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Isomorphisms between diffeomorphism groups

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Abstract. It is shown that any group isomorphism between the C^k diffeomorphism groups of two finite-dimensional, boundaryless paracompact manifolds is induced by a C^k diffeomorphism between the manifolds ($1 \le k \le \infty$).

I. *Introduction*

In [**1**] Whittaker proves that for a certain class of topological spaces any group isomorphism between the homeomorphism groups of two such spaces is induced by a homeomorphism between the spaces themselves. Since the class of spaces he considers contains all topological manifolds it is natural to ask whether the same result holds in the differentiable category. We will answer this question in the affirmative by means of the following theorem:

THEOREM. *Let M and N be smooth (i.e. C^∞) manifolds without boundary and let* $\mathrm{Diff}^p(M)$ *and* $\mathrm{Diff}^q(N)$ *for $1 \le p, q \le \infty$ denote the groups of C^p diffeomorphisms of M and C^q diffeomorphisms of N. If*

$$\phi : \mathrm{Diff}^p(M) \to \mathrm{Diff}^q(N)$$

is a group isomorphism then $p = q$ and there is C^p diffeomorphism $w : M \to N$ such that

$$\phi(f)(n) = wfw^{-1}(n)$$

for all $f \in \mathrm{Diff}^p(M)$ and $n \in N$.

The idea behind the proof is the same as that of Whittaker in that we will show that the isomorphism ϕ induces a bijection between the stabilizer subgroups of M and N. In other words if we consider the subgroup

$$S^p_m = \{f \in \mathrm{Diff}^p(M) | f(m) = m\}$$

for $m \in M$, we will show that there is a point $w(m) \in N$ such that

$$\phi(S^p_m) = S^q_{w(m)}.$$

The mapping $w : M \to N$ so constructed is easily seen to be a homeomorphism inducing ϕ. Takens showed in [**2**] that if $p = q = \infty$ then w must be a C^∞ diffeomorphism. To complete the proof we will appeal to a deep result of Montgomery and Zippin [**3**] concerning Lie groups acting by diffeomorphisms on manifolds to show that $p = q$ and w is a C^p diffeomorphism.

In order to handle the differentiable case several of the arguments of § 2 of [**1**] have to be modified to avoid the infinite patching methods employed by Whittaker to classify the minimal normal subgroups of the homeomorphism group of a space. These, weaker, results are presented in § 2. In order to keep this paper as self-contained as possible we reproduce, in § 3, the proofs of the key lemmas used in [**1**] to prove the main result. Finally, in § 4, the theorem stated above is proved.

Independently, and using completely different methods, in [**4**] M. Rubin has shown the existence of a homeomorphism inducing an isomorphism between diffeomorphism groups. However in this paper he was unable to show that the homeomorphism was differentiable. Subsequently he has managed to achieve this but I have not yet seen the proof [**5**].

Notation.

(i) M and N will denote connected, paracompact, differentiable manifolds without boundary with $\dim(M)$, $\dim(N)<\infty$. We will assume, in addition, that both M and N are smooth (i.e. C^∞). This involves no loss of generality as any differentiable manifold has a compatible C^∞ structure.

(ii) If $1\le k\le\infty$ then $\mathrm{Diff}^k(M)$ will denote the group of C^k diffeomorphisms of M and $\mathrm{Diff}_0^k(M)$ the subgroup of $\mathrm{Diff}^k(M)$ consisting of those diffeomorphisms compactly isotopic to the identity.

(iii) If $g\in\mathrm{Diff}^k(M)$ then

$$\mathrm{Supp}(g)=\overline{\{x\in M|g(x)\ne x\}}$$
$$\mathrm{Fix}(g)=\{x\in M|g(x)=x\}.$$

(iv) D_a^n will denote the closed ball of radius a in $\mathbb{R}^n$ and ∂D_a^n its boundary.

(v) $\mathbb{B}(M)$ will denote the collection of open subsets $U\subset M$ such that $U=h(\mathrm{Int}(D_1^n))$ for some embedding $h:D_{1+\varepsilon}^n\to M$ with $\varepsilon>0$.

(vi) If $U\subset M$ is open then $\mathrm{Diff}_0^k(U)$ will denote the subgroup: $\{g\in\mathrm{Diff}^k(M)|g$ is compactly isotopic to the identity by an isotopy whose support lies in $U\}$.

(vii) If $F\subset\mathrm{Diff}^k(M)$ is a subgroup let $\mathbb{B}(F)$ denote the subcollection $\{U\subset\mathbb{B}(M)|\mathrm{Diff}_0^k(U)\subset F\}$ of $\mathbb{B}(M)$.

(viii) The commutator subgroup of a group H will be denoted $[H,H]$ and if $f,g\in H$ then $[f,g]=fgf^{-1}g^{-1}$. The identity element of any group will be denoted 'e'.

II. *Minimal normal subgroups*

We will now proceed to prove two theorems which are weaker analogues of the results of § 2 of [**1**]. The first states that if $F\subset\mathrm{Diff}^k(M)$ is a subgroup containing enough local subgroups $\mathrm{Diff}_0^k(U_x)$ with U_x a neighbourhood of $x\in M$ then F contains almost all of $\mathrm{Diff}_0^k(M)$.

LEMMA 2.1. *Let $\mathscr{C}$ be a covering of D_1^n by open subsets of $\mathbb{R}^n$. Then if $a\in(0,1]$ there are $f_i, g_i\in\mathrm{Diff}_0^\infty(\mathbb{R}^n)$ for $1\le i\le r$ such that:*

(a) *For each i there is an element $C_i\in\mathscr{C}$ such that $f_i, g_i\in\mathrm{Diff}_0^\infty(C_i)$.*

(b) $[f_r,g_r]\circ\cdots\circ[f_1,g_1](D_1^n)\subset D_a^n$.

Proof. Let $A=\{a\in(0,1]|$ lemma 2.1 is true for $a\}$. Then $A\ne\phi$ since $1\in A$. If the lemma is true for $a_1<1$ then it is true for all $a\in[a_1,1]$; hence A is an interval.

Let $a_0 = \text{Inf}(A)$: we wish to show that $a_0 = 0$. Assume that it is not and let $V_1, \ldots, V_s$ be a finite covering of $\partial D^n_{a_0}$ by elements of $\mathscr{C}$ and let $\varepsilon > 0$ be less than its Lebesgue number. For each $x \in \partial D^n_{a_0}$ choose $g_x \in \text{Diff}^\infty_0(B(x, \varepsilon))$ and a neighbourhood U_x of x in $\mathbb{R}^n$ such that:

(i) $U_x \subset D^n_{a_0+\varepsilon/2} - D^n_{a_0-\varepsilon/2}$ and $U_x \subset B(x, \varepsilon)$.

(ii) $g_x(U_x) \subset D^n_{a_0-\varepsilon/2}$ and $g_x^{-1}(U_x) \subset \mathbb{R}^n - D^n_{a_0+\varepsilon/2}$.

Let $\{U_i\}_{i=1}^r$ be a finite covering of $\partial D^n_{a_0}$ by these neighbourhoods with corresponding diffeomorphisms $\{g_i\}$ and let $\varepsilon' < \varepsilon/2$ be such that

$$\overline{(D^n_{a_0+\varepsilon'} - D^n_{a_0-\varepsilon'})} \subset \bigcup_{i=1}^r U_i.$$

Choose

$$f \in \text{Diff}^\infty_0(\text{Int}(D^n_{a_0+\varepsilon'} - D^n_{a_0-\varepsilon'}))$$

such that

$$f(D^n_{a_0+\varepsilon'/2}) \subset D^n_{a_0-\varepsilon'/2}.$$

If ε' is small enough and f is sufficiently near the identity in the C^∞ topology we can use the techniques of lemma 3.1 of Palis and Smale [6] to factor f as a product $f = f_r \circ \cdots \circ f_1$ with $f_i \in \text{Diff}^\infty_0(U_i)$ for $1 \le i \le r$. If we now set $h_i = [f_i, g_i]$ we have:

$$h_i(x) = \begin{cases} f_i(x) & \text{if } x \in U_i, \\ g_i f_i^{-1} g_i^{-1} & \text{if } x \in g_i(U_i), \\ x & \text{otherwise,} \end{cases}$$

since by (i) and (ii) the sets U_i, $g_i(U_i)$, $g_i^{-1}(U_i)$ are pairwise disjoint. It follows that if

$$x \in D^n_{a_0+\varepsilon'} - D^n_{a_0-\varepsilon'}$$

we have

$$h_r \circ \cdots \circ h_1(x) = f_r \circ \cdots \circ f_1(x) = f(x);$$

hence

$$h_r \circ \cdots \circ h_1(D^n_{a_0}) \subset D^n_{a_0-\varepsilon'/2}.$$

Now, by construction, for each i there is an $x_i \in \partial D^n_{a_0}$ such that

$$f_i, g_i \in \text{Diff}^\infty_0(B(x, \varepsilon))$$

so, since ε is smaller than the Lebesgue number of the covering $\mathscr{C}$, it follows that there exists $C_i \in \mathscr{C}$ with

$$f_i, g_i \in \text{Diff}^\infty_0(C_i)$$

for $1 \le i \le r$. Hence $a_0 \le a_0 - \varepsilon'/2 < a_0$ which is a contradiction. □

THEOREM 2.2. *Let F be a subgroup of* $\text{Diff}^k(M)$ *and suppose that for each $x \in M$ there is a neighbourhood U_x of x in $\mathbb{B}(F)$ such that*

$$[\text{Diff}^k_0(U_x), \text{Diff}^k(U_x)] \subset F.$$

Then

$$[\text{Diff}^k_0(M), \text{Diff}^k_0 M)] \subset F.$$

Proof. Let H be the subgroup of $[\text{Diff}^k_0(M), \text{Diff}^k_0(M)]$ generated by the groups $[\text{Diff}^k_0(U), \text{Diff}^k_0(U)]$ as U ranges over $\mathbb{B}(M)$. Then H is normal in $\text{Diff}^k(M)$ so by Epstein [7]

$$H = [\text{Diff}^k_0(M), \text{Diff}^k_0(M)].$$

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Hence we need only show that if $W \in \mathbb{B}(M)$ then

$$[\mathrm{Diff}_0^k(W), \mathrm{Diff}_0^k(W)] \subset F.$$

Let $W \in \mathbb{B}(M)$ be the image of $\mathrm{Int}(D_1^n)$ under an embedding $h: D_{1+\varepsilon}^n \to M$. Then, by hypothesis, we can cover $\bar{W}$ by $U_1, \ldots, U_s \in \mathbb{B}(M)$ such that

$$[\mathrm{Diff}_0^k(U_i), \mathrm{Diff}^k(U_i)] \subset F.$$

If we set

$$V_i = h^{-1}(U_i)$$

then the V_i cover D_1^n and we assume that $0 \in V_1$. For some $a > 0$ we have $D_a^n \subset V_1$ and so by lemma 2.1 we can choose commutators $[f_j, g_j]$ $(1 \leq j \leq r)$ with

$$f_j, g_j \in \mathrm{Diff}_0^\infty(V_{i(j)})$$

and such that

$$[f_r, g_r] \circ \cdots \circ [f_1, g_1](D_1^n) \subset D_a^n.$$

Now, since

$$f_j, g_j \in \mathrm{Diff}_0^\infty(\mathrm{Int}(D_{1+\varepsilon}^n))$$

for $1 \leq j \leq r$ we can define

$$\hat{f}_j, \hat{g}_j \in \mathrm{Diff}_0^k(M)$$

as follows:

$$\hat{f}_j(x) = \begin{cases} hf_jh^{-1}(x) & \text{if } x \in h(D_{1+\varepsilon}^n), \\ x & \text{otherwise.} \end{cases}$$

$$\hat{g}_j(x) = \begin{cases} hg_jh^{-1}(x) & \text{if } x \in h(D_{1+\varepsilon}^n), \\ x & \text{otherwise.} \end{cases}$$

Then

$$[\hat{f}_j, \hat{g}_j] \in [\mathrm{Diff}_0^k(U_{i(j)}), \mathrm{Diff}_0^k(U_{i(j)})]$$

and, if we define ψ to be $[\hat{f}_r, \hat{g}_r] \circ \cdots \circ [\hat{f}_1, \hat{g}_1]$, we have $\psi \in F$, by hypothesis, and $\psi(W) \subset U_1$. It follows that

$$[\mathrm{Diff}_0^k(W), \mathrm{Diff}_0^k(W)] \subset \psi^{-1}[\mathrm{Diff}_0^k(U_1), \mathrm{Diff}_0^k(U_1)]\psi \subset F. \qquad \square$$

Following Whittaker we will now attempt to classify the minimal normal subgroups of a subgroup F of $\mathrm{Diff}^k(M)$. Let $B(F)$ denote the union of all the open sets in $\mathbb{B}(F)$. We define a relation R on $B(F)$ as follows:

If $x, y \in B(F)$ then xRy iff there are $U_i \in \mathbb{B}(F)$ with $1 \leq i \leq k$ such that $x \in U_1$, $y \in U_k$ and $U_i \cap U_{i+1} \neq \varnothing$ for $1 \leq i \leq k-1$.

This is clearly an equivalence relation. The equivalence classes, which are open and connected, will be called F-components.

LEMMA 2.3. *Let C be an F-component and $\{x_i\}_{i=1}^k$, $\{y_i\}_{i=1}^k$ two collections of k distinct points of C. If $\dim(M) = 1$ assume, in addition, that $x_i < x_j$ implies $y_i < y_j$. Then there is an $f \in F$ such that $f(x_i) = y_i$ for $1 \leq i \leq k$.*

Proof. It is clear that F acts transitively on C. The result follows by induction on k (replacing M by $M - \{x_1, \ldots, x_k\}$). $\square$

THEOREM 2.4. *Let F be a subgroup of $\mathrm{Diff}^k(M)$, G a normal subgroup of F, and C an F-component. If there is an $x_0 \in C$ and a $g_0 \in G$ such that $g_0(x_0) \neq x_0$ then*

$$[\mathrm{Diff}_0^k(C), \mathrm{Diff}_0^k(C)] \subset G.$$

Proof. If $g_0(x_0) \notin C$ we must have $g_0(C) \cap C = \emptyset$ since $g_0(C)$ is an F-component. We choose an $f \in F$ with Supp $(f) \subset C$ and $f(x_0) \neq x_0$. Then if we set

$$\tilde{g} = fg_0^{-1}f^{-1}g_0$$

we have $\tilde{g} \in G$, since G is normal in F, and

$$\tilde{g}(x_0) = f(x_0) \neq x_0.$$

Hence we can assume, without loss of generality, that

$$g_0(x_0) \in C.$$

We now wish to show that G is transitive on C. Let

$$y_0 = g_0(x_0)$$

and let $y \in C$ be point distinct from y_0. If $\dim(M) \neq 1$ then by lemma 2.3 there is an $f \in F$ such that $f(y) = x_0$ and $f(y_0) = y_0$. Then

$$f^{-1}g_0^{-1}fg_0(x_0) = y.$$

If $\dim(M) = 1$ we have two cases:

Case 1: ($x_0 < y_0$ and $y < y_0$) or ($x_0 > y_0$ and $y > y_0$).
By lemma 2.3 there is an $f \in F$ such that $f(y_0) = y_0$ and $f(y) = x_0$. Then

$$f^{-1}g_0^{-1}fg_0(x_0) = y.$$

Case 2: ($x_0 < y_0$ and $y > y_0$) or ($x_0 > y_0$ and $y < y_0$).
By lemma 2.3 there is an $f \in F$ such that $f(x_0) = x_0$ and $f(y) = y_0$. Then

$$f^{-1}g_0f(x_0) = y.$$

Since G is normal in F we have shown that for all $y \in C$ there is a $g \in G$ such that $g(x_0) = y$; hence G is transitive on C.

Let x_1, x_2 be points of C distinct from x_0 and choose $g_1, g_2 \in G$ with $g_i(x_0) = x_i$ for $i = 1, 2$. Let U_0 be a neighbourhood of x_0 such that U_0, $g_1(U_0)$, $g_1^{-1}(U_0)$, $g_2(U_0)$, $g_2^{-1}(U_0)$ are pairwise disjoint. If

$$h_1, h_2 \in \mathrm{Diff}_0^k(U_0)$$

then, setting $c_i = [h_i, g_i]$ for $i = 1, 2$ we have

$$c_i(x) = \begin{cases} h_i(x) & \text{if } x \in U_0, \\ g_i h_i^{-1} g_i^{-1}(x) & \text{if } x \in g_i(U_0), \\ x & \text{otherwise.} \end{cases}$$

Hence $[c_1, c_2] = [h_1, h_2]$. Therefore

$$[\mathrm{Diff}_0^k(U_0), \mathrm{Diff}_0^k(U_0)] \subset G.$$

Since G is transitive on C it follows from the previous paragraph that for each $x \in C$ there is a neighbourhood U_x of x in C such that

$$[\mathrm{Diff}_0^k(U_x), \mathrm{Diff}_0^k(U_x)] \subset G.$$

From theorem 2.2, since C is an open submanifold of M, we conclude that

$$[\mathrm{Diff}_0^k(C), \mathrm{Diff}_0^k(C)] \subset G. \qquad \square$$

We finish this section with the following refinement of the results of Epstein in [**7**].

THEOREM 2.5. $[\mathrm{Diff}_0^k(M), \mathrm{Diff}_0^k(M)]$ *is the* unique *minimal normal subgroup of* $\mathrm{Diff}^k(M)$.

Proof. Clearly by Epstein [7] $[\mathrm{Diff}_0^k(M), \mathrm{Diff}_0^k(M)]$ is a minimal normal subgroup of $\mathrm{Diff}^k(M)$ so it remains to show that if H is any normal subgroup of $\mathrm{Diff}^k(M)$ then

$$H \cap [\mathrm{Diff}_0^k(M), \mathrm{Diff}_0^k(M)] \neq \{e\}.$$

To see this choose $h \in H$ and $U \in \mathbb{B}(M)$ such that U, $h(U)$, $h^{-1}(U)$ are pairwise disjoint. If $f, g \in \mathrm{Diff}_0^k(U)$ are such that $[f, g] \neq e$ we set $\hat{f} = [f, h]$ and $\hat{g} = [g, h]$. It is clear that

$$\hat{f}, \hat{g} \in H \cap \mathrm{Diff}_0^k(M)$$

and, by the choice of U,

$$[\hat{f}, \hat{g}](x) = [f, g](x)$$

for all $x \in U$. Thus $[\hat{f}, \hat{g}] \neq e$ and

$$[\hat{f}, \hat{g}] \in H \cap [\mathrm{Diff}_0^k(M), \mathrm{Diff}_0^k(M)]. \qquad \square$$

III. *Characterization of stabilizer subgroups*

In this section we will reproduce the proofs of a number of lemmas due to Whittaker [1]. This will enable us to give an algebraic characterization of the stabilizer subgroup of a point of M.

Definition. For $x \in M$ the group defined by

$$S_x^k = \{g \in \mathrm{Diff}^k(M) \mid g(x) = x\}$$

is called the stabilizer of x.

LEMMA 3.1. *Let $f, g, h \in \mathrm{Diff}^k(M) - S_x^k$ for some $x \in M$. Then:*

(*a*) $\mathrm{Diff}^k(M) - S_x^k = S_x^k f S_x^k \cup S_x^k f^{-1} S_x^k$.

(*b*) *$g \in S_x^k f S_x^k$ and $fg, gf \notin S_x^k$ implies that $fg, gf \in S_x^k f S_x^k$.*

(*c*) *$g, h \in S_x^k f S_x^k$ implies that there are $s_1, t_1 \in f^{-1}S_x^k g \cap S_x^k$ and $s_2, t_2 \in f^{-1}S_x^k h \cap S_x^k$ such that $s_1 s_2 = t_2 t_1$.*

Proof. Since S_x^k is n-transitive on $M - \{x\}$ for all n if $\dim(M) > 1$ and this is not true for $\dim(M) = 1$ the proof will be done in two parts.

(i) $\dim(M) > 1$. Choose $s \in S_x^k$ such that $sg(x) = f(x)$. Then

$$f^{-1}sg(x) = x$$

so $f^{-1}sg = t \in S_x^k$ and $g = s^{-1}ft$. Thus

$$\mathrm{Diff}^k(M) - S_x^k = S_x^k f S_x^k$$

which implies both (*a*) and (*b*).

Setting $u = f^{-1}(x)$, $v = g^{-1}(x)$, $w = h^{-1}(x)$ we have:

$$f^{-1}S_x^k g \cap S_x^k = \{s \in S_x^k \mid s(v) = u\}$$
$$f^{-1}S_x^k h \cap S_x^k = \{s \in S_x^k \mid s(w) = u\}.$$

Now choose $y \in M$ such that $y \neq x$, $y = v$ iff $w = u$, $y = w$ iff $v = u$. Choose

$$s_2 \in f^{-1}S_x^k h \cap S_x^k$$

such that $s_2(y) = v$ and

$$t_1 \in f^{-1}S_x^k g \cap S_x^k$$

such that $t_1(y) = w$. If we set $t_2 = s_1 s_2 t_1^{-1}$ for arbitrary $s_1 \in f^{-1}S_x^k g \cap S_x^k$ then $t_2 \in S_x^k$ and $t_2(w) = u$ which shows (*c*).

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(ii) $\dim(M)=1$. If $f(x)$, $g(x)$ lie in the same component of $M-\{x\}$ we can apply the method of (i)(a) above to show that $g \in S_x^k f S_x^k$. If they lie in different components then $f^{-1}(x)$, $g(x)$ lie in the same component so we can find $s \in S_x^k$ such that

$$sg(x)=f^{-1}(x).$$

Then $t=sfg \in S_x^k$ so

$$g=s^{-1}f^{-1}t \in S_x^k f^{-1} S_x^k.$$

This proves (a).

To prove (b) we need only remark that by the proof of (a) above $g \in S_x^k f S_x^k$ means that $f(x)$, $g(x)$ lie in the same component of $M-\{x\}$ and hence so do the pairs $(f(x), fg(x))$ and $(f(x), gf(x))$.

To demonstrate (c) we first note that M is diffeomorphic to the real line or the circle. If M is diffeomorphic to the real line then $g, h \in S_x^k f S_x^k$ implies that $g^{-1}(x)$, $f^{-1}(x)$, $h^{-1}(x)$ all lie in the same component of $M-\{x\}$. If M is diffeomorphic to the circle then we choose an orientation of M to induce an ordering of $M-\{x\}$. The proof of (i)(c) will now work as before if we add the condition: $y>v$ iff $w>u$ and $y>w$ iff $v>u$. □

For the rest of this section let

$$\phi : \mathrm{Diff}^p(M) \to \mathrm{Diff}^q(N)$$

be a group isomorphism. The next two lemmas will show that if S_y^q is the stabilizer of a point $y \in N$ then $\phi^{-1}(S_y^q)$ behaves very much like the stabilizer of some point $x \in M$.

LEMMA 3.2. *Let $F=\phi^{-1}(S_y^q)$ for some $y \in N$ and let A be a proper closed subset of M. If $f(A)=A$ for every $f \in F$ then $A=\{x\}$ and $F=S_x^p$ for some $x \in M$.*

Proof. If $A-\mathrm{Int}(A)$ consists of a single point x then $f(x)=x$ for all $f \in F$ and hence $F \subset S_x^p$. By lemma 3.1(a) and the fact that ϕ is a group isomorphism we see that both F and S_x^p are maximal subgroups of $\mathrm{Diff}^p(M)$; hence $F=S_x^p$.

Now assume that there are two distinct points

$$x_1, x_2 \in A-\mathrm{Int}(A).$$

Choose $U_1, U_2 \in \mathbb{B}(M)$ with $x_i \in U_i$ for $i=1, 2$ and $U_1 \cap U_2=\varnothing$. Since $x_i \notin \mathrm{Int}(A)$ we have

$$U_i - A \neq \varnothing$$

so we can choose

$$h_i \in \mathrm{Diff}_0^p(U_i)$$

such that $h_i(x_i) \notin A$ for $i=1, 2$. By hypothesis $h_i \notin F$ for $i=1, 2$ so lemma 3.1(a) implies that

$$g_2 \in F h_1 F,$$

where $g_2=h_2$ or h_2^{-1}. We will set $g_1=h_1$. Now $g_1 g_2$ and $g_2 g_1 \notin F$ so from lemma 3.1(b), since we have $g_2 \in F g_1 F$ and $g_1 \in F g_2 F$, it follows that

$$g_3=g_1 g_2 \in F g_i F$$

i.e. $g_i \in F g_3 F$ for $i=1, 2$.

Let g_i for $i=1, 2, 3$ be as in the previous paragraph and set

$$A_i = A \cap g_i^{-1}(M-A).$$

Then $A_1 \neq \emptyset$ since $x_1 \in A_1$, $A_i \subset U_i$ for $i=1,2$ and $A_3 = A_1 \cup A_2$ since $U_1 \cap U_2 = \emptyset$. Choose

$$t_i \in g_3^{-1} F g_i \cap F.$$

Then

$$g_3 t_i(A_i) = f_i g_i(A_i) \subset M-A$$

for some $f_i \in F$ and $i=1, 2$. Now $t_i \in F$ means that

$$t_i(A) = A$$

so $t_i(A_i) \subset A_3$ for $i=1, 2$. Similarly

$$g_i t_i^{-1}(A_3) = f_i^{-1} g_3(A_3) \subset M-A$$

so $t_i^{-1}(A_3) \subset A_i$. Thus $t_i(A_i) = A_3$ and so, since we know that $A_1 \neq \emptyset$, it follows that the three sets A_1, A_2, A_3 are non-empty.

By lemma 3.1(c) there are

$$s_i, t_i \in g_3^{-1} F g_i \cap F$$

such that

$$s_2^{-1} s_1^{-1} = t_1^{-1} t_2^{-1}$$

and so, by the results above we have

$$s_2^{-1} s_1^{-1}(A_3) = s_2^{-1}(A_1) \subset s_2^{-1}(A_3) = A_2$$

and

$$t_1^{-1} t_2^{-1}(A_3) = t_1^{-1}(A_2) \subset t_1^{-1}(A_3) = A_1,$$

which contradicts the fact that

$$A_1 \cap A_2 = \emptyset.$$

Hence $A - \mathrm{Int}\,(A)$ must reduce to a single point. □

LEMMA 3.3. *Let $F = \phi^{-1}(S_y^q)$ for some $y \in N$. Then there is an $f \in F$, $f \neq e$, such that* $\mathrm{Int}(\mathrm{Fix}\,(f)) \neq \emptyset$.

Proof. Let $U_i \in \mathbb{B}(M)$ for $1 \le i \le 4$ be such that

$$\bar{U}_i \cap \bar{U}_j = \emptyset \qquad \text{for } i \neq j.$$

Assume that

$$\mathrm{Int}\,(\mathrm{Fix}\,(f)) = \emptyset$$

for all $f \in F$ and choose $h_i \neq e$ in $\mathrm{Diff}^p\,(U_i)$ such that

$$\mathrm{Fix}\,(h_i) = (M - U_i) \cup \{x_i\}$$

for some point $x_i \in U_i$ $(1 \le i \le 4)$. It is clear that h_1 and $h_5 = h_3 h_4$ are not conjugate since their fixed point sets are not homeomorphic. Similarly the pairs (h_1, h_5^{-1}), (h_2, h_5), and (h_2, h_5^{-1}) are not conjugates.

Since $\operatorname{Int}(\operatorname{Fix}(h_i)) \neq \varnothing$ for $1 \le i \le 5$ we have $h_i \notin F$. By lemma 3.1(a) there are s_i, $t_i \in F$ for $i = 1, 2$ such that $s_i h_5 t_i = g_i$ where $g_i = h_i$ or h_i^{-1}. If there is an

$$x \in t_i^{-1}(M - \bar{U}_5) \cap (M - \bar{U}_i)$$

then $s_i t_i(x) = s_i h_5 t_i(x) = g_i(x) = x$ so

$$\operatorname{Int}(\operatorname{Fix}(s_i t_i)) \neq \varnothing.$$

Since g_i and h_5 are not conjugate we have $s_i t_i \neq e$ contradicting the assumption. It then follows that

$$t_i^{-1}(M - \bar{U}_5) \subset \bar{U}_i$$

whenever

$$t_i \in h_5^{-1} F g_i \cap F.$$

By lemma 3.1(c) there are s_i, $t_i \in h_5^{-1} F g_i \cap F$ such that

$$s_2^{-1} s_1^{-1} = t_1^{-1} t_2^{-1}.$$

But then we have

$$s_2^{-1} s_1^{-1}(M - \bar{U}_5) \subset s_2^{-1}(\bar{U}_1) \subset s_2^{-1}(M - \bar{U}_5) \subset \bar{U}_2$$

and

$$t_1^{-1} t_2^{-1}(M - \bar{U}_5) \subset t_1^{-1}(\bar{U}_2) \subset t_1^{-1}(M - \bar{U}_5) \subset \bar{U}_1,$$

which contradicts the choice of $\bar{U}_1 \cap \bar{U}_2 = \varnothing$. Hence we must have $\operatorname{Int}(\operatorname{Fix}(f)) \neq \varnothing$ for some $f \neq e$. □

IV. *Main theorem*

THEOREM 4. *Let*

$$\phi : \operatorname{Diff}^p(M) \to \operatorname{Diff}^q(N)$$

be a group isomorphism. Then $p = q$ and there is a C^p-diffeomorphism $w : M \to N$ inducing ϕ. In other words $\phi(h)(n) = whw^{-1}(n)$ for all $h \in \operatorname{Diff}^p(M)$ and $n \in N$.

Proof. The proof will proceed in three steps. First we will show that there are points $m_0 \in M$ and $n_0 \in N$ such that

$$\phi(S^p_{m_0}) = S^q_{n_0}.$$

Then w will be constructed and shown to be a homeomorphism. Finally, using some arguments from the theory of Lie groups, we will show that $p = q$ and w is a C^p diffeomorphism.

Step 1. For $n_0 \in N$ consider the subcollection $\mathscr{C}_{n_0}$ of $\mathbb{B}(M)$ defined by:

$$\mathscr{C}_{n_0} = \{U \in \mathbb{B}(M) | [\operatorname{Diff}^p_0(U), \operatorname{Diff}^p_0(U)] \subset \phi^{-1}(S^q_{n_0}) = F_{n_0}\}.$$

If $\mathscr{C}_{n_0}$ covers M then by theorem 2.2

$$[\operatorname{Diff}^p_0(M), \operatorname{Diff}^p_0(M)] \subset F_{n_0}$$

and so, since theorem 2.5 implies that ϕ maps $[\operatorname{Diff}^p_0(M), \operatorname{Diff}^p_0(M)]$ onto $[\operatorname{Diff}^q_0(N), \operatorname{Diff}^q_0(N)]$, we must have

$$[\operatorname{Diff}^q_0(N), \operatorname{Diff}^q_0(N)] \subset S^q_{n_0}.$$

But $[\mathrm{Diff}_0^q(N), \mathrm{Diff}_0^q(N)]$ is normal in $\mathrm{Diff}^q(N)$ and stabilizers are sent to stabilizers by conjugation; hence it follows that

$$[\mathrm{Diff}_0^q(N), \mathrm{Diff}_0^q(N)] \subset S_n^q \quad \text{for all } n \in N.$$

This is a contradiction since

$$\bigcap_{n \in N} S_n^q = \{e\}$$

and $\mathrm{Diff}_0^q(N)$ is not abelian. Thus $\mathscr{C}_{n_0}$ cannot cover M. Now let $U \in \mathscr{C}_{n_0}$ and set $V = f(U)$ for some $f \in F_{n_0}$. Then

$$V \in \mathbb{B}(M)$$

and

$$[\mathrm{Diff}_0^p(V), \mathrm{Diff}_0^p(V)] = f[\mathrm{Diff}_0^p(U), \mathrm{Diff}_0^p(U)]f^{-1} \subset F_{n_0}.$$

Hence $V \in \mathscr{C}_{n_0}$ and $\mathscr{C}_{n_0}$ is invariant under F_{n_0}. If we now set

$$C_{n_0} = M - \bigcup_{U \in \mathscr{C}_{n_0}} U$$

we see that C_{n_0} is a closed non-empty subset of M invariant under F_{n_0}. If $\mathscr{C}_{n_0} \neq \varnothing$ then $C_{n_0} \neq M$ and so, by lemma 3.2, we must have $F_{n_0} = S_{m_0}^p$ for some $m_0 \in M$ i.e.

$$\phi(S_{m_0}^p) = S_{n_0}^q.$$

A similar argument to the preceding paragraph shows that if the subcollection $\mathscr{D}_{m_0}$ or $\mathbb{B}(N)$ defined by:

$$\mathscr{D}_{m_0} = \{V \in \mathbb{B}(N) \mid [\mathrm{Diff}_0^q(V), \mathrm{Diff}_0^q(V)] \subset \phi(S_{m_0}^p)\}$$

is non-empty for some $m_0 \in M$ there is an $n_0 \in N$ such that

$$\phi(S_{m_0}^p) = S_{n_0}^q.$$

It remains to show, therefore, that $\mathscr{C}_{n_0} \neq \varnothing$ for some $n_0 \in N$ or $\mathscr{D}_{m_0} \neq \varnothing$ for some $m_0 \in M$.

Choose $n_0 \in N$. By lemma 3.3 we can find $g_0 \neq e$ in F_{n_0} such that $\mathrm{Int}(\mathrm{Fix}(g_0)) = A \neq \varnothing$. Let

$$B = \mathrm{Fix}(\phi(g_0))$$

and define subgroups H and K of $\mathrm{Diff}^p(M)$ as follows:

$$H = \phi^{-1}\{h \in \mathrm{Diff}^q(N) \mid h(B) = B\}$$

$$K = \phi^{-1}\{h \in \mathrm{Diff}^q(N) \mid B \subset \mathrm{Fix}(h)\}.$$

Now $B \neq \varnothing$ since $n_0 \in B$ and both H and K are non-trivial since they contain g_0. Also K is clearly normal in H. If

$$g \in \mathrm{Diff}_0^p(A)$$

then

$$g g_0 g^{-1} = g_0.$$

Hence

$$\begin{aligned}\phi(g)(B) &= \phi(g)(\mathrm{Fix}\,(\phi(g_0)))\\ &= \mathrm{Fix}\,(\phi(g)\phi(g_0)\phi(g^{-1}))\\ &= \mathrm{Fix}\,(\phi(g_0))\\ &= B\end{aligned}$$

and so

$$\mathrm{Diff}_0^p\,(A) \subset H.$$

Thus $\mathbb{B}(H) \neq \varnothing$. We now have the following two cases:

Case (i): If there exist $x \in A$ and $k \in K$ such that $k(x) \neq x$ then by theorem 2.4 we can find a non-empty open subset $U \subset A$ such that

$$[\mathrm{Diff}_0^p\,(U), \mathrm{Diff}_0^p\,(U)] \subset K.$$

But $\phi(K) \subset S_{n_0}^q$, since $n_0 \in B$ and so

$$[\mathrm{Diff}_0^p\,(U), \mathrm{Diff}_0^p\,(U)] \subset F_{n_0}$$

so $\mathscr{C}_{n_0} \neq \varnothing$.

Case (ii): If for all $k \in K$ we have $A \subset \mathrm{Fix}\,(k)$ then, since

$$h \in \mathrm{Diff}_0^q\,(N - B)$$

implies that

$$\phi^{-1}(h) \in K,$$

the collection $\mathscr{D}_x$ is non-empty for all $x \in A$.

Thus we have shown that either $\mathscr{C}_{n_0} \neq \varnothing$ or there exists $x \in M$ such that $\mathscr{D}_x \neq \varnothing$ as required.

Step 2. Let m_0, n_0 be the points defined in step 1 above and let $g_1, g_2 \in \mathrm{Diff}^p\,(M)$ be such that $g_1(m_0) = g_2(m_0)$. Then we have

$$g_1^{-1} g_2 S_{m_0}^p g_2^{-1} g_1 = S_{m_0}^p$$

and so

$$\phi(g_1^{-1} g_2) S_{n_0}^q \phi(g_2^{-1} g_1) = S_{n_0}^q;$$

hence

$$\phi(g_1)(n_0) = \phi(g_2)(n_0).$$

We can therefore define $w: M \to N$ as follows: If $m \in M$ choose $g \in \mathrm{Diff}^p\,(M)$ such that $g(m_0) = m$ and set

$$w(m) = \phi(g)(n_0).$$

It is easy to see that w is a bijection between M and N inducing ϕ. Furthermore, since w induces ϕ, we have

$$\mathrm{Fix}\,(\phi(g)) = w(\mathrm{Fix}\,(g)) \quad \text{for all } g \in \mathit{Diff}^p\,(M)$$

and so, since w is a bijection,

$$N - \mathrm{Fix}\,(\phi(g)) = w(M - \mathrm{Fix}\,(g)).$$

But these sets form the bases for the topologies of N and M respectively so w^{-1} is continuous. Similarly w is continuous.

Step 3. First note that if w is a C^k diffeomorphism and $p<q$ then $k<p$ otherwise w would induce an isomorphism between $\mathrm{Diff}^p(M)$ and

$$\mathrm{Diff}^p(N) \neq \mathrm{Diff}^q(N).$$

Similarly if $p>q$ then, considering ϕ^{-1}, we see that $k<q$. Hence if $p\neq q$ we must have $k<\min(p,q)$. We will show, however, that $\min(p,q)\leq k$ and so $p=q$ and w is a C^p diffeomorphism.

For each $m_0 \in M$ a result of de Rham (proposition 1, § 15 of [**8**]) gives the existence of an embedding $h:\mathbb{R}^d \to U \subset M$ with the following properties ($d=\dim(M)$):

(a) $h(0)=m_0$;

(b) We set $a(x,m)=hT_xh^{-1}$ if $m\in U, x\in\mathbb{R}^d$ and $a(x,m)=m$ if $m\notin U$ ($T_x:\mathbb{R}^d\to\mathbb{R}^d$ denotes translation by x). Then a is a C^∞ action of $\mathbb{R}^d$ on M.

If w is the homeomorphism defined in step 2 the induced action $\tilde{a}(x,n)$ of $\mathbb{R}^d$ on N defined by: $\tilde{a}(x,n)=w(a(x,w^{-1}(n)))$ is continuous. If $x_0\in\mathbb{R}^d$ is fixed the map $m\to a(x_0,m)$ is in $\mathrm{Diff}^p(M)$ and so, since w induces an isomorphism between $\mathrm{Diff}^p(M)$ and $\mathrm{Diff}^q(N)$, the map $n\to\tilde{a}(x_0,n)$ is in $\mathrm{Diff}^q(N)$. Therefore $\tilde{a}$ is a continuous action of $\mathbb{R}^d$ by diffeomorphisms. Since $\mathbb{R}^d$ is a Lie group we can apply theorem 3 of § 5.2 of Montgomery and Zippin [**3**] to conclude that a is a C^q action. Set $n_0=w(m_0)$. It follows that the map $x\to a(x,m_0)$ is a C^∞ chart around m_0 and the map $x\to\tilde{a}(x,n_0)$ is a C^q chart around n_0. We have $h(x)=a(x,m_0)$; hence

$$\begin{aligned} h^{-1}w^{-1}(\tilde{a}(x,n_0)) &= h^{-1}w^{-1}w(a(x,m_0)) \\ &= h^{-1}(a(x,m_0)) \\ &= x \end{aligned}$$

so w^{-1} is C^q. Reversing the roles of M and N a similar argument shows that w is C^p. Hence w is a C^k diffeomorphism with $k=\min(p,q)$. □

We conclude by remarking that the following questions remain open:

Question 1. Is theorem 4 true if M and N are allowed to have non-empty boundaries? The problem here lies in the fact that the minimal normal subgroup is now the commutator of the group of diffeomorphisms compactly isotopic to the identity whose support lies in the interior of M.

Question 2. Is theorem 4 true in the analytic case? The methods in this paper fail completely in this case.

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