off, and hence the map is right minimal. □

THEOREM 4.6. *Let $v, w \in W$. Let Q be non-Dynkin, and assume that at least one of the following conditions holds:*

(i) $\mathscr{C}(I_v) \subseteq \mathscr{C}(I_w)$;
(ii) $\mathrm{Fac}\,\mathscr{C}(I_v) \subseteq \mathrm{Fac}\,\mathscr{C}(I_w)$;
(iii) ${}^{\perp}I_v \subseteq {}^{\perp}I_w$;

then it follows that $I_v \subseteq I_w$.

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

Proof. It has already been remarked that condition (i) is equivalent to $\mathrm{filt}(\mathscr{C}(I_v)) \subseteq \mathrm{filt}(\mathscr{C}(I_w))$, and similarly that (ii) is equivalent to $\mathrm{filt}(\mathrm{Fac}\,\mathscr{C}(I_v)) \subseteq \mathrm{filt}(\mathrm{Fac}\,\mathscr{C}(I_w))$. We may therefore replace (i) and (ii) by these conditions.

We consider the following exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{c^n} & \xrightarrow{\alpha} & I_v & \longrightarrow & I_v/I_{c^n} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\ 0 & \longrightarrow & I_{c^n} & \xrightarrow{\beta} & I_w & \longrightarrow & I_w/I_{c^n} & \longrightarrow & 0 \end{array}$$

By Corollary 4.5 they are universal extensions, and by assumption the sets we are universally extending by are contained one in the other. Hence we obtain the factorization indicated in the diagram.

Let $x \in W$. Consider the short exact sequence

$$0 \longrightarrow I_x \xrightarrow{\iota_x} \Pi \longrightarrow \Pi/I_x \longrightarrow 0.$$

Since Π/I_x is finite dimensional, we have $\mathrm{Ext}^i_\Pi(\Pi/I_x, \Pi) = 0$ for $i \in \{0, 1\}$. Thus the above sequence induces an isomorphism

$$\mathrm{Hom}_\Pi(I_x, \Pi) \cong \mathrm{Hom}_\Pi(\Pi, \Pi).$$

In other words, any map $I_x \longrightarrow \Pi$ factors uniquely through the embedding ι_x.

We then observe that $\iota_w \varphi \in \mathrm{Hom}_\Pi(I_v, \Pi)$, and hence $\iota_w \varphi = \lambda \iota_v$ for some $\lambda \in \Pi$. By the commutativity of the left square above we have

$$\lambda \iota_{c^n} = \lambda \iota_v \alpha = \iota_w \varphi \alpha = \iota_w \beta = \iota_{c^n},$$

so $\lambda = 1$. Hence we have a commutative triangle

$$\begin{array}{ccc} I_v & \overset{\iota_v}{\hookrightarrow} & \\ {\scriptstyle \varphi}\downarrow & & \Pi \\ I_w & \underset{\iota_w}{\hookrightarrow} & \end{array}$$

and thus φ is an inclusion of ideals of Π. □

We state here a lemma, essentially in [BIRS09], which we will need to refer to in the following. As usual, for $w \in W$, we denote by $\ell(w)$ the length of a reduced expression for w.

Lemma 4.7. *Suppose that Q is a non-Dynkin quiver. Let w be an element of the Weyl group W. Assume that $\ell(s_i w) > \ell(w)$. Then we have the following:*

(i) *$I_{s_i w}$ is properly contained in I_w;*

(ii) *$\mathrm{Tor}_1^\Pi(I_w, S_i) = 0$;*

(iii) *the natural map $I_w \otimes I_i \longrightarrow I_{s_i w}$ is an isomorphism.*

Proof. (i) is part of [BIRS09, Proposition III.1.10].

Consider the short exact sequence $0 \longrightarrow I_i \longrightarrow \Pi \longrightarrow S_i \longrightarrow 0$. Tensoring with I_w we obtain the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^\Pi(I_w, S_i) \longrightarrow I_w \otimes_\Pi I_i \longrightarrow I_w \longrightarrow I_w \otimes_\Pi S_i \longrightarrow 0.$$

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

From [BIRS09, Proposition III.1.1], we know that at least one of $\operatorname{Tor}_1^{\Pi}(I_w, S_i)$ and $I_w \otimes_{\Pi} S_i$ is zero. The image of $I_w \otimes I_i$ inside I_w is $I_{s_i w}$, which is properly contained in I_w. Thus $I_w \otimes_{\Pi} S_i$ is non-zero, and it follows that $\operatorname{Tor}_1^{\Pi}(I_w, S_i)$ is zero. This establishes (ii), and (iii) also follows. □

5. Describing Π/I_w

In this section, we show that I_w can be constructed from each of the categories $\mathscr{C}(\Pi/I_w)$ and $\mathscr{C}(\operatorname{Sub}\Pi/I_w)$. This will be done by investigating the annihilators of the categories $\operatorname{filt}(\mathscr{C}(\Pi/I_w))$ and $\operatorname{filt}(\mathscr{C}(\operatorname{Sub}\Pi/I_w))$. The results of this section hold for arbitrary quivers, but will be applied only in the Dynkin case in the following section.

Lemma 5.1. *The category*

$$\mathcal{Q} = \{X \in \operatorname{mod}\Pi \mid X \text{ is a } kQ\text{-split quotient of an object in } \operatorname{Sub}\Pi/I_w\}$$

is closed under kQ-split extensions.

Proof. Let

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be a kQ-split short exact sequence, with X and Z in $\mathcal{Q}$. Then there are kQ-split epimorphisms $X' \twoheadrightarrow X$ and $Z' \twoheadrightarrow Z$ with X' and Z' in $\operatorname{Sub}\Pi/I_w$.

First consider the pullback along $Z' \twoheadrightarrow Z$ as indicated in the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y' & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow\!\!\downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

The map $Y' \longrightarrow Y$ is a kQ-split epimorphism, since it is a pullback of the kQ-split epimorphism $Z' \twoheadrightarrow Z$.

Now note that, if we denote by K the kernel of the map $X' \twoheadrightarrow X$, then right exactness of $\underline{\operatorname{Ext}}_{\Pi}^1$ from Proposition 3.2 implies that we obtain the following pullback diagram, and moreover that the lower short exact sequence is also kQ-split.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow g & & \downarrow f & & \\
0 & \longrightarrow & K & \longrightarrow & Y'' & \longrightarrow & Y' & \longrightarrow & 0
\end{array}$$

The map f is a monomorphism with cokernel Z' by the first diagram. It follows from the second diagram that g is also a monomorphism with cokernel Z'. Thus Y'' is an extension of Z' by X'.

Finally, by [BIRS09, Proposition III.2.3], the category $\operatorname{Sub}\Pi/I_w$ is extension closed, so $Y'' \in \operatorname{Sub}\Pi/I_w$, and hence $Y \in \mathcal{Q}$. □

Proposition 5.2. *For $w \in W$ we have*

$$\begin{aligned} I_w &= \operatorname{Ann}(\operatorname{filt}(\mathscr{C}(\Pi/I_w))) \\ &= \operatorname{Ann}(\operatorname{filt}(\mathscr{C}(\operatorname{Sub}\Pi/I_w))). \end{aligned}$$

In particular $\mathscr{C}(\Pi/I_w) \subseteq \mathscr{C}(\Pi/I_v)$ implies $I_v \subseteq I_w$ for any $v, w \in W$.

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

Proof. Since $\Pi/I_w \in \operatorname{filt}(\mathscr{C}(\Pi/I_w)) \subseteq \operatorname{filt}(\mathscr{C}(\operatorname{Sub}\Pi/I_w))$, and $\operatorname{Ann}\Pi/I_w = I_w$, we clearly have

$$I_w \supseteq \operatorname{Ann}(\operatorname{filt}(\mathscr{C}(\Pi/I_w))) \supseteq \operatorname{Ann}(\operatorname{filt}(\mathscr{C}(\operatorname{Sub}\Pi/I_w))).$$

Thus it only remains to see that I_w annihilates $\operatorname{filt}(\mathscr{C}(\operatorname{Sub}\Pi/I_w))$.

Now note that $\mathscr{C}(\operatorname{Sub}\Pi/I_w)$ is contained in the set $\mathcal{Q}$ of Lemma 5.1 above. Since this set is closed under kQ-split extensions by Lemma 5.1 it follows that also $\operatorname{filt}(\mathscr{C}(\operatorname{Sub}\Pi/I_w)) \subseteq \mathcal{Q}$. Thus it suffices to see that I_w annihilates $\mathcal{Q}$. This however is clear, since the objects in $\mathcal{Q}$ are subquotients of $\operatorname{add}\Pi/I_w$ by definition. $\square$

We also record here a lemma which we will need later.

Lemma 5.3. *The subcategory $\mathscr{C}(\operatorname{Sub}\Pi/I_w)$ is subclosed.*

Proof. Let M be a submodule of some module in $\mathscr{C}(\operatorname{Sub}\Pi/I_w)$. That means that M is a submodule of $(\Pi/I_w)^n_{kQ}$ for some n. Note that Π/I_w is a graded Π-module, and $(\Pi/I_w)_{kQ}$ is just the sum of the graded pieces $(\Pi/I_w)_d$. Thus M is a submodule of $\bigoplus_{d=0}^{\infty}(\Pi/I_w)_d^{n_d}$, for suitable n_d. It follows that in the upper line of the following diagram, M is embedded into the degree 0 part of the Π-module on the right, where we have written (d) to indicate a shift of the grading by d.

$$\begin{array}{ccccc} M & \hookrightarrow & \bigoplus_{d=0}^{\infty}(\Pi/I_w)_d^{n_d} & \hookrightarrow & \bigoplus_{d=0}^{\infty}(\Pi/I_w)(d)^{n_d} \\ \downarrow & & & \nearrow & \\ M \otimes_{kQ} \Pi & & - - - & & \end{array}$$

By Hom-tensor adjointness we obtain a degree-preserving Π-linear map as indicated by the dashed arrow above. In particular its image Y is a graded Π-submodule of $\bigoplus_{d=0}^{\infty}(\Pi/I_w)(d)^{n_d}$. Looking at degree 0, we see that $Y_0 \cong M$, and clearly $Y_0 \in \mathscr{C}(Y) \subseteq \mathscr{C}(\operatorname{Sub}\Pi/I_w)$. $\square$

6. Connection between the ideals I_w and the quotients Π/I_w in the Dynkin case

In § 4 we have seen that $\mathscr{C}(I_v) \subseteq \mathscr{C}(I_w)$ implies $I_v \subseteq I_w$ for Q non-Dynkin. In order to prove the same result in the Dynkin case as well, we prove in this section that each I_w is dual to some $\Pi/I_{w'}$. This allows us to work with $\Pi/I_{w'}$ instead of I_w, so that we can use the results from § 5 describing $\Pi/I_{w'}$ to achieve the desired result.

Throughout this section let Π be the preprojective algebra of a Dynkin quiver. We write $\operatorname{mod}\Pi^{\mathrm{op}}$ for the category of right Π^{op}-modules (or equivalently left Π-modules).

The following lemma is the Dynkin analogue of Lemma 4.7. For $w \in W$ we denote as usual by $\ell(w)$ the length of a shortest expression for w.

Lemma 6.1. *Let Q be Dynkin, and let w be an element of the Weyl group W. Assume that $\ell(s_i w) > \ell(w)$. Then we have the following:*

(i) *$I_{s_i w}$ is properly contained in I_w;*
(ii) *$\operatorname{Tor}_1^{\Pi}(I_w, S_i) = 0$;*
(iii) *the natural map $I_w \otimes I_i \longrightarrow I_{s_i w}$ is an isomorphism.*

Proof. Let $\widehat{Q}$ be a non-Dynkin quiver containing Q as a full subquiver. Let Π be the preprojective algebra for Q, and $\widehat{\Pi}$ the preprojective algebra for $\widehat{Q}$. Denote by $\widehat{I}_w$ and $\widehat{I}_i$ the corresponding

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

ideals in $\widehat{\Pi}$. We have the exact sequence:

$$\mathrm{Tor}_1^{\widehat{\Pi}}(\widehat{I}_w, S_i) \longrightarrow \widehat{I}_w \otimes \widehat{I}_i \longrightarrow \widehat{I}_w \longrightarrow \widehat{I}_w \otimes S_i,$$

and $\widehat{I}_{s_iw}$ is the image of $\widehat{I}_w \otimes \widehat{I}_i$ inside $\widehat{I}_w$. By Lemma 4.7, we know $\mathrm{Tor}_1^{\widehat{\Pi}}(\widehat{I}_w, S_i) = 0$, and that $\widehat{I}_{s_iw}$ is properly contained in $\widehat{I}_w$.

(i) Since $\widehat{I}_{s_iw}$ is properly contained in $\widehat{I}_w$, we have a proper epimorphism $\widehat{\Pi}/\widehat{I}_{s_iw} \longrightarrow \widehat{\Pi}/\widehat{I}_w$. Since $\widehat{\Pi}/\widehat{I}_{s_iw} \cong \Pi/I_{s_iw}$ and $\widehat{\Pi}/\widehat{I}_w \cong \Pi/I_w$, we have a proper epimorphism $\Pi/I_{s_iw} \longrightarrow \Pi/I_w$. Hence we have that I_{s_iw} is properly contained in I_w.

(ii) As discussed above, $\mathrm{Tor}_1^{\widehat{\Pi}}(\widehat{I}_w, S_i) = 0$. Further,

$$\begin{aligned}\mathrm{Tor}_1^{\widehat{\Pi}}(\widehat{I}_w, S_i) &\cong \mathrm{Tor}_2^{\widehat{\Pi}}(\widehat{\Pi}/\widehat{I}_w, S_i) \cong \mathrm{Tor}_2^{\widehat{\Pi}}(\Pi/I_w, S_i) \cong D(\mathrm{Ext}^2_{\widehat{\Pi}}(\Pi/I_w, S_i)) \\ &\cong \mathrm{Hom}_{\Pi}(S_i, \Pi/I_w) \cong \underline{\mathrm{Hom}}_{\Pi}(S_i, \Pi/I_w),\end{aligned}$$

by using [CE56] and the 2-CY property for finite length $\widehat{\Pi}$-modules.

On the other hand, we have

$$\mathrm{Tor}_1^{\Pi}(I_w, S_i) \cong \mathrm{Tor}_2(\Pi/I_w, S_i) \cong D(\mathrm{Ext}^2_{\Pi}(\Pi/I_w, S_i)) \cong \underline{\mathrm{Hom}}(S_i, \Pi/I_w),$$

using [CE56] and the 2-CY property for $\underline{\mathrm{mod}}(\Pi)$.

Since $\mathrm{Tor}_1^{\widehat{\Pi}}(\widehat{I}_w, S_i) = 0$, we then have $\mathrm{Hom}_{\Pi}(S_i, \Pi/I_w) = 0$, hence $\underline{\mathrm{Hom}}_{\Pi}(S_i, \Pi/I_w) = 0$, and so $\mathrm{Tor}_1^{\Pi}(I_w, S_i) = 0$.

(iii) Consider the exact sequence

$$\mathrm{Tor}_1^{\Pi}(I_w, S_i) \longrightarrow I_w \otimes_{\Pi} I_i \longrightarrow I_w \longrightarrow I_w \otimes_{\Pi} S_i.$$

By definition, I_{s_iw} is the image of $I_w \otimes I_i$ in I_w. The result now follows from (ii). □

Lemma 6.2. *Let $w \in W$.*

(i) *If $\ell(s_iw) > \ell(w)$ then $\mathrm{Ext}^1_{\Pi}(I_w, S_i) = 0 = \mathrm{Ext}^1_{\Pi}(S_i, I_w)$.*
(ii) *If $\ell(ws_i) > \ell(w)$ then $\mathrm{Ext}^1_{\Pi^{\mathrm{op}}}(I_w, S_i) = 0 = \mathrm{Ext}^1_{\Pi^{\mathrm{op}}}(S_i, I_w)$.*

Proof. We only prove (i), since (ii) is dual.

Consider the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\Pi}(I_w, S_i) \longrightarrow I_w \otimes_{\Pi} I_i \longrightarrow I_w \longrightarrow I_w \otimes_{\Pi} S_i \longrightarrow 0.$$

We know that if $\ell(s_iw) > \ell(w)$ then $\mathrm{Tor}_1^{\Pi}(I_w, S_i) = 0$ by Lemma 6.1.

Now note that $\mathrm{Tor}_1^{\Pi}(I_w, S_i) = D\,\mathrm{Ext}^1_{\Pi}(I_w, \underbrace{DS_i}_{=S_i})$ (see [CE56]).

Hence the claim that $\mathrm{Ext}^1_{\Pi}(S_i, I_w) = 0$ follows from the 2-CY property of the stable category $\underline{\mathrm{mod}}\Pi$. □

Lemma 6.3. (i) *Let $M \subseteq N$ in $\mathrm{mod}\,\Pi^{\mathrm{op}}$. Then $I_iM = I_iN$ if and only if $N/M \cong S_i^n$ for some n.*
(ii) *Let $M \subseteq N$ in $\mathrm{mod}\,\Pi$. Then $MI_i = NI_i$ if and only if $N/M \cong S_i^n$ for some n.*

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

Proof. We only prove (i), since (ii) is dual.

Consider $0 \longrightarrow M \longrightarrow N \longrightarrow N/M \longrightarrow 0$ in $\operatorname{mod} \Pi^{\mathrm{op}}$. Multiplying with I_i we obtain the complex

$$0 \longrightarrow I_i M \longrightarrow I_i N \longrightarrow I_i N/M \longrightarrow 0$$

whose homology is concentrated in the middle.

If we assume $I_i M = I_i N$ then it follows that $I_i(N/M) = 0$, and hence $N/M \cong S_i^n$ for some n.

Assume now conversely that $N/M \cong S_i^n$ for some n. Then $I_i \otimes_\Pi N/M = 0$, since $I_i^2 = I_i$, and hence the map $I_i \otimes_\Pi M \longrightarrow I_i \otimes_\Pi N$ is onto. It follows that the inclusion map $I_i M \hookrightarrow I_i N$ is also onto, and hence $I_i M = I_i N$. $\square$

We write w_0 for the longest element in W (which only exists when W is finite).

Proposition 6.4. *Let $w \in W$. Then:*

(i) $DI_w \cong \Pi/I_{w_0 w^{-1}}$ *in* $\operatorname{mod} \Pi^{\mathrm{op}}$;

(ii) $DI_w \cong \Pi/I_{w^{-1} w_0}$ *in* $\operatorname{mod} \Pi$.

Proof. We only prove (i), since (ii) is dual.

Since Π is self-injective, we have $\Pi \cong D\Pi$ in $\operatorname{mod} \Pi^{\mathrm{op}}$. Hence we have an epimorphism $\Pi \twoheadrightarrow DI_w$ of Π^{op}-modules. Let $\widetilde{I}_w$ be its kernel. (So $\widetilde{I}_w$ is a left ideal.) We want to show that $\widetilde{I}_w = I_{w_0 w^{-1}}$.

Since $I_w \cdot I_{w_0 w^{-1}} = 0$, and hence also $I_{w_0 w^{-1}} \cdot DI_w = 0$, we have $I_{w_0 w^{-1}} \subseteq \widetilde{I}_w$. We now show by induction on $\ell(w)$ that $I_{w_0 w^{-1}} = \widetilde{I}_w$. For $w = e$ we have $I_{w_0 w^{-1}} = 0 = \widetilde{I}_w$.

Assume the claim holds for w, so that $\widetilde{I}_w = I_{w_0 w^{-1}}$ and assume $\ell(s_i w) > \ell(w)$. Then we have an inclusion $I_{s_i w} = I_w I_{s_i} \hookrightarrow I_w$ of Π-modules. By Lemma 6.3(ii) we have that $I_w / I_{s_i w} \cong S_i^n$ for some n as Π-modules. Dualizing we obtain the following short exact sequence in $\operatorname{mod} \Pi^{\mathrm{op}}$:

$$0 \longrightarrow S_i^n \longrightarrow DI_w \longrightarrow DI_{s_i w} \longrightarrow 0.$$

Then we obtain the following diagram in $\operatorname{mod} \Pi^{\mathrm{op}}$.

$$\begin{array}{ccccccccc}
 & & & & & & S_i^n & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \widetilde{I}_w & \longrightarrow & \Pi & \longrightarrow & DI_w & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \widetilde{I}_{s_i w} & \longrightarrow & \Pi & \longrightarrow & DI_{s_i w} & \longrightarrow & 0
\end{array}$$

By the snake lemma the left vertical map is an inclusion with cokernel S_i^n. Thus by Lemma 6.3(i) we have $I_i \widetilde{I}_{s_i w} \subseteq \widetilde{I}_w$. By our induction assumption, we have $\widetilde{I}_w = I_{w_0 w^{-1}}$. Since

$$\ell(w_0 w^{-1} s_i) = \ell(w_0) - \ell(w^{-1} s_i) = \ell(w_0) - \ell(s_i w) < \ell(w_0) - \ell(w) = \ell(w_0 w^{-1})$$

we have $I_i I_{w_0 w^{-1} s_i} = I_{w_0 w^{-1}}$. Putting all these together we have $I_i \widetilde{I}_{s_i w} \subseteq I_i I_{w_0 w^{-1} s_i}$, and clearly $I_i I_{w_0 w^{-1} s_i} \subseteq I_i \widetilde{I}_{s_i w}$. So, by Lemma 6.3(i) the quotient $\widetilde{I}_{s_i w} / I_{w_0 w^{-1} s_i}$ is isomorphic to S_i^m for some m as Π^{op}-modules.

Now note that by Lemma 6.2(ii) we have $\mathrm{Ext}^1_{\Pi^{\mathrm{op}}}(S_i, I_{w_0 w^{-1} s_i}) = 0$. Hence $\widetilde{I}_{s_i w}$ is the direct sum of $I_{w_0 w^{-1} s_i}$ and a left ideal R of Π which is isomorphic to S_i^m. If $R = 0$, then we are done, so assume $R \neq 0$. The only non-zero left ideal of Π which is isomorphic to S_i^m for some m is $I_{w_0 s_i}$. By assumption $R \cap I_{w_0 w^{-1} s_i} = 0$, so $I_{w_0 s_i} \not\subseteq I_{w_0 w^{-1} s_i}$. But $I_{w_0 s_i} = I_{w_0 w w_0 w_0 w^{-1} s_i} = I_{w_0 w^{-1} s_i} I_{w_0 w w_0}$, since $\ell(w_0 w^{-1} s_i) + \ell(w_0 w w_0) = [\ell(w_0) - \ell(w) - 1] + \ell(w) = \ell(w_0) - 1 = \ell(w_0 s_i)$, and hence $I_{w_0 w^{-1} s_i} \supseteq I_{w_0 s_i}$. This is a contradiction. So $R = 0$, and hence $\widetilde{I}_{s_i w} = I_{w_0 w^{-1} s_i}$. This finishes the induction step, and hence the proof of the lemma. □

For v, w in W, it is said that v is less than w in the Bruhat order, written $v \leqslant w$, if, fixing a reduced word for w, it is possible to find a subexpression of that word which equals v.

Lemma 6.5. *We have I_v contains I_w if and only if $v \leqslant w$ in the Bruhat order.*

Proof. This is known in the extended Dynkin case [IR08]. To see that it follows in the Dynkin case, denote by $\widehat{\Pi}$ the preprojective algebra of the corresponding Euclidean quiver. Observe that $\Pi = \widehat{\Pi}/\widehat{I}_{w_0}$. Since w_0 is the maximal element of W under Bruhat order, the ideal $\widehat{I}_{w_0}$ is contained in $\widehat{I}_w$ for any $w \in W$. Thus, $I_{i_1} \ldots I_{i_r} = (\widehat{I}_{i_1} \ldots \widehat{I}_{i_r})/\widehat{I}_{w_0}$, so containment relations agree in the two algebras. □

Lemma 6.6. *Let Q be a Dynkin quiver, and $v, w \in W$. If $\mathscr{C}(I_v) \subseteq \mathscr{C}(I_w)$ then $I_v \subseteq I_w$.*

Proof. By Proposition 6.4 we have that (as Π-modules) $I_v \cong D\Pi/I_{w_0 v^{-1}}$ and $I_w \cong D\Pi/I_{w_0 w^{-1}}$. Thus the assumption may be rewritten as

$$\mathscr{C}(D\Pi/I_{w_0 v^{-1}}) \subseteq \mathscr{C}(D\Pi/I_{w_0 w^{-1}}).$$

Now note that dualizing commutes with restricting to the path algebra, so the above inclusion is equivalent to the inclusion

$$\mathscr{C}_{kQ^{\mathrm{op}}}(\Pi/I_{w_0 v^{-1}}) \subseteq \mathscr{C}_{kQ^{\mathrm{op}}}(\Pi/I_{w_0 w^{-1}}).$$

By Proposition 5.2 (for kQ^{op}) this implies that $I_{w_0 w^{-1}} \subseteq I_{w_0 v^{-1}}$. By Lemma 6.5 we deduce that $w_0 w^{-1} \geqslant w_0 v^{-1}$ in Bruhat order. The map $u \longmapsto u^{-1}$ is clearly an automorphism of Bruhat order, while the map $u \longmapsto w_0 u$ is an anti-automorphism of Bruhat order [BB05, Proposition 2.3.4], so we deduce that $v \geqslant w$. Thus, by Lemma 6.5 again, we deduce that $I_v \subseteq I_w$. □

7. Proof of main results

In this section, we prove our main results, stated in § 2 as Theorems 2.3 and 2.4. In particular, we establish that the cofinite quotient closed subcategories of the category of preprojective kQ-modules are exactly those of the form $\mathscr{C}(I_w)$ for some element w in the Weyl group W.

Lemma 7.1. *For $w \in W$, and i a source, we have that $\mathscr{C}(I_w)$ contains S_i, the simple projective at i, if and only if w has no reduced expression as $w = s_i v$.*

Proof. Observe first that a Π-module M has $S_i \in \mathscr{C}(M)$ if and only if it has S_i in its top, since otherwise (because i is a source), whatever is 'sitting over' S_i will also be sitting over it as a kQ-module. So the problem reduces to asking when I_w has S_i in its top.

Consider the exact sequence

$$\mathrm{Tor}_1(I_w, S_i) \longrightarrow I_w \otimes I_i \longrightarrow I_w \longrightarrow I_w \otimes S_i \longrightarrow 0.$$

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

Suppose first that $\ell(s_iw) > \ell(w)$. The image of $I_w \otimes I_i$ in I_w is I_{s_iw}, which is strictly contained in I_w, by Lemma 4.7 or Lemma 6.1. Thus we get a non-zero quotient, which is necessarily semisimple as a Π-module, since it is annihilated by I_i. Thus it is of the form S_i^n for some $n > 0$, and we deduce that I_w has S_i in its top.

Now suppose that $\ell(s_iw) < \ell(w)$. We can write $w = s_iv$ as a reduced product. The image of $I_{s_iv} \otimes I_i$ in I_{s_iv} is $I_{s_iv}I_i = I_{s_iv}$. Thus the map $I_{s_iv} \otimes I_i \longrightarrow I_{s_iv}$ is surjective. It follows that $I_{s_iv} \otimes S_i$ is zero, and thus that I_{s_iv} does not have S_i in its top. □

Lemma 7.2. *Let i be a source, so S_i is a simple projective. Assume $\ell(s_iw) > \ell(w)$. Then $\mathscr{C}(I_w) = \mathscr{C}(I_{s_iw}) \cup \{S_i\}$.*

Proof. We have already established that $\mathscr{C}(I_w)$ contains S_i while $\mathscr{C}(I_{s_iw})$ does not. Observe that we have a short exact sequence

$$0 \longrightarrow I_{s_iw} \longrightarrow I_w \longrightarrow I_w \otimes S_i \longrightarrow 0.$$

View this sequence as a sequence of kQ-modules. Since the right-hand side is projective as a kQ-module, the sequence splits. Thus the summands of $(I_w)_{kQ}$ other than those surviving in $(I_w \otimes S_i)_{kQ}$ coincide with those in $(I_{s_iw})_{kQ}$. □

Number the vertices of Q from 1 to n so that if there is an arrow $i \longrightarrow j$ then $i < j$.

Theorem 7.3. *Any cofinite, quotient closed subcategory $\mathcal{A}$ of the preprojective kQ-modules appears as $\mathscr{C}(I_w)$ for some $w \in W$ such that $\ell(w)$ is the number of missing preprojective modules.*

Such a w can be found as described in § 2. Number the vertices of Q from 1 to n, so that if there is an arrow $i \longrightarrow j$, then $i < j$. Order the indecomposable preprojective modules as

$$P_1, \dots, P_n, \tau^{-1}P_1, \dots, \tau^{-1}P_n, \tau^{-2}P_1, \dots.$$

Let $\mathcal{X}$ be the indecomposable modules missing from $\mathcal{A}$. Take these indecomposables in the induced order, and read $\tau^{-j}P_i$ as s_i. The result is the leftmost word for w in $\underline{c}^\infty$, where $\tau^{-j}P_i$ is identified with the jth instance of s_i in $\underline{c}^\infty$.

Before we prove this theorem, we first state and prove two lemmas.

Let $\mathcal{A}$ be a cofinite, quotient closed subcategory of the preprojective kQ-modules. Let S_1 be the simple projective kQ-module associated to the vertex 1, and let P be the sum of the other indecomposable projectives. Define $T = P \oplus \tau^{-1}(S_1)$. Let $Q' = \mu_1(Q)$. The associated reflection functor is $R_1^+ = \mathrm{Hom}_{kQ}(T, \ )$. If Q is non-Dynkin, then let $\mathcal{A}'$ be the subcategory of kQ'-modules given by $\mathrm{Hom}(T, \mathcal{A})$. If Q is Dynkin, define $\mathcal{A}'$ to consist of the additive subcategory of $\mathrm{mod}\, kQ'$ generated by $\mathrm{Hom}(T, \mathcal{A})$ together with S_1', the new simple kQ'-module.

Lemma 7.4. *Let $\mathcal{A}'$, Q', S_1' be as defined above. Then:*

(i) *$\mathcal{A}'$ is quotient closed; and*
(ii) *if we have a short exact sequence of kQ'-modules*

$$0 \longrightarrow Y' \longrightarrow Z' \longrightarrow (S_1')^r \longrightarrow 0,$$

with $Y' \in \mathcal{A}'$ and Z' preprojective, then $Z' \in \mathcal{A}'$.

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

Proof. (i) Let $X \in \mathcal{A}$, so $\operatorname{Hom}(T, X) \in \mathcal{A}'$. Denote $\operatorname{Hom}(T, X)$ by X', and suppose that there is an epimorphism from X' to Y', with Y' a preprojective kQ'-module. We want to show that $Y' \in \mathcal{A}'$.

Since $S_1' \in \mathcal{A}'$ if it is preprojective, by construction, we may assume that Y' has no summands isomorphic to S_1'. By assumption, we have a short exact sequence in kQ'-mod,

$$0 \longrightarrow K' \longrightarrow X' \longrightarrow Y' \longrightarrow 0.$$

Since R_1^+ is an equivalence of categories from the additive hull of the indecomposable objects kQ-mod other than S_1 to the additive hull of the indecomposable objects of kQ'-mod other than S_1', and our short exact sequence lies in the latter subcategory, there is a corresponding short exact sequence in kQ-mod, which shows that there is an epimorphism from X to Y, and thus that $Y \in \mathcal{A}$, so $Y' \in \mathcal{A}'$, as desired.

(ii) We may assume that Y' and Z' have no summands isomorphic to S_1', so we may write $Y' = \operatorname{Hom}(T, Y)$ with $Y \in \mathcal{A}$, and $Z' = \operatorname{Hom}(T, Z)$. The given short exact sequence of kQ'-modules then implies the existence of a short exact sequence of kQ-modules $0 \longrightarrow S_1^r \longrightarrow Y \longrightarrow Z \longrightarrow 0$. Since $\mathcal{A}$ is closed under surjections, $Z \in \mathcal{A}$, so $Z' \in \mathcal{A}'$. □

LEMMA 7.5. *Let $\mathcal{A}$ be a cofinite, quotient closed subcategory of the preprojective kQ-modules. Let $\mathcal{A}'$ be defined as above.*

Suppose that Theorem 7.3 holds for $\mathcal{A}'$. Then it also holds for $\mathcal{A}$.

Proof. The assumption that the theorem holds for $\mathcal{A}'$ tells us that $\mathcal{A}' = \mathscr{C}_{Q'}(I_w)$ for w obtained by reading the AR-quiver for kQ' (starting with $P_2, P_3, \ldots$) and recording s_i for each indecomposable object which is not in $\mathcal{A}'$.

We claim that $\ell(s_1 w) > \ell(w)$. Seeking a contradiction, suppose that $w = s_1 v$ is a reduced expression for w. Thus $I_w = I_v I_{s_1}$. We claim that $\mathscr{C}(I_v) \subseteq \mathscr{C}(I_w)$. The reason is that modules in $\mathscr{C}(I_v)$ are extensions of S_1 by some object from $\mathscr{C}(I_w)$, but $\mathscr{C}(I_w)$ is closed under such extensions by Lemma 7.4(ii). So $\mathscr{C}(I_v) \subseteq \mathscr{C}(I_w)$. At the same time, I_v strictly contains I_w, by Lemma 4.7 or Lemma 6.1. These two statements together contradict Theorem 4.6 or Lemma 6.6. Therefore we conclude that $\ell(s_1 w) > \ell(w)$.

Write $\mathcal{A}^+$ for the additive category generated by $\mathcal{A}$ together with S_1, and write $\mathcal{A}^-$ for the additive category generated by the indecomposables of $\mathcal{A}$ excluding S_1.

We now have either $\mathcal{A} = \mathcal{A}^-$ or $\mathcal{A} = \mathcal{A}^+$. Since $\mathcal{A}' = R_1^+(\mathcal{A}^+) = R_1^+(\mathcal{A}^-)$, both $\mathcal{A}^+$ and $\mathcal{A}^-$ satisfy our hypotheses, so we need to prove that the theorem holds for both of them. We first treat $\mathcal{A}^-$.

We recall that Theorem 3.4 says that the following diagram commutes.

$$\begin{array}{ccc} \operatorname{Mod}\Pi & \xrightarrow{-\otimes_{\Pi} I_1} & \operatorname{Mod}\Pi \\ \Big\downarrow{\scriptstyle \text{res}} & & \Big\downarrow{\scriptstyle \text{res}} \\ \operatorname{Mod} kQ' & \xrightarrow{-\otimes_{kQ'} T} & \operatorname{Mod} kQ \end{array}$$

If we start with I_w in the upper left-hand corner, we get $\mathcal{A}'$ in the bottom left, and thus $\mathcal{A}^-$ in the bottom right. On the other hand, in the upper right corner, we have $I_w \otimes I_1$. Since $\ell(s_1 w) > \ell(w)$, this is $I_{s_1 w}$ by Lemma 4.7 or Lemma 6.1. Therefore, $\mathscr{C}(I_{s_1 w}) = \mathcal{A}^-$.

Now we establish the link to the leftmost word for $s_1 w$. Since $s_1 w$ admits s_1 on the left, the leftmost word for $s_1 w$ begins with s_1 (corresponding in the AR-quiver to the simple projective S_1). The rest of the leftmost word for s_1w is the leftmost word for w in the AR-quiver for Q', and by assumption, this corresponds to the indecomposable objects not in $\mathcal{A}'$.

Now we consider $\mathcal{A}^+$. Using the result which we have already established for $\mathcal{A}^-$, Lemma 7.2 tells us that $\mathscr{C}(I_w) = \mathcal{A}^+$. Since w does not admit s_1 as a leftmost factor, the leftmost word for w in the AR-quiver of kQ is the same as the leftmost word for s_1w with the initial s_1 removed. This establishes the desired result for $\mathcal{A}^+$. □

Proof of Theorem 7.3. We establish the theorem by induction on m, the number of indecomposable objects missing from $\mathcal{A}$, and on p, the position of the first indecomposable missing from $\mathcal{A}$ in the order on the indecomposable objects of $\mathcal{A}$.

The statement is clear if $\mathcal{A}$ has no missing indecomposables. (The prescription for finding w gives us the empty word, which is the unique reduced word for the identity element e, and $\mathscr{C}(I_e)$ is the whole preprojective component.)

Now, let $\mathcal{A}$ be some cofinite, quotient closed subcategory of the preprojective kQ-modules, with m missing indecomposables, and with the first missing indecomposable in position p. Suppose that we already know that the theorem holds for any quotient closed subcategory with fewer than m missing indecomposables, or with exactly m missing indecomposables and with the first missing indecomposable in a position earlier than p. Define $\mathcal{A}'$ as above.

If $p = 1$, then $\mathcal{A}$ does not contain S_1. In this case, $\mathcal{A}'$ has fewer missing indecomposables than $\mathcal{A}$ does. If $p > 1$, then $\mathcal{A}$ and $\mathcal{A}'$ have the same number of missing indecomposables, but the first missing indecomposable is earlier in $\mathcal{A}'$ than it is in $\mathcal{A}$.

Thus, in either case, the induction hypothesis tells us that the statement of the theorem holds for $\mathcal{A}'$, and Lemma 7.5 tells us that it also holds for $\mathcal{A}$. □

We also have a converse.

Theorem 7.6. *The subcategory $\mathscr{C}(I_w)$ is quotient closed for any $w \in W$.*

Proof. If Q is non-Dynkin, we proceed as follows. Consider $\operatorname{Fac}_{\mathrm{pp}} \mathscr{C}(I_w)$, where we write $\operatorname{Fac}_{\mathrm{pp}} \mathcal{A}$ for the part of $\operatorname{Fac} \mathcal{A}$ consisting of preprojective modules. $\operatorname{Fac}_{\mathrm{pp}} \mathscr{C}(I_w)$ is quotient closed and clearly cofinite, so by Theorem 7.3, $\operatorname{Fac}_{\mathrm{pp}} \mathscr{C}(I_w) = \mathscr{C}(I_v)$ for some v, and hence $\operatorname{Fac} \mathscr{C}(I_w) = \operatorname{Fac} \mathscr{C}(I_v)$. Thus $I_w = I_v$, by Theorem 4.6. Thus $\mathscr{C}(I_w)$ is quotient closed.

Now, in the Dynkin case, for $x \in W$, $\mathscr{C}(\operatorname{Sub}(\Pi/I_x))$ is subclosed by Lemma 5.3, so must equal $\mathscr{C}(\Pi/I_v)$ for some $v \in W$ (applying Theorem 7.3 after dualizing). By Proposition 5.2, $\operatorname{Ann}(\mathscr{C}(\operatorname{Sub}(\Pi/I_x))) = I_x$, and $\operatorname{Ann}(\mathscr{C}(\Pi/I_v)) = I_v$, so $I_x = I_v$, and thus $x = v$ by [BIRS09, Propositions III.1.9, III.3.5]. It follows that $\mathscr{C}(\Pi/I_x)$ is subclosed for any x, and by Proposition 6.4, we conclude that $\mathscr{C}(I_w)$ is quotient closed for all $w \in W$. □

Proof of Theorems 2.3 and 2.4. Theorem 7.3 shows that the correspondences of Theorem 2.3 yield a bijection between the cofinite quotient closed subcategories of $\mathcal{P}$ and some subset X of W. Theorem 7.6 shows that for any $w \in W$, $\mathscr{C}(I_w)$ is quotient closed. It therefore can also be written as $\mathscr{C}(I_x)$ for some $x \in X$. From the fact that $\mathscr{C}(I_w) = \mathscr{C}(I_x)$ we conclude that $I_w = I_x$ (using Theorem 4.6 in the non-Dynkin case, and Lemma 6.6 in the Dynkin case). It then follows from [BIRS09, Propositions III.1.9, III.3.5] that $w = x$. Therefore, $w \in X$. Thus $X = W$, and Theorem 2.3 is proved.

Theorem 2.4 now also follows. □

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

8. Infinite words

In this section we extend our bijection between the Weyl group and the cofinite quotient closed subcategories of $\mathcal{P}$, to a bijection between a specific class of subwords of $\underline{c}^\infty$ and arbitrary (i.e., not necessarily cofinite) quotient closed subcategories of $\mathcal{P}$.

Let Q be non-Dynkin. Fix the (one-way) infinite word $\underline{c}^\infty = \underline{c}\,\underline{c}\,\underline{c}\dots$.

We say that an infinite subword $\underline{w}$ of $\underline{c}^\infty$ is *leftmost* if, for all n, the subword of $\underline{c}^\infty$ consisting of the first n letters of $\underline{w}$ is leftmost (among all reduced words in $\underline{c}^\infty$ for that element of W, i.e., in the usual sense).

Theorem 8.1. *There is a bijective correspondence between the leftmost subwords of $\underline{c}^\infty$ and the quotient closed subcategories of the preprojective component of* $\operatorname{mod} kQ$*: the reflections in the word correspond to indecomposable objects not in the subcategory.*

Proof. We need only worry about infinite words and non-cofinite subcategories. Let $\mathcal{C}$ be a non-cofinite quotient closed subcategory. It determines a sequence $\mathcal{C}_1, \mathcal{C}_2, \dots$ where $\mathcal{C}_i$ consists of the indecomposable kQ-modules except for the i leftmost indecomposables missing from $\mathcal{C}$. Clearly, $\mathcal{C}_i$ is cofinite and quotient closed. It therefore determines a word w_i. By construction, w_{i-1} is a prefix of w_i, and thus they together define an infinite word all of whose prefixes are leftmost, which means that it is, by definition, an (infinite) leftmost word in $\underline{c}^\infty$.

The argument in the converse direction works in essentially the same way. Each finite prefix determines a subcategory which is quotient closed; the intersection of all these is a non-cofinite quotient closed subcategory. □

Our main theorem, Theorem 7.3, relates quotient closed subcategories, elements of W, and certain ideals in Π. One might therefore wonder if it is possible to find a class of ideals of Π in bijection with the subcategories and words of the previous theorem. The obvious way to do this fails. Specifically, let w_i be the element of W corresponding to the first i symbols of an infinite subword $\underline{w}$ of $\underline{c}^\infty$, and then take $I_{\underline{w}} = \bigcap I_{w_i}$. It can happen that if $\underline{w}$ and $\underline{v}$ are distinct leftmost subwords of $\underline{c}^\infty$, then nonetheless $I_{\underline{w}} = I_{\underline{v}}$. For example, any leftmost subword $\underline{w}$ with the property that each simple reflection occurs an infinite number of times, will yield $I_{\underline{w}} = 0$. (Consider the case of the quiver $\widetilde{A}_1$, with two arrows from vertex 1 to vertex 2. Now consider the infinite words $s_1 s_2 s_1 s_2 s_1 \dots$ and $s_2 s_1 s_2 s_1 s_2 \dots$. The subcategory corresponding to the first of these words is empty, while that corresponding to the second word contains the simple projective. Both words nonetheless define the 0 ideal. Further, note that there is no ideal I of Π such that $\mathscr{C}(I)$ is the additive hull of the simple projective.)

Theorem 8.1 gives a correspondence between certain subcategories and certain subwords of $\underline{c}^\infty$. This bijection seems somewhat different from the bijection in Theorem 7.3: in that theorem, we biject subcategories with elements of W, rather than with subwords of $\underline{c}^\infty$. However, Theorem 7.3 could be formulated in a fashion parallel to that of Theorem 8.1, because every element of W has a unique leftmost expression as a subword of $\underline{c}^\infty$.

We will now proceed instead to reformulate Theorem 8.1 in a way which is closer to Theorem 7.3. In order to do so, we will replace the leftmost subwords of $\underline{c}^\infty$ which appear in Theorem 8.1 by certain equivalence classes of subwords of $\underline{c}^\infty$, such that the equivalence classes of finite subwords are just the reduced subwords in $\underline{c}^\infty$ for a given $w \in W$.

We say that an infinite word is *reduced* if any prefix of it is reduced. We restrict our attention to such words.

Say that the infinite reduced word $\underline{v}$ is a *braid limit* of the infinite reduced word $\underline{w}$ if there is some, possibly infinite, sequence of braid moves $B_1, \dots$, which transforms $\underline{w}$ into $\underline{v}$, such that, for

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

any particular position n, there is some $N(n)$ such that B_j for $j > N(n)$ only affects positions greater than n. (This is a rephrasing of the definition in [LP13].)

Note that it is possible for $\underline{v}$ to be a braid limit of $\underline{w}$ even if the converse is not true. An example in $\widetilde{A}_2$ (from [LP13]) is as follows. Let $\underline{w} = s_2s_1s_2s_3s_1s_2s_3\ldots$ and let $\underline{v} = s_1s_2s_3s_1s_2s_3\ldots$. Transform $\underline{w} \longrightarrow s_1s_2s_1s_3s_1s_2s_3\ldots \longrightarrow s_1s_2s_3s_1s_3s_2s_3\ldots$. After i steps, the first $2i$ positions agree with $\underline{v}$. However, there are no braid moves applicable to $\underline{v}$, thus $\underline{w}$ is certainly not a braid limit of $\underline{v}$.

We then have the following proposition, analogous to the statement that any element of W has a unique leftmost expression in $\underline{c}^\infty$.

Proposition 8.2. *Any infinite reduced word has a unique leftmost braid limit.*

Proof. Let s_1 be the first reflection in $\underline{c}^\infty$. [LP13, Lemma 4.8] (an extension to infinite words of the usual exchange lemma from Coxeter theory) states that, given an infinite word $\underline{w}$, one of two things will happen when we consider the infinite word $s_1\underline{w}$: either it will be reduced in turn, or there will be a unique reflection from $\underline{w}$ which cancels with s_1, leaving some $\underline{\hat{w}}$.

In the first case, no finite prefix of $\underline{w}$ is equivalent under braid moves to a word beginning s_1. Since a finite number of braid moves can only alter a finite prefix of $\underline{w}$, it follows that no braid limit for $\underline{w}$ can begin with s_1. In the second case, $\underline{w}$ is equivalent, after a finite number of braid moves, to $s_1\underline{\hat{w}}$. Clearly, in this case, $\underline{w}$ admits a braid limit which begins with s_1.

Therefore, if the first case holds, no braid limit for $\underline{w}$ can involve the initial s_1, so it plays no role and we can continue on to consider the next simple reflection in $\underline{c}^\infty$. In the second case, a finite number of braid moves suffice to bring s_1 to the front of $\underline{w}$, and we can now go on to find a braid limit for $\underline{\hat{w}}$ beginning with the second reflection in $\underline{c}^\infty$. □

Say that two (possibly infinite) reduced words in the simple generators of W are equivalent if they have the same leftmost braid limit in $\underline{c}^\infty$. (The equivalence classes in which the words are of finite length correspond naturally to elements of W.) Then we can restate Theorem 8.1 in the following way.

Corollary 8.3. *There is a bijection between equivalence classes of reduced words in the simple generators of W and quotient closed subcategories of the preprojective component of mod kQ.*

9. Quotient closed subcategories of modules over hereditary algebras over finite fields

In this section, we show how to extend our analysis of quotient closed subcategories of modules for path algebras over algebraically closed fields to quotient closed subcategories of modules for arbitrary hereditary algebras over finite fields.

Let $\mathbb{F}_q$ be a finite field with q elements, and let $\overline{\mathbb{F}}_q$ be its algebraic closure. Let H be a hereditary algebra over $\mathbb{F}_q$. We first recall (see § 9.1 below) that there is a quiver Q and an $\mathbb{F}_q$-endomorphism F of $\overline{\mathbb{F}}_qQ$ called a Frobenius morphism, such that $H \cong (\overline{\mathbb{F}}_qQ)^F$, the $\mathbb{F}_q$-subspace of $\overline{\mathbb{F}}_qQ$ fixed under F. We then invoke a theorem which says that the module category of $(\overline{\mathbb{F}}_qQ)^F$ is equivalent to the category of F-stable representations of $\overline{\mathbb{F}}_qQ$. (The F-stable representations of $\overline{\mathbb{F}}_qQ$, defined below, are a certain non-full subcategory of the representations of $\overline{\mathbb{F}}_qQ$.) We apply our analysis of quotient closed categories for path algebras over algebraically closed fields to analyze quotient closed subcategories of the F-stable representations of $\overline{\mathbb{F}}_qQ$.

The Frobenius morphism F comes from a certain quiver automorphism σ of Q. There is a Weyl group $W_{Q,\sigma}$ (typically not simply laced) associated to the quotient of Q by σ. We prove Theorem 9.7, an analogue of our main theorem, which applies to cofinite quotient closed subcategories of H-modules.

Our main reference for the techniques specific to this section is [DDPW08, ch. 2 and 3].

9.1 Recollection of results on Frobenius twisting

Let V be an $\overline{\mathbb{F}}_q$ vector space. An $\mathbb{F}_q$-linear isomorphism $F : V \longrightarrow V$ is called a Frobenius map on V if:

– $F(\lambda v) = \lambda^q v$ for $\lambda \in \overline{\mathbb{F}}_q$;

– for any $v \in V$, $F^t(v) = v$ for some $t \geqslant 1$.

Let A be an $\overline{\mathbb{F}}_q$-algebra. A Frobenius morphism on A is a Frobenius map on the underlying vector space of A which also satisfies $F(ab) = F(a)F(b)$ for a, b in A. We write A^F for the elements of A fixed by F. It is an $\mathbb{F}_q$-subalgebra of A.

For example, let Q be a quiver with an automorphism σ. Define a Frobenius morphism $F = F_\sigma : \overline{\mathbb{F}}_q Q \longrightarrow \overline{\mathbb{F}}_q Q$ by sending $\sum x_i p_i$ to $\sum x_i^q \sigma(p_i)$, for $x_i \in \overline{\mathbb{F}}_q$ and p_i paths in Q. In fact, this example plays an important role.

Theorem 9.1 [DDPW08, Theorem 3.40]. *Any hereditary algebra H over $\mathbb{F}_q$ can be realized as $(\overline{\mathbb{F}}_q Q)^{F_\sigma}$ for a suitable quiver Q and quiver automorphism σ.*

Let (A, F_A) be an algebra with a Frobenius morphism. Let M be an A-module, with a Frobenius map F_M (i.e., a Frobenius map on the underlying vector space of M).

Define a new A-module, $M^{[F_M]}$, which is called the *Frobenius twist* of M, by letting the underlying vector space be that of M, and defining the A-action by

$$m * a = F_M(F_M^{-1}(m)F_A^{-1}(a)).$$

Up to isomorphism, this A-module is independent of the choice of F_M.

A module M is called F-stable if for some Frobenius map F_M (or, equivalently, for all of them) we have $M \cong M^{[F_M]}$. If M is F-stable, it is possible to choose F_M so that $M = M^{[F_M]}$, or equivalently

$$F_M(ma) = F_M(m)F_A(a). \tag{9.1}$$

The F-stable modules form a category: its objects are pairs (M, F_M) with F_M satisfying (9.1), and the morphisms from (M, F_M) to (N, F_N) are maps $f \in \operatorname{Hom}(M, N)$ satisfying $f \circ F_M = F_N \circ f$.

Now we can state the following theorem.

Theorem 9.2 [DDPW08, Theorem 2.16]. *The category of F-stable A-modules is equivalent to the category of A^F-modules.*

We want to understand the indecomposable F-stable A-modules. It turns out that they can be constructed as follows. Let M be an A-module with Frobenius morphism F_M. Let r be the maximum possible so that M, $M^{[F_M]}$, $M^{[F_M]^2}, \ldots, M^{[F_M]^{r-1}}$ are pairwise non-isomorphic. (Such an r exists by [DDPW08, Proposition 2.13].) We can choose F_M so that $M = M^{[F_M]^r}$ (note that here we want equality, not just isomorphism). Let

$$\widetilde{M} = M \oplus M^{[F_M]} \oplus M^{[F_M]^2} \oplus \cdots \oplus M^{[F_M]^{r-1}},$$

and define $F_{\widetilde{M}}$ by

$$F_{\widetilde{M}}(m_0, \ldots, m_{r-1}) = (F_M(m_{r-1}), F_M(m_0), \ldots, F_M(m_{r-2})).$$

Theorem 9.3 [DDPW08, Theorem 2.20]. *Let M be an indecomposable A-module. Then $\widetilde{M}$ is an indecomposable F-stable module, and every indecomposable F-stable module arises in this way.*

The AR-sequences of A-mod and A^F-mod are closely related.

Theorem 9.4 [DDPW08, Theorem 2.48]. *Every almost split sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in A-mod gives rise to an almost split sequence

$$0 \longrightarrow \widetilde{X}^F \longrightarrow \widetilde{Y}^F \longrightarrow \widetilde{Z}^F \longrightarrow 0$$

in A^F-mod and every almost split sequence in A^F-mod arises in this way.

In particular, in the case we considered above, where $A = \overline{\mathbb{F}}_q Q$ and $F = F_\sigma$ for σ an automorphism of Q, there is a nice description of the preprojective component of A^F-mod.

Theorem 9.5 [DDPW08, Theorem 3.30]. *The preprojective component of the AR-quiver of $(\overline{\mathbb{F}}_q Q)^F$ is (up to multiplicities of arrows) a translation quiver on Q/σ, whose vertices are labelled by σ-orbits of preprojective indecomposable representations of $\overline{\mathbb{F}}_q Q$.*

The cited theorem provides a more precise statement about multiplicities of arrows, but we will not need it.

9.2 Quotient closed subcategories of hereditary algebras over $\mathbb{F}_q$

Let H be a hereditary algebra over $\mathbb{F}_q$. Fix a quiver Q, a quiver automorphism σ, and a Frobenius map $F = F_\sigma$ as in Theorem 9.1, so $H \cong (\overline{\mathbb{F}}_q Q)^F$. By Theorem 9.2, the category of modules over H is equivalent to the category of F-stable modules over $\overline{\mathbb{F}}_q Q$.

Proposition 9.6. *There is a numbering of the vertices of Q which has the following properties:*

(i) *if $\mathrm{Hom}(P_i, P_j) \neq 0$ then $i < j$; and*

(ii) *the labels assigned to any σ-orbit of vertices form a consecutive sequence of numbers.*

Proof. A labelling f satisfying (i) exists because Q has no oriented cycles. Define a new labelling $\widetilde{f}$, by averaging f over σ-orbits. Now $\widetilde{f}$ is constant on σ-orbits and still satisfies (i), since if there is an arrow from $[i]$ to $[j]$ in Q/σ, then $\widetilde{f}([j]) - \widetilde{f}([i])$ is the average difference between the head and the tail of the arrows in the σ orbit of this arrow. Number the vertices of Q in an order such that $\widetilde{f}$ weakly increases at each step, and such that once we assign a label to an element of a σ-orbit, we go on to assign consecutive labels to the remainder of the σ-orbit before we assign any further labels. This labelling satisfies (i) and (ii). □

Fix a numbering for Q as in the proposition. Consider the subgroup of the Weyl group W_Q generated by the elements $s_{[i]}$, where $[i]$ denotes the σ-orbit of i, and $s_{[i]}$ is the product of the simple reflections labelled by elements of $[i]$, in arbitrary order. Since the corresponding vertices are not adjacent in Q, $s_{[i]}$ does not depend on the choice of the order in which the product is taken.

The subgroup of W_Q generated by the $s_{[i]}$ is again a Weyl group [Ste08]; we denote it by $W_{Q,\sigma}$. It is typically not simply laced. Let $\widetilde{c}$ denote the Coxeter element in $W_{Q,\sigma}$ corresponding to this ordering.

We can now state the main theorem of this section.

Theorem 9.7. *Let H be a hereditary algebra over $\mathbb{F}_q$. Let Q and σ be defined as above. Then quotient closed cofinite subcategories of H-modules correspond to lexicographically first subwords of $\underline{\widetilde{c}}^\infty$ in $W_{Q,\sigma}$.*

Let $\mathcal{C}$ be a cofinite quotient closed subcategory of H-modules. We think of $\mathcal{C}$ as a category of F-stable modules over $\overline{\mathbb{F}}_q Q$.

Write $\widehat{\mathcal{C}}$ for the category of modules over $\overline{\mathbb{F}}_q Q$ obtained from $\mathcal{C}$ by forgetting the F structure, and then taking direct sums of direct summands. We now want to describe $\widehat{\mathcal{C}}$.

Proposition 9.8. (i) *The indecomposable objects of $\widehat{\mathcal{C}}$ are a union of σ-orbits of indecomposable representations of Q.*

(ii) *In addition, $\widehat{\mathcal{C}}$ is a cofinite, quotient closed subcategory of $\overline{\mathbb{F}}_q Q$-mod.*

Proof. The first claim follows from Theorem 9.3.

Note that the second claim is not obvious, because $\widehat{\mathcal{C}}$ has more morphisms than $\mathcal{C}$.

So, suppose that we have a surjection f from X to Y, with $X \in \widehat{\mathcal{C}}$. We may assume that X and Y are (sums of) preprojective representations of Q. Let K be the kernel of f.

Since X is a sum of preprojective representations, it can be lifted from an $\mathbb{F}_q$-representation, and thus admits a Frobenius map F_X with finite period r.

We then have an F-stable module

$$\widetilde{X} = X \oplus X^{[F_X]} \oplus \cdots \oplus X^{[F_X]^{r-1}}.$$

Consider the submodule defined by $\widetilde{K} = K \oplus K^{[F_X]} \oplus \cdots \oplus K^{[F_X]^{r-1}}$. This is also F-stable. We now see that Y is a summand of $\widetilde{X}/\widetilde{K}$. This implies that $\widehat{\mathcal{C}}$ is quotient closed. □

We also have the following converse.

Proposition 9.9. *Any cofinite quotient closed subcategory of modules over $\overline{\mathbb{F}}_q Q$, whose indecomposables consist of a union of σ-orbits, arises as $\widehat{\mathcal{C}}$ for some cofinite quotient closed category $\mathcal{C}$ of F-stable modules.*

Proof. By Theorem 9.3, there is a subcategory $\mathcal{C}$ of F-stable modules such that the associated subcategory of $\mathbb{F}_q Q$-modules is $\widehat{\mathcal{C}}$. Since $\widehat{\mathcal{C}}$ is quotient closed, so is $\mathcal{C}$. □

We now prove the main theorem of this section.

Proof of Theorem 9.7. A cofinite quotient closed subcategory of H-mod corresponds, as above, to a cofinite, quotient closed subcategory of $\overline{\mathbb{F}}_q Q$-mod. Thus, when we read the word in W_Q corresponding to it, it is a leftmost word in $\underline{c}^\infty$. (Here c is the Coxeter element in W_Q, and $\underline{c}$ is the corresponding word.) Because the indecomposables in the subcategory of $\overline{\mathbb{F}}_q Q$-mod are unions of σ-orbits, the factors in the word can be grouped into generators of $W_{Q,\sigma}$, so we can view it as a word in $\underline{\widetilde{c}}^\infty$, which is automatically also leftmost.

Conversely, suppose we take some $w \in W_{Q,\sigma}$. Because the leftmost word for w inside $\underline{c}^\infty$ can be obtained greedily, for each σ-orbit of reflections in $\underline{c}^\infty$, either all of them will be used in the leftmost word for w, or none of them will. Thus the leftmost word for w in $\underline{c}^\infty$ corresponds to a word for w in $\underline{\widetilde{c}}^\infty$, and therefore to the leftmost word there. The subcategory of $\overline{\mathbb{F}}_q Q$-mod corresponding to w satisfies the hypotheses of Proposition 9.9, so it corresponds to a cofinite, quotient closed subcategory of F-stable $\mathbb{F}_q Q$-modules, as desired. □

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

10. Subclosed subcategories

We have seen that $\mathscr{C}$ induces a bijection between the ideals of the form I_w in Π and the cofinite quotient closed subcategories of $\operatorname{mod} kQ$. It is natural to ask if $\mathscr{C}$ similarly induces a bijection between the quotients Π/I_w and certain subcategories of $\operatorname{mod} kQ$, and further if one can explicitly describe the subcategories of $\operatorname{mod} kQ$ which are of the form $\mathscr{C}(\Pi/I_w)$ for some w.

We start by observing that in case Q is Dynkin the situation is as good as one could have hoped.

Theorem 10.1. *Let Q be a Dynkin quiver. Then the map*

$$\begin{aligned} W &\longrightarrow \{\textit{subcategories of } \operatorname{mod} kQ\} \\ w &\longmapsto \mathscr{C}(\Pi/I_w) \end{aligned}$$

induces a bijection between W and the subclosed subcategories of $\operatorname{mod} kQ$.

Proof. We have

$$\mathscr{C}(\Pi/I_w) = \mathscr{C}(DI_{w_0 w^{-1}}) = D\mathscr{C}_{\text{left}}(I_{w_0 w^{-1}})$$

from § 6. Now the claim follows since $w \longmapsto w_0 w^{-1}$ is a bijection from W to itself, while $w \longmapsto \mathscr{C}_{\text{left}}(I_w)$ is a bijection from W to the set of quotient closed subcategories of $\operatorname{mod} kQ^{\text{op}}$, and D induces a bijection between quotient closed subcategories of $\operatorname{mod} kQ^{\text{op}}$ and subclosed subcategories of $\operatorname{mod} kQ$. □

The most obvious guess would be that for Q arbitrary, the map $w \longrightarrow \mathscr{C}(\Pi/I_w)$ might be a bijection from W to the subclosed subcategories of $\operatorname{mod} kQ$ containing only finitely many indecomposables. However, this is not the case: there are subclosed subcategories of $\operatorname{mod} kQ$ with finitely many indecomposable objects which do not appear as $\mathscr{C}(\Pi/I_w)$ for any w. For instance, let Q be the Kronecker quiver. The subcategories consisting of direct sums of copies of one quasi-simple regular module and the simple projective module are subclosed, but not of the form $\mathscr{C}(\Pi/I_w)$, as we will see in Proposition 10.6.

We however suspect that the following statement in the converse direction holds.

Conjecture 10.2. For $w \in W$, the subcategory $\mathscr{C}(\Pi/I_w)$ of $\operatorname{mod} kQ$ is subclosed.

By Theorem 10.1 the conjecture holds for Q Dynkin. It is also easy to verify this conjecture in the case that Q has two vertices by a direct calculation. See Proposition 10.6 below for an explicit description of the categories that arise.

Recall that we have seen in Lemma 5.3 that the categories $\mathscr{C}(\operatorname{Sub}\Pi/I_w)$ are always subclosed. It follows that $\operatorname{Sub}\mathscr{C}(\Pi/I_w) = \mathscr{C}(\operatorname{Sub}\Pi/I_w)$, and hence that Conjecture 10.2 is equivalent to

$$\mathscr{C}(\operatorname{Sub}\Pi/I_w) = \mathscr{C}(\Pi/I_w).$$

We now formulate two conjectures on the description of the subcategories $\mathscr{C}(\Pi/I_w)$. In the first one we restrict to the affine case, where the combinatorial description is somewhat simpler.

Conjecture 10.3. Suppose that Q is affine. A full subcategory $\mathcal{Z}$ of kQ-mod arises as $\mathscr{C}(\Pi/I_w)$ for some $w \in W$ if and only if

- $\mathcal{Z}$ has a finite number of indecomposables;
- $\mathcal{Z}$ is subclosed; and
- for any tube, there is at least one ray in the tube which does not intersect $\mathcal{Z}$.

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

In order to generalize this conjecture to arbitrary quivers, we introduce the following notion. Define the reduced Ext-quiver of a full subcategory $\mathcal{Z}$ of kQ-mod as follows: the vertices are the indecomposable objects of $\mathcal{Z}$. There is an arrow from Y to X if the simple $\mathcal{Z}$-module $\mathrm{Hom}_{kQ}(-,X)/\mathrm{Rad}(-,X)$ is a direct summand of the socle of the $\mathcal{Z}$-module $\mathrm{Ext}^1_{kQ}(-,Y)$. Equivalently there is an arrow from Y to X if there is a morphism from $\tau^{-1}Y$ to X which does not factor through any radical map $\mathcal{Z} \longrightarrow X$.

CONJECTURE 10.4. A full subcategory $\mathcal{Z}$ of kQ-mod arises as $\mathscr{C}(\Pi/I_w)$ for some $w \in W$ if and only if:
- $\mathcal{Z}$ has a finite number of indecomposables;
- $\mathcal{Z}$ is submodule-closed; and
- the reduced Ext-quiver of $\mathcal{Z}$ contains no cycles.

This conjecture is clearly true for Q of finite type. (The third condition is vacuous in this case.)

In the tame case, we show that the two conjectures above coincide.

PROPOSITION 10.5. *In the case that Q is affine, Conjecture 10.3 is equivalent to Conjecture 10.4.*

Proof. Suppose that $\mathcal{Z}$ is a subclosed category, and that there is some tube such that $\mathcal{Z}$ contains objects from each ray of the tube. By the fact that $\mathcal{Z}$ is subobject closed, it contains the quasi-simples from the bottom of the tube. Since each is the AR-translation of the next around the tube, the corresponding extensions cannot factor through any other element of $\mathcal{Z}$, so they give rise to a cycle in the reduced Ext-quiver of $\mathcal{Z}$, contrary to our assumption.

Conversely, suppose that $\mathcal{Z}$ is subclosed and each tube has a ray such that $\mathcal{Z}$ does not intersect that ray. Suppose Y and X are in the same tube, with X strictly higher than Y. A map from $\tau^{-1}Y$ to X factors through X', the indecomposable in the same ray as X and on the same level as Y, and X' is in $\mathcal{Z}$ since $\mathcal{Z}$ is subclosed. Therefore, there is no arrow in the reduced Ext-quiver from Y to X, so the only arrows in the reduced Ext-quiver go from an object to another object at the same height or lower. Further, there can only be an arrow from Y to some X at the same height as Y if $X = \tau^{-1}Y$, since otherwise the map from $\tau^{-1}Y$ will factor through X'', the indecomposable on the same ray as X which is one level lower.

Thus, a cycle would necessarily involve objects all at the same height, and each would have to be τ^{-1} of the previous one. This would imply that there was no ray in the tube not intersecting $\mathcal{Z}$. □

We now prove Conjecture 10.4 for the case that Q is a quiver with two vertices.

PROPOSITION 10.6. *Conjecture 10.4 holds when Q is a quiver with two vertices.*

Proof. The subcategories that arise as $\mathscr{C}(\Pi/I_w)$ are exactly those of the following form:
- a finite initial segment of the preprojective component; or
- a finite initial segment of the preprojective component together with the simple injective.

It is clear that these subcategories satisfy the conditions of Conjecture 10.4, so it is just a matter of checking that no other subcategory does.

Let $\mathcal{Z}$ be some subcategory satisfying the conditions of Conjecture 10.4. If $\mathcal{Z}$ contains a non-simple injective, then (being subclosed) it would also contain all the predecessors of $\mathcal{Z}$ in the preinjective component, and thus it would not be finite, contradicting our assumption.

Suppose now that $\mathcal{Z}$ contains some regular objects, $R_1, \ldots, R_r$. Since kQ-mod has no rigid regular objects, each of these objects admits a self-extension. Thus, for each R_i, there is a map

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

from $\tau^{-1}R_i$ to R_i. This does not necessarily yield an arrow in the reduced Ext-quiver, but we can conclude that there is *some* arrow in the reduced Ext-quiver of $\mathcal{Z}$ starting at R_i, and further that this arrow goes to a regular indecomposable of $\mathcal{Z}$, since the morphism from $\tau^{-1}R_i$ to R_i factors through the morphism corresponding to this arrow. Thus, the reduced Ext-quiver contains arbitrarily long walks, so it must contain a directed cycle, contradicting our assumption.

Finally, if $\mathcal{Z}$ contains a preprojective object E, it must contain all the predecessors of E, since $\mathcal{Z}$ is subclosed. It follows that the only subcategories $\mathcal{Z}$ which satisfy the conditions of Conjecture 10.4 are those which we have already identified. □

11. Which cofinite quotient closed subcategories are torsion classes?

We have established that the cofinite quotient closed subcategories of kQ-mod can be formed as the additive hull of $\mathscr{C}(I_w)$ together with all non-preprojective indecomposable objects for $w \in W$. It is natural to ask for which w these subcategories are actually torsion classes.

When we have found a torsion class, it is also natural to ask about the corresponding torsion-free class. Since our torsion classes are cofinite, the corresponding torsion-free class will be finite, and it will therefore be useful to recall the correspondence established in [IT09, AIRT12] between torsion-free classes and certain elements of W called c-sortable elements.

As usual, $c = s_1 s_2 \dots s_n$, where the simple reflections (or equivalently, the vertices of Q) are numbered compatibly with the orientation of Q. An element $w \in W$ is called c-sortable if there is an expression for w of the form $c^{(0)}c^{(1)} \dots c^{(r)}$, where each $c^{(i)}$ consists of some subset of the simple reflections, taken in increasing order, and such that the set of reflections appearing in $c^{(i)}$ is contained in the set of reflections appearing in $c^{(i-1)}$ [Rea07a]. It is shown in [AIRT12] that there is a one-to-one correspondence between c-sortable elements of W and finite torsion-free classes for kQ-mod, which takes w to $\mathscr{C}(\Pi/I_w)$.

Therefore, when $\mathscr{C}(I_w)$ together with the non-preprojective indecomposables form a torsion class, we can ask for the element $v \in W$ such that the corresponding torsion-free class is given by $\mathscr{C}(\Pi/I_v)$. In this section we give conjectural answers to both these questions, and we prove that our conjectures hold in the Dynkin case.

Write $\mathrm{sort}_c(w)$ for the longest c-sortable prefix of w.

Conjecture 11.1. The following conditions are equivalent:

(i) the additive category generated by $\mathscr{C}(I_w)$ together with all non-preprojective indecomposable kQ-modules is a torsion class;
(ii) for every i such that $\ell(ws_i) > \ell(w)$, we have that $\mathrm{sort}_c(ws_i)$ is strictly longer than $\mathrm{sort}_c(w)$.

Consider A_2, with S_1 the simple projective, so $c = s_1 s_2$. We find that the elements of W satisfying each of the above conditions are e, s_1, $s_1 s_2$, $s_2 s_1$, $s_1 s_2 s_1$. (For example, s_2 does not satisfy the second condition, because $s_2 s_1$ is longer than s_2, but the longest c-sortable prefix of both $s_2 s_1$ and of s_2 is s_2. On the other hand, note that $s_2 s_1$ satisfies the conditions: its longest c-sortable prefix is s_2, while the only word which can be obtained by lengthening $s_2 s_1$ is $s_2 s_1 s_2 = s_1 s_2 s_1$ which is c-sortable.)

Conjecture 11.2. If the additive hull of $\mathscr{C}(I_w)$ together with the non-preprojective indecomposable objects forms a torsion class, its corresponding torsion-free class is that associated to $\mathrm{sort}_c(w)$.

We will now prove both these conjectures for finite type. In order to do so, we introduce some notation.

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

There is an order on W, called *right weak order*, in which $u \leqslant_{\mathrm{R}} v$ if and only if there is a reduced expression for v with a prefix which is an expression for u. This is a weaker order than Bruhat order, in the sense that if $u \leqslant_{\mathrm{R}} v$, it is also true that $u \leqslant v$ in Bruhat order. (Left weak order, which we shall not need here, is defined similarly, using suffixes instead of prefixes.) For more on weak orders, see [BB05, ch. 3].

In finite type, the map sort_c, which takes W to c-sortable elements, is a lattice homomorphism from W with the right weak order to the c-sortable elements of W, ordered by the restriction of right weak order [Rea07b, Theorem 1.1]. This implies, in particular, that each fibre of sort_c is an interval in W.

Lemma 11.3. *For Q of finite type, the following conditions on $w \in W$ are equivalent:*

(i) *for every simple reflection s_i such that $\ell(ws_i) > \ell(w)$, we have that $\ell(\mathrm{sort}_c(ws_i)) > \ell(\mathrm{sort}_c(w))$;*
(ii) *w is the unique longest element among those $x \in W$ satisfying $\mathrm{sort}_c(w) = \mathrm{sort}_c(x)$;*
(iii) *ww_0 is c^{-1}-sortable.*

Proof. Suppose (ii) does not hold, so there exists some $y >_{\mathrm{R}} w$ such that $\mathrm{sort}_c(y) = \mathrm{sort}_c(w)$. It then follows that the whole interval from y to w has the same maximal c-sortable prefix, and in particular this holds for some element ws_i which covers w. This shows that (i) does not hold.

Now suppose that (ii) holds. Let s_i be a simple reflection such that $\ell(ws_i) > \ell(w)$. By (ii), $\mathrm{sort}_c(ws_i) \neq \mathrm{sort}_c(w)$. Since ws_i lies above w in the right weak order, $\mathrm{sort}_c(ws_i)$ lies above $\mathrm{sort}_c(w)$ in the right weak order, so in particular it is longer. This establishes (i).

The equivalence of (ii) and (iii) follows from [Rea07b, Proposition 1.3]. □

Proposition 11.4. *Conjecture 11.1 holds if Q is of finite type.*

Proof. We denote by $\mathscr{C}_{\mathrm{left}}$ the left module version of $\mathscr{C}$, that is, the map associating to a left Π-module the category of all finite direct sums of direct summands of its restriction to kQ. Note that for a finite dimensional Π-module X we have $D\mathscr{C}(X) = \mathscr{C}_{\mathrm{left}} D(X)$.

By the left module version of [AIRT12] we know that $\mathscr{C}_{\mathrm{left}}(\Pi/I_w)$ is a torsion-free class if and only if w^{-1} is c^{-1}-sortable.

Since $DI_w \cong \Pi/I_{w_0 w^{-1}}$ as left Π-modules by Proposition 6.4(i), we have

$$D\mathscr{C}(I_w) = \mathscr{C}_{\mathrm{left}}(\Pi/I_{w_0 w^{-1}}),$$

and this is a torsion-free class in $\mathrm{mod}\, kQ^{\mathrm{op}}$ if and only if $(w_0 w^{-1})^{-1} = ww_0$ is c^{-1}-sortable.

Dualizing we obtain the claim. □

As was already mentioned, if w is c-sortable, we know that $\mathscr{C}_Q(\Pi/I_w)$ is torsion-free. Write $\mathcal{F}_w$ for this class.

For c-sortable w, write $\hat{w}$ for the unique longest word with the same c-sortable prefix as w. (In order to know that such an element exists, we must continue to assume that Q is Dynkin.) Note, in particular, $\hat{w}$ satisfies the equivalent conditions of Lemma 11.3. Thus, by Proposition 11.4, $\mathscr{C}(I_{\hat{w}})$ is a torsion class. Write $\mathcal{T}_{\hat{w}}$ for this subcategory. From the proof of Proposition 11.4, we also have the further equality

$$\mathcal{T}_{\hat{w}} = \mathscr{C}(I_{\hat{w}}) = \mathscr{C}_{\mathrm{left}}(\Pi/I_{w_0 \hat{w}^{-1}}).$$

Proving Conjecture 11.2 amounts to showing that, for w any c-sortable element, $(\mathcal{T}_{\hat{w}}, \mathcal{F}_w)$ is a torsion pair.

For $w \in W$, choose a reduced expression $w = s_{i_1} \dots s_{i_r}$. Define $\mathrm{Inv}(w)$ to consist of the set of positive roots $\{s_{i_1} \dots s_{i_{t-1}} \alpha_{i_t}\}$. Note that this set does not depend on the chosen expression for w.

Lemma 11.5. *For v, w two c-sortable elements, the following are equivalent:*

(i) $v \leqslant_{\mathrm{R}} w$;
(ii) $\mathrm{Inv}(v) \subseteq \mathrm{Inv}(w)$;
(iii) $\mathcal{F}_v \subseteq \mathcal{F}_w$;
(iv) $\hat{v} \leqslant_{\mathrm{R}} \hat{w}$;
(v) $\hat{v}w_0 \geqslant_{\mathrm{R}} \hat{w}w_0$;
(vi) $\mathcal{T}_v \supseteq \mathcal{T}_w$.

Proof. The equivalence of (i) and (ii) is clear. The equivalence of (ii) and (iii) follows from the fact that the dimension vectors of the indecomposable objects in $\mathcal{F}_w$ are given by $\mathrm{Inv}(w)$, by [AIRT12, Theorem 2.6]. The equivalence of (i) and (iv) follows from [Rea07b, Theorem 1.1] together with its dual. The equivalence of (iv) and (v) follows from the fact that $\mathrm{Inv}(vw_0)$ is the complement of $\mathrm{Inv}(v)$ in the set of positive roots. The equivalence of (v) and (vi) follows in the same way as the equivalence of (i) and (iii), using $\mathcal{T}_{\hat{v}} = \mathscr{C}_{\mathrm{left}}(\Pi/I_{w_0 v^{-1}})$ and the similar description of $\mathcal{T}_{\hat{w}}$. □

Define ϕ to be the map on torsion-free classes that takes $\mathcal{F}_w$ to the torsion-free class associated to $\mathcal{T}_w$. We want to show that ϕ is the identity.

Lemma 11.6. *The map ϕ is a lattice automorphism of the lattice of torsion-free classes.*

Proof. It is clear from the definition that ϕ is invertible. The fact that ϕ is a poset automorphism follows from the equivalence of (iii) and (v) in Lemma 11.5, together with the fact that taking the torsion-free class associated to a torsion class reverses containment. For a finite lattice, being a lattice automorphism is equivalent to being a poset automorphism, because poset relations determine lattice operations and vice versa. □

Lemma 11.7. *For w a c-sortable element, $\mathcal{F}_w$ is splitting if and only if w admits an expression corresponding to an initial segment of the AR-quiver of kQ-mod.*

Proof. If $\mathcal{F}_w$ is splitting it implies that the AR-quiver of $\mathcal{F}_w$ is an initial subquiver of the AR-quiver of $\mathrm{mod}\, kQ$. By [AIRT12], we can read off a word for w from the AR-quiver of $\mathcal{F}_w$, so this shows that w admits an expression corresponding to an initial segment of the AR-quiver of $\mathrm{mod}\, kQ$.

Conversely, suppose w corresponds to an initial subquiver of the AR-quiver with respect to an arbitrary linear extension. Reading this word by slices gives the c-sorting word for w. (This uses the fact, shown in [Arm09], that if we think of the c-sorting word for w_0 contained in $\underline{c}^{\infty}$, the c-sorting word for any w will be contained in the c-sorting word for w_0.) By [AIRT12], it follows that the AR-quiver for the torsion-free class corresponding to w coincides with the given initial subquiver of the AR-quiver. It follows that the objects of $\mathcal{F}_w$ consist of an initial segment of the AR-quiver of kQ-mod. □

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

Lemma 11.8. *The map ϕ is the identity map on the lattice of torsion-free classes.*

Proof. We first show that ϕ fixes splitting torsion-free classes. If $\mathcal{F}_w$ is splitting, then it follows from Lemma 11.7 that w can be read off from an initial segment of the AR-quiver for $\operatorname{mod} kQ$. [PS11, Proposition 2.8] establishes that if w comes from an initial segment of the AR-quiver for $\operatorname{mod} kQ$, then w is the unique element of W whose c-sortable prefix is w. Thus $w = \hat{w}$.

The leftmost word for w inside the word for w_0 is the word read off from the initial segment of the AR-quiver, since we know that this is the c-sorting word. (This is not a complete triviality, because an initial segment of the AR-quiver is not typically an initial segment of our fixed linear order on the indecomposables.) It now follows from Theorem 2.3 that $\mathcal{T}_w$ consist of all sums of indecomposables not in this initial segment. This is precisely the splitting torsion class corresponding to $\mathcal{F}_w$. It follows that $\phi(\mathcal{F}_w) = \mathcal{F}_w$ whenever $\mathcal{F}_w$ is splitting.

Say that a torsion-free class is *principal* if it is of the form $\operatorname{Sub}(E)$ for some indecomposable kQ-module E. We will now show that ϕ fixes principal torsion-free classes.

Observe that principal torsion-free classes can be described in purely lattice-theoretic terms, as the non-zero torsion-free classes which cannot be written as the join of two smaller torsion-free classes. It follows that ϕ takes principal torsion-free classes to principal torsion-free classes.

Let E be an indecomposable object. Let $\mathcal{S}$ be the splitting torsion-free class consisting of the additive hull of the objects up to and including E in our standard linear order on the indecomposable kQ-modules. Let $\mathcal{S}'$ be the additive hull of the indecomposable objects of $\mathcal{S}$ other than E. Then $\mathcal{S}'$ is clearly also a splitting torsion-free class. As we have already seen, ϕ fixes both $\mathcal{S}$ and $\mathcal{S}'$. Since $\operatorname{Sub} E$ is the only principal torsion-free class contained in $\mathcal{S}$ but not in $\mathcal{S}'$, we have that $\phi(\operatorname{Sub}(E)) = \operatorname{Sub}(E)$.

Since any torsion-free class can be written as the join of the principal torsion-free classes corresponding to the indecomposable summands of a cogenerator for the torsion-free class, it follows that ϕ fixes all torsion-free classes, as desired. □

Proposition 11.9. *Conjecture 11.2 holds if Q is of finite type.*

Proof. Suppose that $\mathscr{C}(I_v)$ is a torsion class. By Proposition 11.4, we know that vw_0 is c^{-1}-sortable. Let $w = \operatorname{sort}_c(v)$. Applying the above analysis, we find that $\mathcal{F}_w = \phi(\mathcal{F}_w)$, so $\mathcal{F}_w$ is the torsion-free class associated to $\mathscr{C}(I_v)$, as desired. □

12. Leftmost reduced words and ⅃-diagrams

In this section, we explain how our results applied in type A_n provide an alternative derivation for Postnikov's description of leftmost reduced subwords inside Grassmannian permutations in terms of ⅃-diagrams.

Let W be the Weyl group of type A_n, which is isomorphic to the symmetric group on $n+1$ letters. Fix an integer k such that $1 \leqslant k \leqslant n$. Let $W_{\langle k\rangle}$ be the parabolic subgroup generated by the simple reflections other than s_k, and let $W^{\langle k\rangle}$ be the minimal length coset representatives for $W_{\langle k\rangle}\backslash W$. These are the k-Grassmannian permutations in S_{n+1} (or, depending on a choice of convention, their inverses). The elements of $W^{\langle k\rangle}$ have an essentially unique expression as a product of simple reflections; if $w \in W^{\langle k\rangle}$, then any two reduced expressions for w differ by commutation of commuting reflections.

Leftmost reduced subwords inside a reduced word for $w \in W^{\langle k\rangle}$ are of interest, as they index the cells in the totally non-negative part of the Grassmannian of k-planes in $\mathbb{C}^{n+1}$. In this context, such subwords are referred to as 'positive distinguished subexpressions' of w. For more

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

background on this, and for the equivalence of 'leftmost reduced' and 'positive distinguished,' see [Pos06, § 19].

Postnikov gives a combinatorial criterion to identify the lexicographically first subexpressions in a reduced word for $w \in W^{\langle k \rangle}$, as follows.

Let $w_0^{\langle k \rangle}$ be the longest element of $W^{\langle k \rangle}$. A reduced expression for $w_0^{\langle k \rangle}$ can be written out explicitly as $(s_k s_{k+1} \dots s_n)(s_{k-1} \dots s_{n-1}) \dots (s_1 \dots s_{n-k+1})$. Write out this reduced expression inside a $k \times (n-k+1)$ rectangle, as is done in the example below with $n = 4, k = 2$.

s_2	s_3	s_4
s_1	s_2	s_3

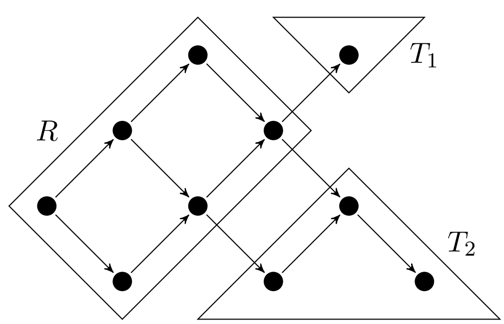
The elements $w \in W^{\langle k \rangle}$ correspond bijectively to partitions that can be drawn inside this rectangle in the usual English notation (that is to say, the parts of the partition are left-justified rows of boxes, with the sizes of the parts weakly decreasing from top to bottom). If λ is a partition, a word for the corresponding element of $W^{\langle k \rangle}$ is given by reading the reflections inside λ from left to right by rows, starting at the top row. We say that a subword (thought of as a subset of the boxes of this partition) has a bad ⅃ if there is some reflection which is used, such that there is some reflection in the column above it which is unused, and some reflection in the row to the left of it which is unused. (The relative position of the three reflections explains the use of the symbol ⅃.)

Then Postnikov shows the following theorem.

Theorem 12.1 [Pos06, Theorem 19.2], see also [LW08]. *A subword of $w \in W^{\langle k \rangle}$ is a leftmost reduced subword if and only if it has no bad ⅃.*

We will recover this result using our description of leftmost reduced words in terms of quotient closed subcategories.

Let Q be the quiver of type A_n, with all arrows oriented away from vertex k. When we consider the AR-quiver for kQ-mod, we observe that it consists of a rectangle R and two triangles, T_1, T_2, as in the picture below, showing the case $n = 4$, $k = 2$.



The rectangle R consists of the representations whose support includes the vertex k. The left-hand corner of the rectangle is the simple projective supported at vertex k, while the right-hand corner is the corresponding injective, the unique sincere indecomposable representation. The triangle T_1 consists of representations supported only on vertices smaller than k, while T_2 consists of representations supported only on vertices greater than k.

If we replace the indecomposables in the AR-quiver by the corresponding simple reflections, and then read them in the order given by the slices, then Theorem 2.3 tells us that we obtain a word for w_0, the longest element of W. Call this our standard word for w_0. Say that a reading order *respects the AR-quiver* if, for any irreducible morphism $A \longrightarrow B$, we read the reflection corresponding to A before the reflection corresponding to B. Then any reading order which respects the AR-quiver will yield a reduced word for w_0 which differs from the standard word by a sequence of commutations of commuting reflections. It follows that the leftmost reduced words for any reading order which respects the AR-quiver will correspond to quotient closed subcategories in exactly the same way.

In particular, for any $w \in W^{\langle k \rangle}$, we can take a reading order which begins by reading the reflections in the corresponding partition along lines sloping from bottom left to top right, followed by reading the remaining reflections in R and the reflections in the two triangles in any order compatible with the AR-quiver. The result is a word for w_0 which begins with a word for w.

By Theorem 2.3, leftmost reduced words inside w therefore correspond to quotient closed subcategories of kQ-mod which contain all the indecomposables outside the partition corresponding to w. We have therefore reduced the combinatorial problem of classifying leftmost reduced words inside w to the problem of classifying quotient closed subcategories which contain all the indecomposables outside a partition λ.

We say that a subcategory has a bad ⅃ if there is some indecomposable X in R which is missing from the subcategory, such that the subcategory contains an object on the line of morphisms leading to X from the top right, and an object on the line of morphisms leading to X from the bottom left. We will therefore recover Postnikov's result once we have established the following proposition.

Proposition 12.2. *The quotient closed subcategories of kQ-mod which contain all the indecomposables outside λ are exactly those which have no bad ⅃ inside λ.*

Proof. Suppose that a subcategory $\mathcal{C}$ has a bad ⅃. Then $\mathcal{C}$ is missing some indecomposable X, and contains an indecomposable Y on the line of morphisms leading to X from the top right, and an indecomposable Z on the line of morphisms leading to X from the bottom right. It is easy to see that there is an epimorphism $Y \oplus Z \longrightarrow X$. Therefore $\mathcal{C}$ is not quotient closed.

Conversely, suppose that $\mathcal{C}$ contains all the indecomposables outside λ and is not quotient closed. Then there is some indecomposable X which is not in $\mathcal{C}$, and such that there is an epimorphism from some object of $\mathcal{C}$ onto X. It is easy to see that this is only possible if $\mathcal{C}$ has a bad ⅃ with X at the corner, since all the irreducible morphisms inside R are monomorphisms. □

For this choice of Q, it is possible to use the same approach to describe the explicit combinatorics of the leftmost reduced words inside the word for w_0 which is obtained by replacing the indecomposables in the AR-quiver by the corresponding simple reflections, and then reading them in any order compatible with the AR-quiver.

Specifically, we have the following representation-theoretic result.

Proposition 12.3. *A subcategory $\mathcal{C}$ of kQ is quotient closed provided that:*

- *$\mathcal{C}$ has no bad ⅃ inside R;*
- *if any indecomposable from R appears in $\mathcal{C}$, then so do all the elements of T_1 on the same diagonal running from bottom left to top right, and so do all the elements of T_2 on the same diagonal running from top left to bottom right;*
- *along any line of morphisms running from bottom left to top right, if any indecomposable from T_1 is in $\mathcal{C}$, all subsequent indecomposables along the diagonal also lie in $\mathcal{C}$;*
- *along any line of morphisms running from top left to bottom right, if any indecomposable from T_2 is in $\mathcal{C}$, all subsequent indecomposables along the diagonal also lie in $\mathcal{C}$.*

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

Proof. We leave the proof of these elementary facts to the reader. □

By Theorem 2.3, this yields the following consequence. We think of the simple reflections in our word for w_0 as positioned at the vertices of the AR-quiver. In particular, this means that where one usually refers to rows and columns, we will refer to diagonals.

Corollary 12.4. *A leftmost reduced word inside w_0 is one which has the following properties:*

- *it has no bad $\daleth$ inside the reflections coming from R;*
- *if any simple reflection s inside R is skipped, then all subsequent reflections in T_1 and T_2 on the diagonals through that s must also be skipped;*
- *if any simple reflection s inside T_1 is skipped, then all subsequent reflections inside T_1 on the same upward-pointing diagonal must be skipped;*
- *if any simple reflection s inside T_2 is skipped, then all subsequent reflections inside T_2 on the same downward-pointing diagonal must be skipped.*

13. Connection to the work of Armstrong

In this section, we explain the link to Armstrong's work [Arm09], which provided the initial motivation for our investigations. We restrict to the case that Q is Dynkin for simplicity; on the whole, that is the setting in which combinatorialists have worked.

Let E be a finite ground set and let $\mathcal{A}$ be a collection of subsets of E. The sets in $\mathcal{A}$ are referred to as *feasible* sets. We say that the set system $\mathcal{A}$ is *accessible* if, for every $\emptyset \neq A \in \mathcal{A}$, there exists some $x \in A$ such that $A \backslash \{x\} \in \mathcal{A}$.

An accessible set system $\mathcal{A}$ is called an *antimatroid* if it satisfies the condition that if $A, B \in \mathcal{A}$ with $B \not\subseteq A$, then there exists some $x \in B \backslash A$ such that $A \cup \{x\} \in \mathcal{A}$.

An antimatroid is called *supersolvable* [Arm09] if E is equipped with a total order such that, if A, B in $\mathcal{A}$, with $B \not\subseteq A$, then $A \cup \{x\} \in \mathcal{A}$, where x is the minimum element of B not in A (with respect to the total order).

Let W be a Coxeter group, which we assume to be finite. Fix an arbitrary word $\underline{w} = s_{i_1} \dots s_{i_N}$ in the simple reflections of W. For $v \in W$, consider the subwords of $\underline{w}$ which define reduced words for v, and, if there is at least one such subword, define A_v to be the subset of $\{1, \dots, N\}$ corresponding to the positions occupied by the leftmost such word. Then define $\mathcal{A}_{\underline{w}}$ to consist of the collection of all the A_v (for those v such that A_v is defined). One of the main results of [Arm09], Theorem 4.4, says that $\mathcal{A}_{\underline{w}}$ is a supersolvable antimatroid (with respect to the usual order on the ground set $\{1, \dots, N\}$).

Using our results, we can recover this result of Armstrong for particular choices of word $\underline{w}$. Suppose that W is a simply-laced Weyl group, so that it corresponds to a Dynkin diagram. Choose an arbitrary orientation for the diagram, obtaining a quiver Q. Now consider the word for the element $w_0 \in W$ obtained by reading the AR-quiver for kQ, as described in § 2. We call this word $\underline{w}_{\mathrm{AR}}$. Using our correspondence between leftmost words and quotient closed subcategories, $A \in \mathcal{A}_{\underline{w}_{\mathrm{AR}}}$ if and only if A is the set of indecomposables missing from a quotient closed subcategory of kQ-mod. In this case we write A^c for the corresponding quotient closed subcategory.

Let $A, B \in \mathcal{A}_{\underline{w}_{\mathrm{AR}}}$, with $B \not\subseteq A$. Let x be the first indecomposable (with respect to our fixed total order) in B which is not in A. To show that $\mathcal{A}_{\underline{w}_{\mathrm{AR}}}$ is a supersolvable antimatroid, we must show that $A \cup \{x\} \in \mathcal{A}_{\underline{w}_{\mathrm{AR}}}$. This is equivalent to saying that x can be removed from the quotient closed category A^c, without destroying quotient closedness.

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press

To see this, write X for the full subcategory of $\operatorname{mod} kQ$ whose indecomposable objects properly precede x in our fixed total order. Since $x \in B$, we know that x is not a quotient of any object of B^c. Since x is the first element of B not in A, the indecomposable objects of B which precede x all lie in A. Thus $B^c \cap X \supseteq A^c \cap X$, so $A^c \backslash \{x\}$ is still quotient closed, as desired. (As noted in [Arm09], we do not have to check the fact that $\mathcal{A}_{\underline{w}_{AR}}$ is accessible separately, since it follows from the condition we have already checked.)

Acknowledgements

SO was supported by FRINAT grants 19660 and 221893 from the Norwegian Research Council. IR was supported by FRINAT grant 19660 from the Norwegian Research Council. HT was supported by an NSERC Discovery Grant. He also gratefully acknowledges the hospitality of the Mathematics Institute at NTNU. We thank the referees for their helpful comments, and especially for asking what could be said over ground fields which are not algebraically closed.

References

AIRT12 C. Amiot, O. Iyama, I. Reiten and G. Todorov, *Preprojective algebras and c-sortable words*, Proc. Lond. Math. Soc. (3) **104** (2012), 513–539.

Arm09 D. Armstrong, *The sorting order on a Coxeter group*, J. Combin. Th. Series A **116** (2009), 1285–1305.

APR79 M. Auslander, M. I. Platzeck and I. Reiten, *Coxeter functors without diagrams*, Trans. Amer. Math. Soc. **250** (1979), 1–46.

AS93 M. Auslander and Ø. Solberg, *Relative homology and representation theory I-III*, Comm. Algebra **21** (1993), 2995–3097.

BGL87 D. Baer, W. Geigle and H. Lenzing, *The preprojective algebra of a tame hereditary algebra*, Comm. Algebra **15** (1987), 425–457.

BK12 P. Baumann and J. Kamnitzer, *Preprojective algebras and MV polytopes*, Represent. Theory **16** (2012), 152–188.

BGP73 I. N. Bernšteĭn, I. M. Gel'fand and V. A. Ponomarev, *Coxeter functors, and Gabriel's theorem*, Uspekhi Mat. Nauk **28** (1973), 19–33 (in Russian).

BB05 A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231 (Springer, New York, 2005).

BIRS09 A. B. Buan, O. Iyama, I. Reiten and J. Scott, *Cluster structures for 2-Calabi–Yau categories and unipotent groups*, Compositio Math. **145** (2009), 1035–1079.

CE56 H. Cartan and S. Eilenberg, *Homological algebra* (Princeton University Press, Princeton, NJ, 1956).

DDPW08 B. Deng, J. Du, B. Parshall and J. Wang, *Finite dimensional algebras and quantum groups*, Mathematical Surveys and Monographs, vol. 150 (American Mathematical Society, Providence, RI, 2008).

GLS07 Ch. Geiß, B. Leclerc and J. Schröer, *Semicanonical bases and preprojective algebras. II. A multiplication formula*, Compositio Math. **143** (2007), 1313–1334.

IT09 C. Ingalls and H. Thomas, *Noncrossing partitions and representations of quivers*, Compositio Math. **145** (2009), 1533–1562.

IR08 O. Iyama and I. Reiten, *Fomin–Zelevinsky mutation and tilting modules over Calabi–Yau algebras*, Amer. J. Math. **130** (2008), 1087–1149.

KP13 H. Krause and M. Prest, *The Gabriel–Roiter filtration of the Ziegler spectrum*, Q. J. Math. **64** (2013), 891–901.

LP13 T. Lam and P. Pylyavskyy, *Total positivity for loop groups II: Chevalley generators*, Transform. Groups **18** (2013), 179–231.

LW08 T. Lam and L. Williams, *Total positivity for cominuscule Grassmannians*, New York J. Math. **14** (2008), 53–99.

PS11 V. Pilaud and C. Stump, *Brick polytopes of spherical subword complexes: A new approach to generalized associahedra*, Preprint (2011), arXiv:1111.3349.

Pos06 A. Postnikov, *Total positivity, Grassmannians, and networks*, Preprint (2006), arXiv:math/0609764.

Rea07a N. Reading, *Clusters, Coxeter-sortable elements and noncrossing partitions*, Trans. Amer. Math. Soc. **359** (2007), 5931–5958.

Rea07b N. Reading, *Sortable elements and Cambrian lattices*, Algebra Universalis **56** (2007), 411–437.

Rin98 C. Ringel, *The preprojective algebra of a quiver*, in *Algebras and Modules II*, Can. Math. Soc. Conf. Proc., 24 (1998), 467–480.

Rin12 C. Ringel, *Minimal infinite submodule-closed subcategories*, Bull. Sci. Math. **136** (2012), 820–830.

Ste08 J. Stembridge, *Folding by automorphisms*, Preprint (2008); available at math.lsa.umich.edu/~jrs/papers/folding.ps.gz.

Steffen Oppermann steffen.oppermann@math.ntnu.no
Department of Mathematics, NTNU, Trondheim, Norway

Idun Reiten idunr@math.ntnu.no
Department of Mathematics, NTNU, Trondheim, Norway

Hugh Thomas hugh@math.unb.ca
Department of Mathematics and Statistics, UNB, PO Box 4400, Fredericton, Canada

https://doi.org/10.1112/S0010437X14007623 Published online by Cambridge University Press