Obstructions to Approximating Tropical Curves in Surfaces Via Intersection Theory

Dedicated to the memory of Mikael Passare

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Abstract. We provide some new local obstructions to approximating tropical curves in smooth tropical surfaces. These obstructions are based on a relation between tropical and complex intersection theories, which is also established here. We give two applications of the methods developed in this paper. First we classify all locally irreducible approximable 3-valent fan tropical curves in a fan tropical plane. Secondly, we prove that a generic non-singular tropical surface in tropical projective 3-space contains finitely many approximable tropical lines if it is of degree 3, and contains no approximable tropical lines if it is of degree 4 or more.

1 Introduction/Main Results

1.1 Background

Tropical geometry is a recent topic in mathematics that provides powerful tools for the study of classical algebraic varieties. The most striking example is undoubtedly the use of tropical methods in real and complex enumerative geometry initiated by Mikhalkin in [Mik05]. Tropical varieties are piecewise polyhedral objects and satisfy the so-called balancing condition (see [Mik06] or [MS] for a precise definition). One possible way, among others, to relate tropical geometry to classical algebraic geometry is via *amoebas* of complex varieties. Given a family $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ of algebraic subvarieties of the complex torus $(\mathbb{C}^*)^N$, one may consider the corresponding family of amoebas $\mathrm{Log}_t(\mathcal{X}_t)$ in $\mathbb{R}^N$, where Log_t is the map defined by

$$\mathrm{Log}_t: \quad \begin{array}{ccc} (\mathbb{C}^*)^N & \longrightarrow & \mathbb{R}^N \\ (z_i) & \longmapsto & \left(\frac{\log|z_i|}{\log t}\right) \end{array} \quad .$$

When the family $\mathrm{Log}_t(\mathcal{X}_t)$ converges (in the sense of the Hausdorff metric on compact sets of $\mathbb{R}^N$) as t goes to infinity, it is known that the limit set X is a rational polyhedral complex (see [BG84]); moreover, the facets of X come naturally equipped with positive integer weights making X balanced (see [Spea] or [MS]).

In this paper, we call a positively weighted, balanced, rational, polyhedral complex a *tropical variety*. We say a tropical variety is *approximable* when it is the limit of

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amoebas of a family of complex algebraic varieties in the above sense. Not all tropical varieties are approximable. The first example was given by Mikhalkin [Mik05] who constructed a spatial elliptic tropical cubic C that is not tropically planar. By the Riemann–Roch Theorem any classical spatial elliptic cubic is planar, therefore the tropical curve C cannot be approximable. One of the challenging problems in tropical geometry is to understand which tropical varieties are approximable. It follows from the works of Viro, Mikhalkin, and Rullgård (see [Vir01, Mik04, Rul01]; see also [Kap00]) that any tropical hypersurface in $\mathbb{R}^N$ is approximable. In addition, many nice partial results about approximation of tropical curves in $\mathbb{R}^N$ have been proved by different authors (see [Mik05, Mik06, Speb, NS06, Mik, Nis, Tyo12, Kat, BMb, BBM]).

Tropical varieties in $\mathbb{R}^N$ are related to classical subvarieties of toric varieties. When considering non-toric varieties, or when working in tropical models of the torus different from $\mathbb{R}^N$, one is naturally led to the approximation problem for pairs. That is to say, given $X \subset Y$ two tropical varieties in $\mathbb{R}^N$, do there exist two families $\mathcal{X}_t \subset \mathcal{Y}_t \subset (\mathbb{C}^*)^N$ of complex varieties approximating X and Y respectively?

Non-approximable pairs of tropical objects show up even in very simple situations. Some well known pathological examples of such pairs were given by Vigeland [Vig09], who constructed examples of generic non-singular tropical surfaces in $\mathbb{R}^3$ of any degree $d \geq 3$ containing infinitely many tropical lines (called *Vigeland lines* throughout this text). Moreover, the surfaces constructed by Vigeland form an *open* subset of the space of all tropical surfaces of the given degree d, which means that these families of lines survive when perturbing the coefficients of a tropical equation of the surface. Vigeland's construction dramatically contrasts with Segre's Theorem (see [Seg43]) asserting that any non-singular complex surface of degree $d \geq 3$ in $\mathbb{C}P^3$ can contain only finitely many lines.

A very important feature in tropical geometry is the so-called *initial degeneration* or *localization* property (see [MS, Chapter 2]): let X be a tropical variety approximated by a family of amoebas $\text{Log}_t(\mathcal{X}_t)$; then given any point p of X we can also produce from $\mathcal{X}_t$ an approximation of $\text{Star}_p(X)$ by a *constant* family of complex algebraic varieties. Recall that the *star* $\text{Star}_p(X)$ of X at p is the fan composed of all vectors $v \in \mathbb{R}^N$ such that $p + \varepsilon v$ is contained in X for ε a small enough positive real number. In other words, any approximable tropical variety is locally approximable by constant families. This already produces non-trivial obstructions to globally approximating tropical subvarieties of dimension and codimension greater than one in $\mathbb{R}^N$. For example, tropical linear fans are in correspondence with matroids (see [Spe08]), and it is well known that there exist matroids that are not realisable over any field.

It follows from the same argument used in the case of a single tropical variety that a globally approximable pair is locally approximable by constant families. Therefore there are local obstructions to globally approximating pairs. The main motivation for this paper is to provide combinatorial local obstructions in the case of curves in surfaces. Results in this direction were previously obtained by the first author and Mikhalkin (see [BMb] and Theorem 1.10), by Bogart and Katz (see [BK12]), and subsequently by Gathman, Schmitz, and Winstel (see [GSW]).

Our strategy in this paper is to use the relation between tropical and complex intersection theories in order to translate classical results (*e.g.,* adjunction formula) into

combinatorial formulas involving only tropical data. In the case of stable intersections, such a relation has been previously obtained in [Rab12, BLdM12, OR, Kat12]. Our situation is reduced to the induced intersections of [Kat12] thanks to [Sha13, Lemma 2.23], which states that any fan tropical divisor in a matroidal fan can be expressed as the divisor of a tropical rational function.

1.2 Overview of the Paper

We now describe more precisely the main results of this paper. As mentioned above, our main goal is to provide local obstructions to the global approximability of a pair (S, C), where C is a tropical curve contained in a non-singular tropical surface S. By definition, a non-singular tropical variety is locally a tropical linear space, and a tropical curve is locally a fan. This motivates the following problem.

Question 1.1 Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a linear space, and let $C \subset \mathrm{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ be a fan tropical curve. Does there exist a complex algebraic curve $\mathcal{C} \subset \mathcal{P} \subset (\mathbb{C}^*)^N$ such that $\mathrm{Trop}(\mathcal{C}) = C$?

Precise definitions of fan tropical curves and of *tropicalisations* are given in Section 2. For the moment, given an algebraic subvariety $\mathcal{X}$ of $(\mathbb{C}^*)^N$, one can think of $\mathrm{Trop}(\mathcal{X})$ as $\lim_{t \to \infty} \mathrm{Log}_t(\mathcal{X})$.

When the answer to Question 1.1 is positive, we say that the tropical curve C is *coarsely approximable* in $\mathcal{P}$. If in addition the curve $\mathcal{C}$ is irreducible and reduced, we say that C is *finely approximable*.

As it appears in many works by different authors, it is more natural to consider the approximation problem from the point of view of parameterised tropical curves instead of embedded tropical curves. Hence we refine Question 1.1 as follows.

Question 1.2 Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a linear space, and $f \colon C \to \mathrm{Trop}(\mathcal{P})$ be a tropical morphism from an abstract fan tropical curve C. Does there exist a non-singular Riemann surface $\mathcal{C}$ and a proper algebraic map $\mathcal{F} \colon \mathcal{C} \to \mathcal{P}$ such that $\mathrm{Trop}(\mathcal{F}) = f$?

When the answer is positive, we say that the tropical morphism $f \colon C \to \mathrm{Trop}(\mathcal{P})$ is *coarsely approximable* in $\mathcal{P}$. If in addition the morphism $\mathcal{F}$ is irreducible (*i.e.,* does not factor through a holomorphic map of degree at least 2 between Riemann surfaces $\mathcal{F}' \colon \mathcal{C} \to \mathcal{C}'$), we say that f is *finely approximable* in $\mathcal{P}$.

Our results provide combinatorial obstructions to the approximation problems posed in Questions 1.1 and 1.2 when $\mathcal{P}$ is a plane. As mentioned above, they are based on the relation between tropical and complex intersection theories established in Section 3. In particular, in this section we define the tropical intersection product of fan curves in a fan tropical plane.

In Section 4, we combine Theorem 3.8 with the adjunction formula for algebraic curves in surfaces. The following theorem is a weak but easy-to-state version of Theorem 4.1. A plane $\mathcal{P}$ in $(\mathbb{C}^*)^N$ is called *uniform* if its compactification $\overline{\mathcal{P}} \subset \mathbb{C}P^N$ as a projective linear subspace does not meet any $N - k$-coordinate linear space with $k \geq 3$. The geometric genus of a reduced algebraic curve $\mathcal{C}$ (*i.e.,* the genus of its

normalization) is denoted by $g(\mathcal{C})$. Underlying a tropical curve C is a 1-dimensional polyhedral fan equipped with positive integer weights on the edges. We denote by Edge(C) the set of edges of C and by w_e the weight of an edge $e \in$ Edge(C). Each curve C in a uniform plane has a well-defined degree that is described in Definition 3.2.

Theorem 1.3   *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a uniform plane, and let $C \subset \mathrm{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ be a fan tropical curve of degree d. If there exists an irreducible and reduced complex curve $\mathcal{C} \subset \mathcal{P}$ such that $\mathrm{Trop}(\mathcal{C}) = C$, then*

$$C^2 + (N-2)d - \sum_{e_i \subset \mathrm{Edge}(C)} w_{e_i} + 2 \geq 2g(\mathcal{C}).$$

In particular, if the left-hand side is negative, then C is not finely approximable in $\mathcal{P}$.

In Section 5, we combine Theorem 3.8 and intersections of a plane algebraic curve with its Hessian curve. Since the statement is quite technical, we refer to Theorem 5.3 for the precise details. As an application we prove Corollary 5.4, which will be used in Section 7 to prohibit all 4-valent Vigeland lines in a degree $d \geq 4$ non-singular tropical surface.

It is worth stressing that Theorem 4.1 is not contained in Theorem 5.3 and vice versa. This is not surprising, since it is already the case in complex geometry: the adjunction formula prohibits the existence of an irreducible quartic with 4 nodes, while intersection with the Hessian curve does not. On the other hand, intersection with the Hessian curve prohibits the existence of an irreducible quintic with 6 cusps, while the adjunction formula does not. Note also that Theorem 4.1 provides an obstruction to approximate an embedded tropical curve, although Theorem 5.3 provides an obstruction to approximate a tropical morphism.

The last two sections are devoted to applications of the general obstructions proved in Theorems 4.1 and 5.3.

In Section 6, given a non-degenerate plane $\mathcal{P} \subset (\mathbb{C}^*)^N$, we classify all 2 or 3-valent fan tropical curves finely approximable in $\mathcal{P}$ (Theorem 6.9). Here we give two simple instances of this classification.

Theorem 1.4   *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, and let $C \subset \mathrm{Trop}(\mathcal{P})$ be a reduced 2 or 3-valent fan tropical curve.*

(i)   *If $N = 3$, then C is finely approximable in $\mathcal{P}$ if and only if $C^2 = 0$ or $C^2 = -1$.*
(ii)   *If $N \geq 6$ and C is of degree at least 2, then C is not approximable in $\mathcal{P}$.*

In Theorem 6.1, the finely approximable tropical morphisms from point (i) are described. The intermediate cases $N = 4$ and 5 are described in Lemma 6.8. The classification of degree 1 fan tropical curves is given in Lemma 6.7.

Example 1.5   A fan tropical curve from Theorem 1.4(i) inside the standard tropical plane in $\mathbb{R}^3$ is depicted on the left side of Figure 1.
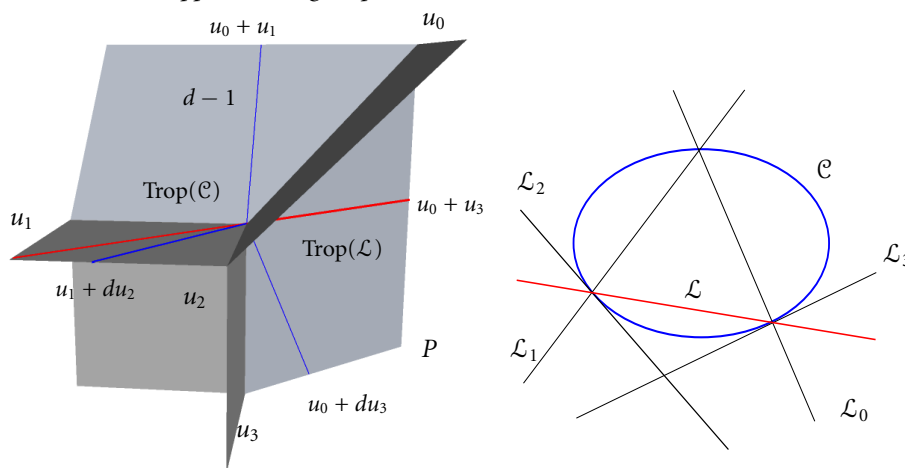
Figure 1: On the left are two tropical curves in the standard plane from Theorem 6.1(i). In red is the 2-valent curve in the case $d = 1$. On the right are the complex line and conic that approximate the tropical curves with their positions drawn relative to the four lines in $\mathbb{C}P^2 \setminus \mathcal{P}$ drawn in $\mathbb{R}P^2$.

At this point, it is interesting to note that all fan tropical curves $C \subset \mathbb{R}^3$ known to us to be finely approximable in a plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ satisfy $C^2 \geq -1$ in $\mathrm{Trop}(\mathcal{P})$. This leads us to the following open question.

Question 1.6 Does there exist a fan tropical curve $C \subset \mathbb{R}^3$ that is finely approximable in a plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ and satisfies $C^2 \leq -2$ in $\mathrm{Trop}(\mathcal{P})$?

Remark 1.7 If the ambient torus has dimension bigger than 3, then such fan tropical curves can exist. Consider an arrangement of 6 lines $\mathcal{L}_1, \ldots, \mathcal{L}_6$ in $\mathbb{C}P^2$ such that the 3 points $\mathcal{L}_1 \cap \mathcal{L}_2$, $\mathcal{L}_3 \cap \mathcal{L}_4$, $\mathcal{L}_5 \cap \mathcal{L}_6$ lie on the same line $\mathcal{L}$; one can embed $\mathbb{C}P^2 \setminus \{\mathcal{L}_1, \ldots, \mathcal{L}_6\}$ as a plane $\mathcal{P}$ in $(\mathbb{C}^*)^5$ (see Section 2), and if we denote $L = \mathrm{Trop}(\mathcal{L})$ we have $L^2 = -2$ in $\mathrm{Trop}(\mathcal{P})$. Note that if $\mathcal{P}' \subset (\mathbb{C}^*)^5$ is the complement in $\mathbb{C}P^2$ of 6 lines chosen generically, the tropical -2-line L is still in $\mathrm{Trop}(\mathcal{P}') = \mathrm{Trop}(\mathcal{P})$, yet it is no longer approximable in $\mathcal{P}'$. See [BK12, Section 7] for another example (though based on the same observation) of approximation problem for pairs $(\mathrm{Trop}(\mathcal{P}), C)$ that cannot be resolved using solely the combinatorial information of $\mathrm{Trop}(\mathcal{P})$.

Finally in Section 7, we apply our methods to the study of tropical lines in tropical surfaces. In [Vig09] and [Vig], Vigeland exhibited generic non-singular tropical surfaces of degree $d \geq 4$ containing tropical lines, and generic non-singular tropical surfaces of degree $d = 3$ containing infinitely many tropical lines. The next theorem shows that when we restrict our attention to the tropical lines that are approximable in the surface, the situation turns out to be analogous to the case of complex algebraic

surfaces. This solves the problem raised in [Vig09] of generic tropical surfaces of degree $d \geq 4$ containing tropical lines, and tropical surfaces of degree $d = 3$ containing infinitely many tropical lines.

Theorem 1.8 *Let S be a generic non-singular tropical surface in $\mathbb{T}P^3$ of degree d. If $d = 3$, then there exist finitely many tropical lines $L \subset S$ such that the pair (S, L) is approximable.*

If $d \geq 4$, then there exist no tropical lines $L \subset S$ such that the pair (S, L) is approximable.

Except in the case of Vigeland lines, all 1-parametric families contained in a generic non-singular tropical surface S in $\mathbb{T}P^3$ are *singular* in S (see Section 7.3). By this we mean that according to the adjunction formula a line in a surface S of degree d should have self-intersection $2 - d$, which is not the case for such tropical lines. This illustrates a strange tropical phenomenon: L and S are both non-singular in $\mathbb{R}^3$, but L is singular as a subvariety of S. In other words, being non-singular does not seem to be an intrinsic property of tropical subvarieties.

Remark 1.9 There are obstructions to approximating tropical curves in surfaces that are not local, as the next two examples show. Using Theorem 1.10 and tropical modifications, it is proved in [BMb] that the tropical curve C depicted in Figure 2 is approximable in the tropical hyperplane P by an algebraic curve of genus 0 with 4 punctures if and only if the vertex of P is the middle of the weight 2 edge of C. Nevertheless, it is easily verified that this tropical curve is locally approximable in P.
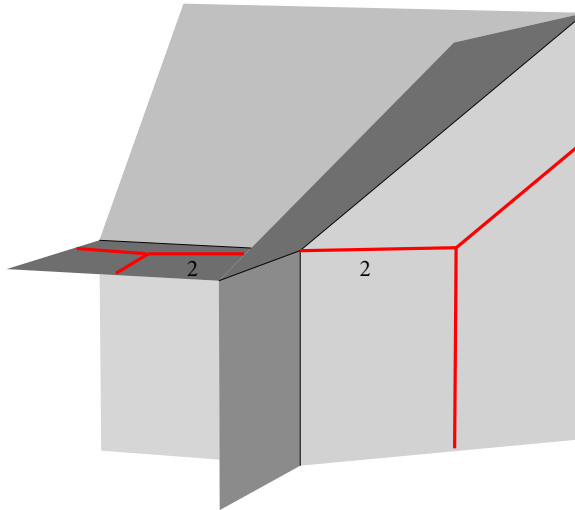

Figure 2: The above tropical curve is approximable in P by a rational curve with 4 punctures if and only if the vertex of the tropical plane is the midpoint of the edge of weight 2.
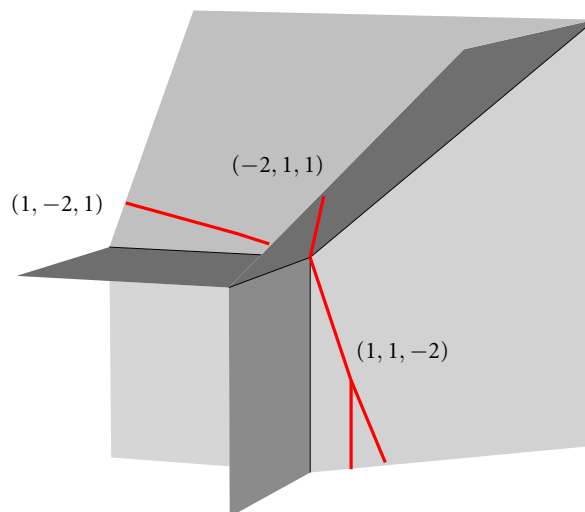
Figure 3: The above tropical curve is not approximable in *P* by a cubic curve since this would contradict the fact that a line passing through two inflection points of a cubic actually intersects this cubic in three inflection points.

Another example of a locally approximable tropical cubic in the standard tropical plane in $\mathbb{R}^3$, which is not globally approximable even by non-rational cubics, is depicted in Figure 3. The directions to infinity are

$$(-2, 1, 1), \quad (1, -2, 1), \quad (0, 0, -1), \quad \text{and} \quad (1, 1, -1)$$

The pair is locally approximable (see Theorem 6.1), but global approximability would imply the existence of an algebraic cubic in the projective plane together with a line passing through exactly two of its inflection points (see Section 1.3).

Notice that the two curves above are both rational. This contrasts with the fact that any tropical rational curve in $\mathbb{R}^N$ is globally approximable (see [Mik06, Mik, BBM, Speb, NS06].

1.3 Related Works

In [BK12], Bogart and Katz used the relation between stable intersections in $\mathbb{R}^n$ and complex intersections to study the realisation problem for pairs in the case of a trivalent fan tropical curve C contained in the standard tropical hyperplane in $\mathbb{R}^3$. By considering the stable intersection of this latter with the classical affine plane spanned by C, they reduce this situation to the approximation problem for a tropical curve in $\mathbb{R}^2$ by a reducible complex algebraic curve. As an application of their method, they proved a particular case of Theorem 1.8, where L is a 3-valent Vigeland line, a particular instance of a 1-parameter family of tropical lines in a non-singular tropical surface.

Gathmann, Schmitz, and Winstel [GSW] proved some general obstructions when $\mathcal{P}$ is a uniform plane in $(\mathbb{C}^*)^3$. These obstructions are of a different nature from the ones presented here, and it seems worthwhile to stress the differences between the two approaches.

The approximation of pairs in the case of a fan tropical curve in a tropical plane is by definition equivalent to the following problem: consider a line arrangement $\mathcal{A} = \{\mathcal{L}_0, \ldots, \mathcal{L}_N\}$ in $\mathbb{C}P^2$; does there exist a complex algebraic curve $\mathcal{C}$ of degree d in $\mathbb{C}P^2$ such that for all lines $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{A}$, the curve $\mathcal{C}$ has a prescribed local Newton polygon (*i.e.,* the polygon $\Gamma^c(\mathcal{C})$ defined at the beginning of Section 5) at $\mathcal{L}_1 \cap \mathcal{L}_2$ in some affine chart of $\mathbb{C}P^2$ for which both $\mathcal{L}_1$ and $\mathcal{L}_2$ are coordinate axes?

One easily reduces this problem to the study of a system (E) of linear equations and inequations in the coefficients of the hypothetic complex algebraic curve. Hence given a specific plane $\mathcal{P} \in (\mathbb{C}^*)^N$ and a specific tropical curve $C \subset \mathrm{Trop}(\mathcal{P})$, the approximation problem of the pair $(\mathrm{Trop}(\mathcal{P}), C)$ is, in principle, solvable. As an illustration, [GSW] classified all realisable tropical curves of low degree in the standard tropical plane in $\mathbb{R}^3$ by solving the corresponding finitely many systems (E). Note that as soon as $N \geq 4$, there are moduli in line arrangements in $\mathbb{C}P^2$ giving rise to the same tropical plane, so the system (E) depends on $\mathcal{A}$ and not only on $\mathrm{Trop}(P)$ (see Remark 1.7).

The obstructions proved in this paper identify tropical curves whose approximation would carry too many singularities: standard tools from algebraic geometry are sometimes sufficient to ensure that the above-mentioned system (E) contains more independent equations than variables, and hence has no solutions. We point out that these obstructions are valid for any N and only depend on $\mathrm{Trop}(\mathcal{P})$.

However, it may happen that the pair $(\mathrm{Trop}(\mathcal{P}), C)$ is not approximable because the set of equations of (E) and the set of inequations are dependent. This second kind of obstruction is more subtle to understand than the previous ones and definitely deserves attention. In particular, studying those non-transversality issues, one cannot only remember $\mathrm{Trop}(P)$ when there are moduli in $\mathcal{A}$. The following classical statement is an example of such a non-transversality: given a non-singular cubic curve $\mathcal{C}$ in $\mathbb{C}P^2$ and a line $\mathcal{L}$ passing through two inflexion points of $\mathcal{C}$, the third intersection point of $\mathcal{C}$ and $\mathcal{L}$ is also an inflexion point of $\mathcal{C}$. The tropical version of this statement is that the spatial tropical fan curve of degree 3 whose directions to infinity are

$$(-2, 1, 1), \quad (1, -2, 1), \quad \text{and} \quad (1, 1, -2)$$

is approximable in the standard tropical plane in $\mathbb{R}^3$, while the spatial tropical fan curve of degree 3 whose directions to infinity are

$$(-2, 1, 1), \quad (1, -2, 1), \quad (0, 0, -1), \quad \text{and} \quad (1, 1, -1)$$

is not. In addition to their classification in low degree, Gathmann, Schmitz, and Winstel [GSW] focused on the study of non-transversality issues in the system (E) when $\mathcal{A}$ is a generic arrangement of $N + 1 = 4$ lines in $\mathbb{C}P^2$. Therefore, among other examples, their methods could generalize the previous observation about cubics to curves of any degree.

In conclusion, this paper and [GSW] study two different kinds of general obstructions to the approximation of pairs, and the results proved in both papers barely

overlap (except for a very particular case of Corollary 3.12). In addition, we also address in this paper the approximation of a tropical morphism to a tropical plane, which is no longer linear. In particular, we do not see how to prove Theorem 5.3 with the methods from [GSW]. Altogether, it is not clear to us that (quoting [GSW, Abstract]) [GSW] "includes and generalizes" the main obstructions presented here, and (quoting an anonymous referee report on a previous version of this paper), [GSW] recovers all of the obstructions presented here as special cases.

Lastly, the techniques presented in this paper should generalise to the study of tropical curves or morphisms in higher dimensional tropical varieties. At present, this problem is widely unexplored. To our knowledge, the following tropical Riemann–Hurwitz condition is the only general obstruction to the approximation of a tropical morphism to a tropical variety. Recall that $g(\mathcal{C})$ denotes the geometric genus of a reduced algebraic curve $\mathcal{C}$.

Theorem 1.10 (Brugallé–Mikahlkin, see [BMb, BBM11, BMa]) *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a linear space of dimension n such that* $\mathrm{Trop}(\mathcal{P})$ *is composed of k faces of dimension n and one face F of dimension $n - 1$. Let $f \colon C \to \mathrm{Trop}(\mathcal{P})$ be a tropical morphism from a fan tropical curve C, let d be the tropical intersection number of $f(C)$ and F in* $\mathrm{Trop}(\mathcal{P})$, *and suppose that C has exactly l edges that are not mapped entirely in F. Then if $f \colon C \to \mathrm{Trop}(\mathcal{P})$ is coarsely approximable by an algebraic map $\mathcal{F} \colon \mathcal{C} \to \mathcal{P}$, one has*

$$g(\mathcal{C}) \geq \frac{d(k-2) - l + 2}{2}.$$

2 Preliminaries

In this section we recall some well known facts in order to make the paper self-contained and to introduce notations used in the following pages.

2.1 Linear Spaces and their Tropicalisations

A *linear space* $\mathcal{P}$ in $(\mathbb{C}^*)^N$ is a subvariety that is given, up to the action of $\mathrm{Aut}((\mathbb{C}^*)^N) = Gl_N(\mathbb{Z})$, by a system of equations of degree one. Equivalently, a subvariety $\mathcal{P} \subset (\mathbb{C}^*)^N$ is a linear space if it is given by a system of equations with support $\Delta(\mathcal{P})$ contained in a primitive simplex Δ. Here, *primitive* indicates that Δ has the same volume as the standard simplex in $\mathbb{R}^N$. Such a simplex Δ induces a toric compactification of $(\mathbb{C}^*)^N$ to $\mathbb{C}P^N$ such that $\mathcal{P}$ compactifies to a projective linear subspace $\overline{\mathcal{P}} = \mathbb{C}P^r$; the simplex Δ will be said to give a degree one compactification of $\mathcal{P} \subset (\mathbb{C}^*)^N$. Note that neither $\Delta(\mathcal{P})$ nor Δ can be uniquely chosen (see Examples 2.2 and 2.3). However once $\Delta(\mathcal{P})$ is fixed, any choice of Δ produces the same pair $(\mathbb{C}P^N, \overline{\mathcal{P}})$ up to toric isomorphism.

A linear space $\mathcal{P} \subset (\mathbb{C}^*)^N$ is said to be *non-degenerate* if it is not contained in any translation of a strict sub-torus of $(\mathbb{C}^*)^N$. If $\mathcal{P} \subset (\mathbb{C}^*)^N$ is a non-degenerate linear space, then for any choice of defining equations we must have $\dim(\Delta(\mathcal{P})) \geq N + 1 - \dim(\mathcal{P})$. A *plane* is a non-degenerate linear space of dimension 2, and in this case $\dim(\Delta(\mathcal{P})) \geq N - 1$.

Definition 2.1   The tropicalisation of a linear space $\mathcal{P}$ in $(\mathbb{C}^*)^N$, denoted by $\text{Trop}(\mathcal{P})$ and called a *tropical linear fan,* is defined as $\lim_{t \to \infty} \text{Log}_t(\mathcal{P})$.

We say that $\mathcal{P}$ approximates $\text{Trop}(\mathcal{P})$. The tropicalisation $\text{Trop}(\mathcal{P})$ is a rational polyhedral fan of pure dimension $\dim \mathcal{P}$. Moreover, it is naturally equipped with a constant weight function equal to 1 on each face of maximal dimension making $\text{Trop}(\mathcal{P})$ into a balanced polyhedral fan (see [MS, AK06]). Note that if $\dim(\Delta(\mathcal{P})) = N - k$ then the fan $\text{Trop}(\mathcal{P})$ contains an affine space of dimension k.

The above mentioned degree one compactification $\overline{\mathcal{P}} = \mathbb{C}P^r$ of a linear space $\mathcal{P} \subset (\mathbb{C}^*)^N$ by way of a simplex Δ defines a hyperplane arrangement $\mathcal{A} = \overline{\mathcal{P}} \setminus \mathcal{P}$ in $\overline{\mathcal{P}} = \mathbb{C}P^r$. Two hyperplane arrangements $\mathcal{A}$ and $\mathcal{A}'$ in $\mathbb{C}P^r$ are isomorphic if there is an automorphism $\phi \in \text{Aut}(\mathbb{C}P^r)$ inducing a bijection $\mathcal{A} \to \mathcal{A}'$. When there is a choice in the simplex $\Delta(\mathcal{P})$, the arrangements defined by the various compactifications may not be isomorphic. However in the case of planes one can easily describe all those different compactifications: if $(\overline{\mathcal{P}}, \mathcal{A})$ and $(\overline{\mathcal{P}}', \mathcal{A}')$ are two non-isomorphic compactifications of $\mathcal{P}$, then $(\overline{\mathcal{P}}', \mathcal{A}')$ is obtained from $(\overline{\mathcal{P}}, \mathcal{A})$ by a standard quadratic transformation σ of $\overline{\mathcal{P}} = \mathbb{C}P^2$ such that any line of $\mathcal{A}$ passes through at least one of the three base points of σ. In particular, the arrangements $\mathcal{A}$ and $\mathcal{A}'$ are combinatorially isomorphic, meaning they determine the same matroid. Also, the pair $(\mathcal{P}, \mathcal{A})$ is unique when $\mathcal{P}$ is a uniform plane.

Example 2.2   Up to a change of coordinates, there exist only two non-degenerate planes in $(\mathbb{C}^*)^3$:

- The plane with equation $z_1 + z_2 + z_3 + 1 = 0$. The corresponding line arrangement is depicted on the left of Figure 4.
- The plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ with equation $z_2 + z_3 + 1 = 0$. The corresponding line arrangement is drawn on the right of Figure 4, and $\text{Trop}(\mathcal{P}) \subset \mathbb{R}^3$ contains the affine line in direction $(1, 0, 0)$. The support of $\mathcal{P}$ is

$$\Delta(\mathcal{P}) = \text{Conv}\{(0,0,0), (0,1,0), (0,0,1)\} \subset \mathbb{R}^3.$$

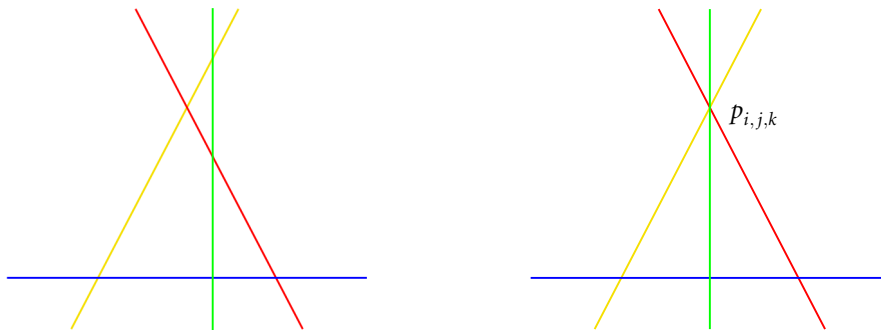*Figure 4*: The line arrangements corresponding to the two types of non-degenerate planes in $(\mathbb{C}^*)^3$

The two different simplicies

$$\Delta = \text{Conv}\{(0,0,0),(0,1,0),(0,0,1),(1,0,0)\},$$
$$\Delta' = \text{Conv}\{(0,0,0),(0,1,0),(0,0,1),(-1,0,0)\}$$

define two degree one compactifications of $\mathcal{P} \subset (\mathbb{C}^*)^3$, and the map $(z_1, z_2, z_3) \mapsto (\frac{1}{z_1}, z_2, z_3)$ induces a toric isomorphism between $(\overline{P}, \mathcal{A})$ and $(\overline{P}', \mathcal{A}')$.

Example 2.3 Consider the plane $\mathcal{P} \subset (\mathbb{C}^*)^4$ defined by the two degree one equations

$$x_3 - x_1 - x_2 = 0 \qquad \text{and} \qquad x_4 - x_1 - x_2 - 1 = 0.$$

The polytope Δ determined by this system of equations is the standard simplex in $\mathbb{R}^4$, and the line arrangement obtained by compactifying by way of Δ is shown on the right of Figure 6.

The plane $\mathcal{P}$ can also be defined by the system of equations:

$$x_3 - x_1 - x_2 = 0 \qquad \text{and} \qquad x_1 x_4 - x_1 x_3 - x_3 - x_2 = 0.$$

The support of this system is a polytope $\Delta' \subset \mathbb{R}^4$ different from the standard simplex but which is also primitive.

Conversely, a hyperplane arrangement $\mathcal{A} = \{\mathcal{H}_0, \ldots, \mathcal{H}_N\}$ in $\mathbb{C}P^r$ satisfying $\bigcap_{i=0}^{N} \mathcal{H}_i = \varnothing$ defines an embedding

$$\phi_{\mathcal{A}}: \quad \mathbb{C}P^r \quad \longrightarrow \quad \mathbb{C}P^N$$
$$z \quad \longmapsto \quad [f_0(z) : \cdots : f_N(z)]$$

where f_i is a linear form defining the hyperplane $\mathcal{H}_i$. Up to a rescaling of each coordinate in $\mathbb{C}P^N$, the map $\phi_{\mathcal{A}}$ depends only on $\mathcal{A}$. The linear space $\mathcal{P} = \phi_{\mathcal{A}}(\mathbb{C}P^r) \cap (\mathbb{C}^*)^N$ is non-degenerate and is the complement $\mathbb{C}P^r \backslash \mathcal{A}$ embedded in $(\mathbb{C}^*)^N$. A hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}P^r$ is *uniform* if any m hyperplanes in $\mathcal{A}$ intersect in a codimension m linear space. So we say a linear space in $(\mathbb{C}^*)^N$ is uniform if its corresponding hyperplane arrangement is. In this paper, all line arrangements $\mathcal{A} = \{\mathcal{L}_0, \ldots, \mathcal{L}_N\}$ are assumed to contain a uniform sub-arrangement of three lines, *i.e.,* $\bigcap_{i=0}^{N} \mathcal{L}_i = \varnothing$.

The tropicalisation $\text{Trop}(\mathcal{P})$ of a linear space $\mathcal{P} \subset (\mathbb{C}^*)^N$ is the Bergman fan of the matroid corresponding to $\mathcal{P}$ and has a very nice combinatorial construction as described in [AK06]. We now recall this construction in the case when $\mathcal{P}$ is a *plane*. Without loss of generality we can assume that $\mathcal{P}$ is non-degenerate.

As previously, let us consider the degree one toric compactification of $(\mathbb{C}^*)^N$ in $\mathbb{C}P^N$ defined by a simplex Δ. Denote by $u_0, \ldots, u_N \in \mathbb{Z}^N$ the outward primitive integer normal vectors to the faces of Δ. There is a natural correspondence between the vectors u_i, the hyperplanes of $\mathbb{C}P^N \setminus (\mathbb{C}^*)^N$, and the lines in the arrangement $\mathcal{A} = \overline{\mathcal{P}} \setminus \mathcal{P}$. In particular we can write $\mathcal{A} = \{\mathcal{L}_0, \ldots, \mathcal{L}_N\}$, where $\mathcal{L}_i$ lies in the coordinate hyperplane corresponding to u_i. A *point* of an arrangement $\mathcal{A}$ is a point in $\overline{\mathcal{P}}$ contained in at least two lines of $\mathcal{A}$. With a point of $\mathcal{A}$ we associate the maximal subset $I \subset \{0, \ldots, N\}$ such that $p = \bigcap_{i \in I} \mathcal{L}_i$, as well as the vector $u_I = \sum_{i \in I} u_i$. Thus we can denote a point of $\mathcal{A}$ by p_I, and denote the set of points of $\mathcal{A}$ by $\mathbf{p}(\mathcal{A})$.

From the construction of the Bergman fan in [AK06], as a set Trop($\mathcal{P}$) is the union of all cones

$$\{\lambda u_i + \mu u_I \mid \lambda, \mu \in \mathbb{R}_{\geq 0}\}$$

where i is contained in I and $I \subset \{0, \ldots, N\}$ corresponds to a point of $\mathcal{A}$. An *edge* of Trop($\mathcal{P}$) is an edge of the coarse polyhedral structure on Trop($\mathcal{P}$) (*i.e.,* it is a ray made of points where Trop($\mathcal{P}$) is not locally homeomorphic to $\mathbb{R}^2$); see [AK06]. The coarse polyhedral structure can be obtained from the Bergman fan structure by removing all rays in the direction $u_i + u_j$ corresponding to points $\mathbf{p}_{i,j}$ and all rays in the direction u_k for all $\mathcal{L}_k$ that contain only two points $\mathbf{p}_I, \mathbf{p}_J$ of the arrangement. In particular the set of edges in the coarse polyhedral structure on Trop($\mathcal{P}$) is contained in $\{u_0, \ldots, u_N, u_I \mid p_I \in \mathbf{p}(\mathcal{A})\}$, this inclusion is usually strict. The tropicalisation Trop($\mathcal{P}$) does not depend on the choice of polytope Δ used to compactify $(\mathbb{C}^*)^N$, therefore neither does this coarse polyhedral structure. However, the fine polyhedral structure on Trop($\mathcal{P}$) does depend on the choice of Δ, when a choice exists.

It follows from this construction that the tropical fan Trop($\mathcal{P}$) depends only on the intersection lattice of the arrangement $\mathcal{A}$. Thus, non-isomorphic line arrangements on $\mathbb{C}P^2$ may have the same tropicalisations. This leads to the phenomena explained in Remark 1.7 and in [BK12, section 7], where the approximation problem of a tropical curve in Trop($\mathcal{P}$) depends on $\mathcal{P}$ and not just on Trop($\mathcal{P}$).

By declaring Log$(0) = -\infty$ we can extend the map Trop continuously to varieties in $\mathbb{C}^N$ and even in $\mathbb{C}P^N$. The images of $\mathbb{C}^N$ and $\mathbb{C}P^N$ under the extended Log map being respectively tropical affine space, $\mathbb{T}^N = [-\infty, \infty)^N$, and tropical projective space $\mathbb{T}P^N$. Here we will use tropical projective space as it appears in [Mik06]. This space is compact and is obtained in accordance with classical geometry. It is equipped with tropical homogeneous coordinates

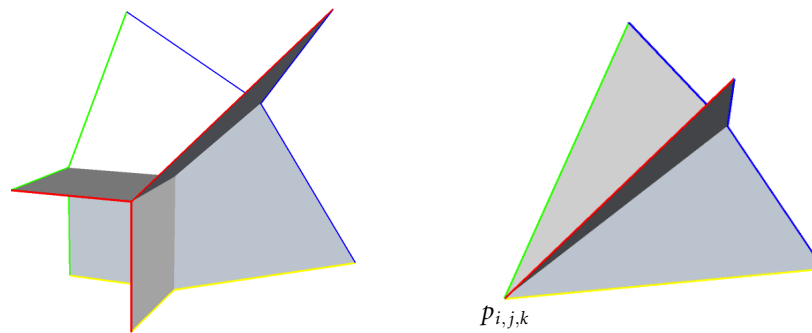$$[x_0 : \cdots : x_N] \sim [x_0 + a : \cdots : x_N + a]$$

Figure 5: The compactifications of two tropical planes in $\mathbb{T}P^3$. On the right there is a corner point $p_{i,j,k}$ corresponding to a triple of lines.
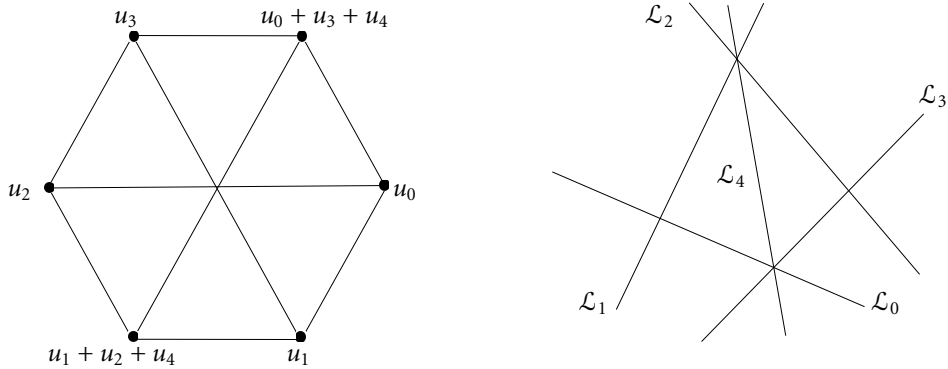
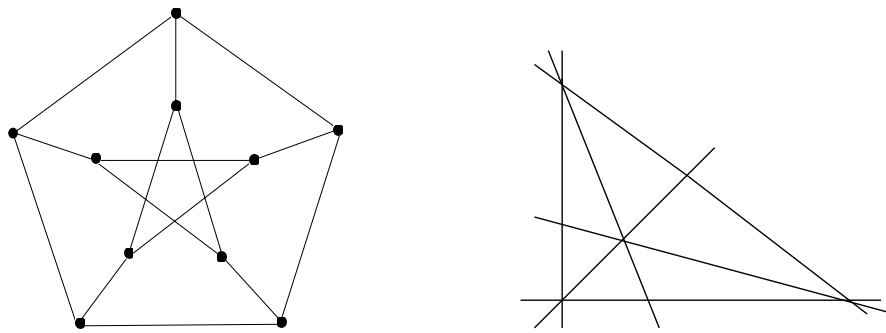*Figure 6*: The link of singularity of a tropical plane in $\mathbb{R}^4$ and the corresponding line arrangement.

Figure 7: The Petersen graph is the link of singularity of $\mathrm{Trop}(\mathcal{M}_{0,5}) \subset \mathbb{R}^5$; it corresponds to the braid arrangement of lines drawn on the right, the directions of the rays are omitted, since they will not be used here. Since the line arrangement is symmetric, we choose not to label it.

where $x_i \neq -\infty$ for at least one $0 \leq i \leq N$, and $a \in \mathbb{R}$. Moreover, it is covered by the affine charts

$$U_i = \{[x_0 : \ldots : x_N] \mid x_i = 0\} \cong \mathbb{T}^N.$$

Given a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$, and a degree one compactification $\overline{\mathcal{P}} = \mathbb{C}P^2$ given by a simplex Δ, the tropicalisation $P = \mathrm{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ can be compactified to $\overline{P} \subset \mathbb{T}P^N$, and by continuity $\mathrm{Trop}(\overline{\mathcal{P}}) = \overline{P}$. Clearly, any line $\mathcal{L}_i \in \overline{\mathcal{P}} \setminus \mathcal{P}$ tropicalises to a boundary component L_i of $\overline{P}$, and $\overline{P} \setminus P = \bigcup L_i$.

Example 2.4 Figure 5 shows the compactifications in $\mathbb{T}P^3$ of the standard tropical plane in $\mathbb{R}^3$, and another plane with only three faces. The first plane corresponds to the complement of a uniform arrangement of four lines in $\mathbb{C}P^2$, whereas the second plane corresponds to the arrangement where three of the four lines belong to the same pencil (see Figure 4).
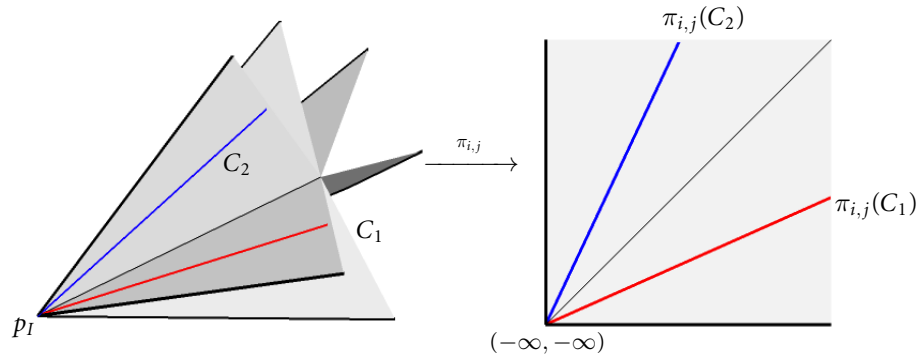
Figure 8: On the left is a neighborhood of a corner point $p_I \subset \overline{P}$ for $|I| = 6$, along with rays of two tropical curves $\overline{C}_1, \overline{C}_2 \subset \overline{P}$ passing through p_I. On the right is the image under the projection $\pi_{i,j} \colon U_l \to \mathbb{T}^2$.

Example 2.5   Figures 6 and 7 are again examples of tropical planes and their corresponding arrangements. In both figures the graphs on the left are the link of singularity of the tropical fan, so that the fans are the cones over the graphs. The rays of the coarse polyhedral structure on the fans correspond to the thick vertices. The line arrangements corresponding to the two planes are drawn on the right in each figure. The arrangement in Figure 7 is known as the Braid arrangement. The complement of this arrangement in $\mathbb{C}P^2$ is $\mathcal{M}_{0,5}$, the moduli space of complex rational curves with 5 marked points. In [AK06], this is also shown to be the moduli space of 5-marked rational tropical curves. Since both arrangements are determined by a generic configuration of four points in $\mathbb{C}P^2$; in each case there is a unique arrangement up to automorphism of $\mathbb{C}P^2$ with the corresponding intersection lattice.

To a point $\mathbf{p}_I$ of $\mathbf{p}(\mathcal{A})$ corresponds a point $p_I = \mathrm{Trop}(\mathbf{p}_I)$ in $\overline{P}$, called a **corner** of $\overline{P}$. Locally at such a point p_I the tropical plane $\overline{P}$ is determined by $|I|$. If $|I| = 2$, a neighborhood of $p_I \in \overline{P}$ is a neighborhood of $(-\infty, -\infty) \in \mathbb{T}^2$. If $|I| = k > 2$, a neighborhood of p_I is the cone over a k-valent vertex, with p_I the vertex of the cone; see the left side of Figure 8. At a point $\mathbf{p}_I \in \overline{\mathcal{P}} \subset \mathbb{C}P^N$ we can choose an affine chart $\mathcal{U}_l \cong \mathbb{C}^N$ for $l \notin I$, then for any pair $i, j \in I$ define the projection $\pi_{i,j} \colon \mathcal{U}_l \to \mathbb{C}^2$ by $(z_0, \ldots, z_{l-1}, z_{l+1}, \ldots z_N) \mapsto (z_i, z_j)$. We use the same notation $\pi_{i,j}$ for the induced projection on $U_l \subset \mathbb{T}P^N$, since the projection commutes with tropicalisation. Then $\pi_{i,j}(\mathbf{p}_I) = (0, 0)$, and of course $\pi_{i,j}(p_I) = (-\infty, -\infty)$. Throughout the article these projections will be applied in order to work locally over $\mathbb{C}^2$ and $\mathbb{T}^2$.

2.2   Tropical Curves and Morphisms

Since we are mainly concerned with the local situation in this paper, for the sake of simplicity we restrict ourselves to *fan* tropical curves. Given a 1-dimensional polyhedral fan $C \subset \mathbb{R}^N$, we denote by $\mathrm{Edge}(C)$ the set of its edges.

Definition 2.6 A fan tropical curve $C \subset \mathbb{R}^N$ is a 1-dimensional rational polyhedral fan such that each edge $e \in \text{Edge}(C)$ is equipped with positive integer weight w_e and satisfying the balancing condition

$$\sum_{e \in \text{Edge}(C)} w_e v_e = 0,$$

where v_e is the primitive integer direction of e pointing away from the vertex of C.

Note that, contrary to some other definitions of tropical curves, the vertex of a fan tropical curve can be 2-valent.

An algebraic curve $\mathcal{C} \subset (\mathbb{C}^*)^N$ produces a fan tropical curve $\text{Trop}(\mathcal{C}) \subset \mathbb{R}^N$, the support of $\text{Trop}(\mathcal{C})$ being $\lim_{t \to \infty} \text{Log}_t(\mathcal{C})$. In order to define the weights on edges of $\text{Trop}(\mathcal{C})$, consider any toric compactification $\mathcal{X}$ of $(\mathbb{C}^*)^N$ such that the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ in $\mathcal{X}$ does not intersect any $\mathcal{D}_i \cap \mathcal{D}_j$, where $\mathcal{D}_i$ and $\mathcal{D}_j$ are any two toric divisors of $\mathcal{X}$. Such a compactification of $(\mathbb{C}^*)^N$ will be called *compatible* with $\mathcal{C}$. For every edge $e \subset \text{Trop}(\mathcal{C})$ there is a corresponding toric divisor $\mathcal{D}_e \subset \mathcal{X}$. The curve $\overline{\mathcal{C}}$ intersects $\mathcal{D}_e$ in a finite number of points, and we define the weight w_e of e as the sum of the multiplicities of these intersection points. The fan $\lim_{t \to \infty} \text{Log}_t(\mathcal{C})$ enhanced with the weights w_e on its edges is a tropical curve $\text{Trop}(\mathcal{C})$.

Definition 2.7 The tropicalisation of a curve $\mathcal{C} \subset (\mathbb{C}^*)^N$ is the tropical curve $\text{Trop}(\mathcal{C})$.

Example 2.8 The tropicalisaton C of the curve $\mathcal{C}$ in $(\mathbb{C}^*)^2$ with equation

$$-Y^2 - X^3 + 4X^2 Y - 5XY^2 + 2Y^3 = 0$$

is depicted in Figure 9. The tropical curve C has 3 rays e_1, e_2, and e_3 with

$$w_{e_1} v_{e_1} = (-2, -3), \quad w_{e_2} v_{e_2} = (-1, 0), \quad \text{and} \quad w_{e_3} v_{e_3} = (3, 3).$$

Note that the presence of the edge e_1 of C is equivalent to the fact that the compactification of the curve $\mathcal{C}$ in $\mathbb{C}P^2$ has a cusp at $[0:0:1]$.

Let us now turn to tropical morphisms. For a graph Γ, let $l(\Gamma)$ denote its set of leaves. Recall that a star graph is a tree with a unique non-leaf vertex.

Definition 2.9 A punctured abstract fan tropical curve C is $\Gamma \backslash l(\Gamma)$ equipped with a complete inner metric, where Γ is a star graph.

A continuous map $f : C \to \mathbb{R}^n$ from a punctured abstract fan tropical curve C is a tropical morphism if

- for any edge e of C with unit tangent vector u_e, the restriction $f_{|e}$ is a smooth map with $df(u_e) = w_{f,e} u_{f,e}$ where $u_{f,e} \in \mathbb{Z}^n$ is a primitive vector, and $w_{f,e}$ is a positive integer;
- it satifsies the balancing condition

$$\sum_{e \in \text{Edge}(C)} w_{f,e} u_{f,e} = 0,$$

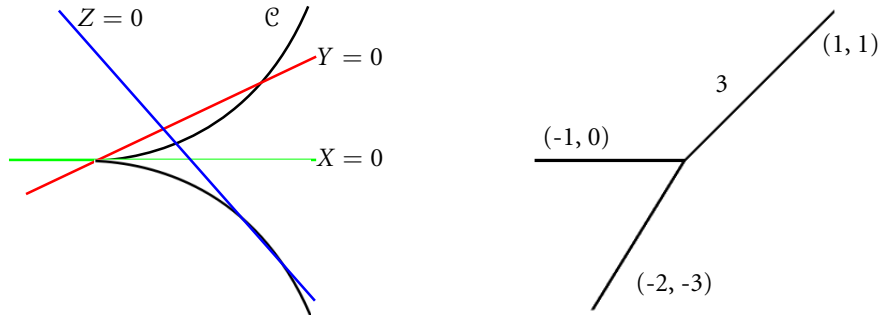where $u_{f,e}$ is chosen so that it points away from the vertex of C.

Figure 9: On the left is the real cubic curve from Example 2.8 with its position with respect to the three coordinate axes. On the right is the corresponding tropical curve in $\mathbb{R}^2$.

Note that there may be several rays of C having the same image in $\mathbb{R}^N$ by f. A tropical morphism $f\colon C \to \mathbb{R}^N$ induces a tropical curve in $\mathbb{R}^N$ in the sense of Definition 2.6: the set $f(C)$ is a 1-dimensional rational fan in $\mathbb{R}^N$, and an edge e of $f(C)$ is equipped with the weight

$$w_e = \sum_{\substack{e' \in \mathrm{Edge}(C) \\ f(e') = e}} w_{f,e'}.$$

Given a proper algebraic map $\mathcal{F}\colon \mathcal{C} \to (\mathbb{C}^*)^N$ from a punctured Riemann surface $\mathcal{C}$, we construct a tropical morphism $f\colon C \to \mathbb{R}^N$ as follows. Consider the punctured abstract fan tropical curve C such that the edges e of C are in one to one correspondence with punctures p_e of $\mathcal{C}$, and set $f(v) = 0$, where v is the vertex of C; for each edge e of C, consider a small punctured disc $D_e \subset \mathcal{C}$ around the corresponding puncture p_e of $\mathcal{C}$; the set $\lim_{t \to \infty} \mathrm{Log}_t(\mathcal{F}(D_e))$ is a half line in $\mathbb{R}^N$ with primitive integer direction v_e, and we can define its weight $w_{f,e}$ as in the case of the tropicalisation of an algebraic curve in $(\mathbb{C}^*)^N$ (in this case $\mathcal{D}_e \cap \overline{\mathcal{F}(D_e)}$ is a single point and $w_{f,e}$ is the intersection multiplicity of the toric divisor $\mathcal{D}_e$ and $\overline{\mathcal{F}(D_e)}$ at that point); we define f on e by $df(u_e) = w_{f,e} v_e$, where u_e is a unit tangent vector on e pointing away from the vertex v.

Definition 2.10 The tropical morphism $f\colon C \to \mathbb{R}^N$ is the tropicalisation of $\mathcal{F}\colon \mathcal{C} \to (\mathbb{C}^*)^N$, and is denoted by $\mathrm{Trop}(\mathcal{F})$.

Note that the definitions of tropicalisations of morphisms and curves are consistent, since we have

$$\mathrm{Trop}(\mathcal{F})(C) = \mathrm{Trop}(\mathcal{F}(\mathcal{C})).$$
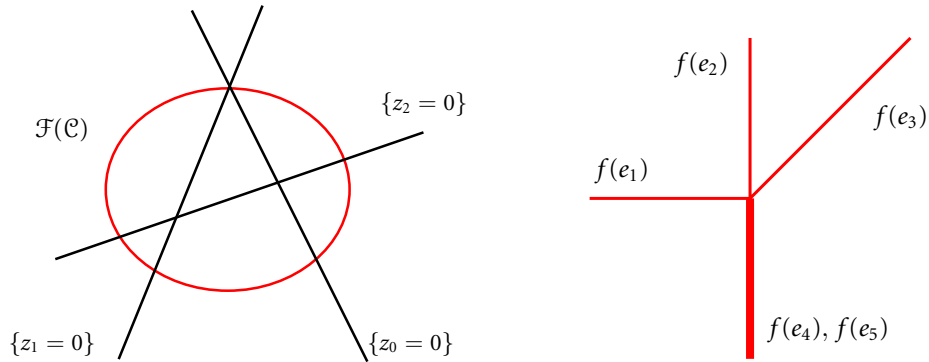
Figure 10: (a) The image of the morphism $\mathcal{F}(\mathcal{C})$ from Example 2.12 drawn with respect to the three lines in $\overline{\mathcal{P}} \backslash \mathcal{P}$.   (b) The tropicalisation of the image $\mathcal{F}(\mathcal{C}) \subset \mathbb{R}^2$.

Example 2.11   The map

$$\mathcal{F}: \quad \mathcal{C} = \mathbb{C}^* \backslash \{-1, 1\} \quad \longrightarrow \quad (\mathbb{C}^*)^2$$
$$z \quad \longmapsto \quad (\tfrac{z^2(z+1)}{z-1}, \tfrac{z^3}{z-1})$$

tropicalises to the tropical morphism $f: C \to \mathbb{R}^2$, where C has 4 edges e_1, e_2, e_3, and e_4 with

$$w_{f,e_1} v_{f,e_1} = (-2, -3), \qquad\qquad w_{f,e_2} v_{f,e_2} = (-1, 0),$$
$$w_{f,e_3} v_{f,e_3} = (1, 1), \qquad\qquad w_{f,e_4} v_{f,e_4} = (2, 2).$$

The image $\mathcal{F}(\mathcal{C})$ is the algebraic curve in $(\mathbb{C}^*)^2$ of Example 2.8 (see Figure 9). The weights of e_3 and e_4 come from the factorization $-X^3 + 4X^2Y - 5XY^2 + 2Y^3 = -(X - 2Y)(X - Y)^2$.

Example 2.12   Consider the map

$$\mathcal{F}: \quad \mathcal{C} = \mathbb{C}^* \backslash \{1, 2, 3\} \quad \longrightarrow \quad (\mathbb{C}^*)^2$$
$$z \quad \longmapsto \quad (-\tfrac{z}{3(z-1)}, -\tfrac{(z-2)(z-3)}{6(z-1)}) \quad .$$

Figure 10 shows the curve $\mathcal{C}$ with respect to the 3 lines in $\mathbb{C}P^2 \backslash (\mathbb{C}^*)^2$. This morphism tropicalises to $f: C \to \mathbb{R}^2$, where C is a tropical curve with 5 edges e_1, e_2, e_3, e_4, and e_5 where,

$$w_{f,e_1} v_{f,e_1} = (-1, 0), \quad w_{f,e_2} v_{f,e_2} = (0, 1), \quad w_{f,e_3} v_{f,e_3} = (1, 1), \quad \text{and}$$
$$w_{f,e_4} v_{f,e_4} = w_{f,e_5} v_{f,e_5} = (0, -1).$$

Now consider another embedding of $\mathcal{C}$ into a torus of higher dimension given by

$$\mathcal{F}': \quad \mathcal{C} = \mathbb{C}^* \backslash \{1, 2, 3\} \quad \longrightarrow \quad (\mathbb{C}^*)^3$$
$$z \quad \longmapsto \quad (-\tfrac{z}{3(z-1)}, -\tfrac{(z-2)(z-3)}{6(z-1)}, -\tfrac{z}{6}) \quad .$$

The curve $\mathcal{F}'(\mathcal{C})$ is contained in the plane $\mathcal{P}$ given by $X + Y + Z + 1 = 0$. In Figure 11 the curve $\mathcal{F}'(\mathcal{C})$ is drawn in $\overline{\mathcal{P}}$ with respect to the 4 lines in $\overline{\mathcal{P}} \backslash \mathcal{P}$.

The map $\mathcal{F}'$ tropicalises to the tropical morphism $f'\colon C \to \mathbb{R}^3$, where C has 5 edges $e_1, e_2, e_3, e_4,$ and e_5 with (see Figure 11)

$$w_{f,e_1} v_{f,e_1} = (-1, 0, -1), \quad w_{f,e_2} v_{f,e_2} = (0, 1, 1), \quad w_{f,e_3} v_{f,e_3,} = (1, 1, 0), \quad \text{and}$$

$$w_{f,e_4} v_{f,e_4} = w_{f,e_5} v_{f,e_5} = (0, -1, 0).$$

3 Intersections of Tropical Curves in Planes

Here we define the intersection of two fan tropical curves C_1, C_2 in a fan tropical plane $\mathrm{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ by defining first intersections at the corner points p_I of a compactification $\overline{\mathrm{Trop}(\mathcal{P})} \subset \mathbb{T}P^N$ defined in Section 2.1. This definition is equivalent to the ones given in [AR10] and [Sha13] by [FR13, Theorem 8.10], but has the advantage of rendering the necessary lemmas more transparent. For a primitive simplex Δ giving a degree one compactification of $\mathcal{P} \subset (\mathbb{C}^*)^N$, let $u_0, \ldots, u_N$ denote the outward primitive integer normal vectors to the faces of Δ.

Definition 3.1 Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and Δ a primitive N-simplex giving a degree one compactification of $\mathcal{P}$. Given two fan tropical curves $C_1, C_2 \subset \mathrm{Trop}(\mathcal{P})$, let $\overline{C}_i$ denote their compactifications in $\overline{\mathrm{Trop}(\mathcal{P})} \subset \mathbb{T}P^N$. Let $p_I \in \overline{\mathrm{Trop}(\mathcal{P})}$ be a corner point and suppose that $\overline{C}_1$ and $\overline{C}_2$ both pass through p_I.

In the case when both curves have each exactly one ray passing through p_I, we define the intersection multiplicity of $\overline{C}_1$ and $\overline{C}_2$ at the corner p_I as follows:

(a) If $I = \{i, j\}$, choose an affine chart U_l for $l \notin I$ and let $\pi_{i,j}\colon U_l \to \mathbb{T}^2$ be the projection from Section 2.1. Suppose the ray of $\pi_{i,j}(\overline{C}_1 \cap U_l) \subset \mathbb{T}^2$ has weight w_1 and primitive integer direction (p_1, q_1), and similarly the ray of $\pi_{i,j}(\overline{C}_2 \cap U_l) \subset \mathbb{T}^2$ has weight w_2 and primitive integer direction (p_2, q_2). Then

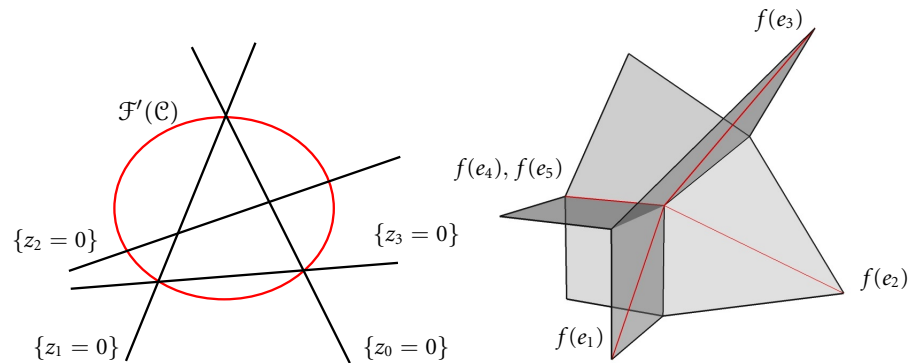$$(\overline{C}_1.\overline{C}_2)_{p_I} = w_1 w_2 \min\{p_1 q_2, q_1 p_2\}.$$

Figure 11: On the left is the conic $\mathcal{F}'(\mathcal{C})$ from Example 2.12 with its position with respect to the 4 lines in $\overline{\mathcal{P}} \backslash \mathcal{P}$. On the right the tropicalisation drawn in the compactification $\overline{P} \subset \mathbb{T}P^3$ of the standard tropical plane.

(b) If $|I| > 2$, choose an affine chart, $U_l \ni p_I$ for $l \notin I$, and a projection $\pi_{i,j}\colon U_l \to \mathbb{T}^2$, where $i, j \in I$ such that the rays of $\overline{C}_1$ and $\overline{C}_2$ are contained in the union of the closed faces generated by u_i, u_j; see Figure 8. Then

$$(\overline{C}_1.\overline{C}_2)_{p_I} = (\pi_{i,j}(\overline{C}_1 \cap U_l).\pi_{i,j}(\overline{C}_2 \cap U_l))_{(-\infty,-\infty)}.$$

We extend this intersection multiplicity by distributivity in the case when $\overline{C}_1$ and $\overline{C}_2$ have several rays passing through p_I.

In part (a) of the above definition, if the two rays of C_1 and C_2 are contained in the same open face generated by u_i and u_I, then we are free to choose j as we wish. In this case even the result of the projection $\pi_{i,j}$ does not depend on the choice of j.

Next we define the degree of a tropical curve $C \subset \mathrm{Trop}(\mathcal{P}) \subset \mathbb{R}^N$. Suppose that a vector $v \in \mathbb{Z}^N$ is contained in $\mathrm{Trop}(\mathcal{P})$. By the construction of $\mathrm{Trop}(\mathcal{P})$ described in Section 2.1, v is contained in a cone generated by u_i, u_I for some $I \in \mathbf{p}(\mathcal{A})$ and $i \in I$. Therefore, there is a unique expression, $v = \rho_i(v)u_i + \rho_I(v)u_I$, where $\rho_i(v), \rho_I(v)$ are non-negative integers. Set $r_i(v) = \rho_i(v) + \rho_I(v)$ if the direction v is contained in a cone generated by u_i and u_I for some $I \ni i$, and $r_i(v) = 0$ otherwise. Given an edge $e \in \mathrm{Edge}(C)$, we denote by v_e the primitive integer vector of e pointing outward from the vertex of C.

Definition 3.2 Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and Δ a primitive N-simplex giving a degree one compactification of $\mathcal{P}$. Let $C \subset \mathrm{Trop}(\mathcal{P})$ be a fan tropical curve and $i \in \{0, \ldots, N\}$. We define the degree of C with respect to Δ as

$$\deg_\Delta(C) = \sum_{e \in \mathrm{Edge}(C)} w_e r_i(v_e).$$

It follows from the balancing condition that the above definition is independent of the choice of u_i. In fact as we will see in the proof of Lemma 3.5, the degree of a curve $C \subset \mathbb{R}^N$ with respect to Δ is the multiplicity of the tropical stable intersection in $\mathbb{R}^N$ of the curve C and the tropical variety dual to Δ. The next examples demonstrate the dependence of $\deg_\Delta(C)$ on the choice of Δ when such a choice exists.

Example 3.3 Recall the plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ from Example 2.2, which admits two compactifications to $\mathbb{C}P^2$ given by polytopes Δ, Δ'. Consider the 4-valent fan tropical curve $C \subset \mathrm{Trop}(\mathcal{P})$ with edges of weight 1 in directions:

$$(1, 1, 1), \quad (-1, 0, 0), \quad (0, -1, 0) \quad \text{and} \quad (0, 0, -1).$$

Calculating the degree using Definition 3.2 we have $\deg_\Delta(C) = 1$ and $\deg_{\Delta'}(C) = 2$.

Example 3.4 Recall the plane $\mathcal{P} \subset (\mathbb{C}^*)^4$ from Example 2.3, which admits two compactifications to $\mathbb{C}P^2$ given by polytopes Δ, Δ'. Consider the curve $\mathcal{C} \subset \mathcal{P}$ given by the additional equation $x_1x_3 - x_2 = 0$. The left-hand side of Figure 12 shows the compactification of this curve $\overline{\mathcal{C}} \subset \mathbb{C}P^2$ given by the standard simplex Δ. The curve $\overline{\mathcal{C}}$ is a conic. The right-hand side of the same figure shows the compactification $\overline{\mathcal{C}}' \subset \mathbb{C}P^2$ given by Δ'. In this compactification $\overline{\mathcal{C}}'$ is a line. It follows from the next lemma that $\deg_\Delta(C) = 2$ and $\deg_{\Delta'}(C) = 1$, where C is the tropicalisation of $\mathcal{C}$.
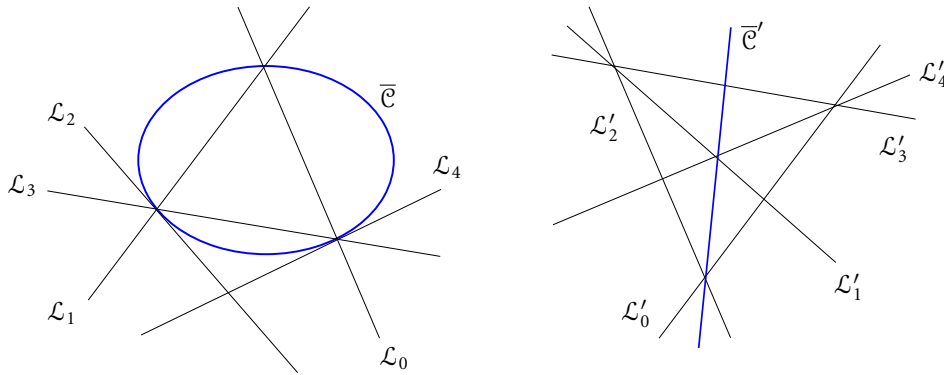
Figure 12: Two compactifications of the plane $\mathcal{P}$ from Example 2.3 along with the compactification of a curve $\mathcal{C} \subset \mathcal{P}$ from Example 3.4.

Lemma 3.5   *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, let $\mathcal{C} \subset (\mathbb{C}^*)^N$ be a complex algebraic curve, and $\Delta \subset \mathbb{R}^N$ a primitive N-simplex giving a degree one compactification of $\mathcal{P}$. Then $\deg(\overline{\mathcal{C}}) = \deg_\Delta(\mathrm{Trop}(\mathcal{C}))$, where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$ in the toric compactification of $(\mathbb{C}^*)^N$ to $\mathbb{C}P^N$ given by Δ.*

Proof   Let H be the tropicalisation of a hypersurface $\mathcal{H}$ of $(\mathbb{C}^*)^N$ with Newton polygon Δ. Then let $\overline{\mathcal{H}} \subset \mathbb{C}P^N$ be the closure of $\mathcal{H}$ in the toric compactification given by Δ. Then $\overline{\mathcal{H}}$ is a hyperplane, thus of degree one. Given $i \in \{0, \dots, N\}$, there is a translation $H' = H + v_i$ where $v_i \in \mathbb{R}^N$ such that H' intersects the tropical curve C in a finite number of points and only in the face of H' orthogonal to u_i. Then each such edge e intersecting H' does so with tropical multiplicity $w_e r_i(v_e)$, in the notation of Definition 3.2. Therefore, by Definition 3.2 $\deg_\Delta(C) = \deg(H' . C) = \deg(H . C)$, where $H' . C$ is the stable intersection from [RGST05].

The translation H' is approximated by the family

$$\mathcal{H}_t = \{(t^{v_1} z_1, \dots t^{v_N} z_N \mid (z_1, \dots, z_N) \in \mathcal{H}\}$$

in the sense that $\lim_{t \to \infty} \mathrm{Log}_t(\mathcal{H}_t) = H'$. The intersection of H' and C is proper, so by [Kat09, Theorem 8.8], for t large enough, $\deg(H' . C)$ is the intersection number of $\mathcal{H}_t$ and $\mathcal{C}$. Compactifying $\mathbb{R}^N$ to $\mathbb{T}P^N$ by way of Δ, the closures $\overline{H'}$ and $\overline{C}$ do not intersect at the boundary for a generic translation. Therefore $\deg_\Delta(C) = \deg(H' . C) = \deg(\overline{\mathcal{H}} . \overline{\mathcal{C}}) = \deg(\overline{\mathcal{C}})$, and the lemma is proved. ∎

We define next the tropical intersection multiplicity of two tropical curves C_1 and C_2 contained in a fan tropical plane. By [FR13, Theorem 8.10] this definition is equivalent to the one from [AR10], [Sha13], and so it is independent of the choice of Δ.

Definition 3.6   Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and Δ a primitive N-simplex giving a degree one compactification of $\mathcal{P}$. Given two fan tropical curves

C_1 and C_2 in $\mathrm{Trop}(\mathcal{P})$, we define their tropical intersection number in $\mathrm{Trop}(\mathcal{P})$ as

$$C_1 \cdot C_2 = \deg_\Delta(C_1) \cdot \deg_\Delta(C_1) - \sum_{\mathbf{p}_l \in \mathbf{p}(\mathcal{A})} (\overline{C}_1 \cdot \overline{C}_2)_{p_l}.$$

In order to interpret the above tropical intersection number in the case of two approximable tropical curves, we must first describe an appropriate compactification of the open space $\mathcal{P}$, in general distinct from $\overline{\mathcal{P}} = \mathbb{C}P^2$. Recall that in Section 2.2 a compactification $\mathcal{X}$ of $(\mathbb{C}^*)^N$ is called *compatible* with a complex curve $\mathcal{C}$ if $\mathcal{X}$ is a toric compactification of $(\mathbb{C}^*)^N$ such that for any two irreducible boundary divisors $\mathcal{D}_i$ and $\mathcal{D}_j$ of $\mathcal{X}$, the intersection $\mathcal{D}_i \cap \mathcal{D}_j \cap \mathcal{C}$ is empty. For two complex algebraic curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$, we call a compactification $\mathcal{X}$ of $(\mathbb{C}^*)^N$ *compatible with $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{P}$* if it is compatible with both curves and the compactification $\widetilde{\mathcal{P}} \subset \mathcal{X}$ of $\mathcal{P}$ is a non-singular surface. We refer to [Tev07] for more details about tropical compactification of subvarieties of $(\mathbb{C}^*)^N$.

Example 3.7 Here we describe a compactification of $(\mathbb{C}^*)^N$ compatible with two given complex curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ that we will use in the rest of the article. First let $\Sigma \subset \mathbb{R}^N$ be the complete unimodular fan dual to a primitive N-simplex Δ giving a degree one compactification of $\mathcal{P}$. Let $\widetilde{\Sigma}$ be a unimodular completion of $\Sigma \cup \mathrm{Trop}(\mathcal{C}_1) \cup \mathrm{Trop}(\mathcal{C}_2) \subset \mathbb{R}^N$, and $\widetilde{\mathcal{X}}(\Delta)$ denote the corresponding compactification of $(\mathbb{C}^*)^N$. Then $\widetilde{\mathcal{X}}(\Delta)$ is compatible with $\mathcal{C}_1, \mathcal{C}_2,$ and $\mathcal{P}$. This compactification $\widetilde{\mathcal{P}}$ of $\mathcal{P}$ is the minimal compactification of $(\mathbb{C}^*)^N$ compatible with $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{P}$. Moreover, it is obtained from $\mathbb{C}P^2$ by blowing up points in $\mathbf{p}(\mathcal{A})$ and points above them which are intersection of boundary divisors (see Figure 13).

The following theorem states the correspondence between the complex and tropical intersection numbers for curves.

Theorem 3.8 *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two algebraic curves in $\mathcal{P}$, and let $C_i = \mathrm{Trop}(\mathcal{C}_i)$. Then $\widetilde{\mathcal{C}}_1 \cdot \widetilde{\mathcal{C}}_2 = C_1 \cdot C_2$, where $\widetilde{\mathcal{C}}_i$ is the compactification of $\mathcal{C}_i$ in the compatible toric compactification $\widetilde{\mathcal{P}} \subset \widetilde{\mathcal{X}}(\Delta)$ from Example 3.7.*

The proof of Theorem 3.8, uses the induced intersection theory of [Kat12] along with [Sha13, Lemma 2.23]. Recall that a tropical rational function on $\mathbb{R}^n$ is a tropical quotient of tropical polynomials, in other words it is a difference of convex piecewise integer affine functions. The divisor of a tropical rational function restricted to a tropical fan plane produces a tropical 1-cycle. For an introduction to the theory of tropical divisors; see [AR10, Mik06]. Since here we are only considering fan curves, we can restrict consideration to tropical rational functions that are simply piecewise linear.

Lemma 3.9 ([Sha13, Lemma 2.23]) *Suppose $P \subset \mathbb{R}^n$ is a fan tropical plane and $C \subset P$ a fan tropical curve, then there exists a tropical rational function $f \colon \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{div}_P(f) = C$.*
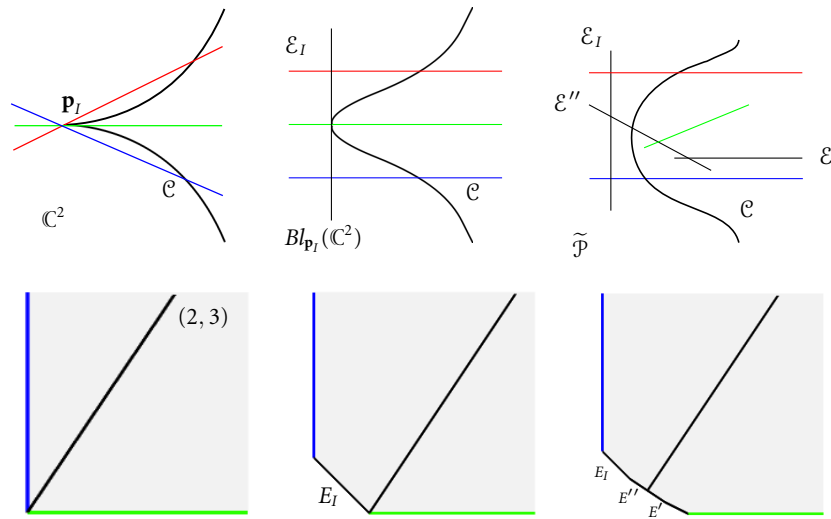
Figure 13: An illustration of the compactification $\widetilde{\mathcal{P}}$ at a single corner point $\mathbf{p}_{i,j,k}$, along with the tropicalisation of a projection $\pi_{i,j}$, and the toric blow ups.

Proof of Theorem 3.8   Let $c_i \in A^1(\widetilde{X}(\Delta))$ denote the cocyle associated with $\widetilde{\mathcal{C}}_i$. Then a tropical rational function f_i such that $\mathrm{div}_P(f_i) = C_i$ from Lemma 3.9 is a lift of c_i in $\widetilde{\mathcal{P}}$ in the sense of [Kat12, Definition 6.2]. The equality of the intersection numbers then follows from [Kat12, Theorem 6.3].   ∎

Note that the above theorem does not depend on the initial choice of simplex Δ used to construct the compatible compactification in Example 3.7. Following Theorem 3.8 we present some examples and immediate corollaries.

Example 3.10   Let P be the tropicalisation of a uniform hyperplane $\mathcal{P} \subset (\mathbb{C}^*)^3$ and let $L \subset P$ be the affine line in the direction $(1, 1, 0)$. It is the red curve depicted in Figure 1. Since $\mathcal{P}$ is uniform there is a unique simplex Δ yielding a degree one compactification of $\mathcal{P}$. Using Definition 3.6 we compute $L \cdot L = -1$. This tropical line is approximable by a line $\mathcal{L} \subset \mathcal{P}$. Viewing $\mathcal{P}$ as a complement of four generic lines $\mathcal{L}_1, \ldots, \mathcal{L}_4$ in $\mathbb{C}P^2$, then for some labeling of these lines, $\mathcal{L}$ is the line passing through the two points $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_3 \cap \mathcal{L}_4$. Again see Figure 1 for a real drawing of the five lines. The blow up of $\mathbb{C}P^2$ at the two points $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_3 \cap \mathcal{L}_4$ gives the desired compactification $\widetilde{\mathcal{P}}$. The proper transform of $\mathcal{L}$ in $\widetilde{\mathcal{P}}$ is indeed a curve of self intersection -1.

Example 3.11   We recall the example presented in [Sha13, Section 4]. Let us consider P and L as in Example 3.10, and let C_d be the trivalent tropical curve of degree
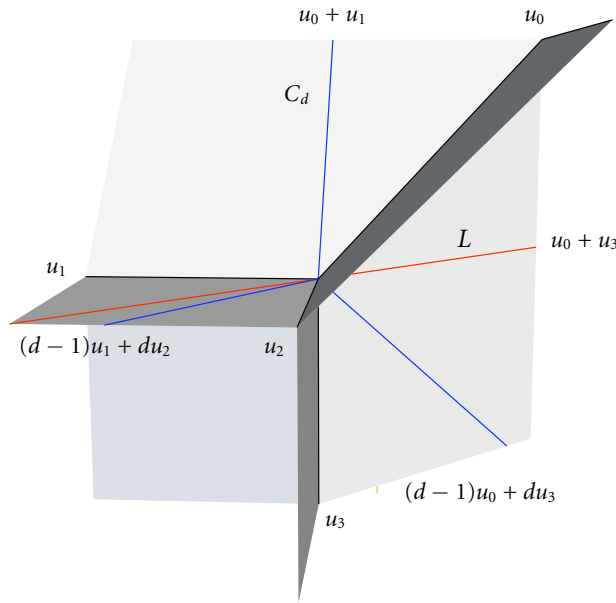
Figure 14: The curve C_d and the line L from Example 3.11.

d centered at the vertex of P with weight one rays in the directions

$$u_0 + u_1, \quad (d-1)u_0 + du_3, \quad \text{and} \quad (d-1)u_1 + du_2;$$

see Figure 14. These curves intersect at two corner points of $\overline{P} \subset \mathbb{T}P^3$. Locally at these two corners the curves appear as rays passing through the corner of $\mathbb{T}^2$, L in the direction $(-1, -1)$ and C_d in the direction $(-d, 1-d)$. Using Definition 3.6 we have

$$C_d \cdot L = d \cdot 1 - 2(d-1) = -d + 2.$$

Corollary 3.12   *Let $\mathcal{P}$ be a non-degenerate plane and $C \subset \mathrm{Trop}(\mathcal{P})$ be a finely approximable fan tropical curve. If $D \subset \mathrm{Trop}(\mathcal{P})$ is also a fan tropical curve such that $D \neq C$ and $C \cdot D < 0$ then D is not finely approximable.*

It follows from this corollary and Example 3.10 that the curve C_d from Example 3.11 is not approximable for $d \geq 3$; this was shown in [Sha13].

Corollary 3.13   *Let $\mathcal{P}$ be a non-degenerate plane and $C \subset \mathrm{Trop}(\mathcal{P})$ be a finely approximable fan tropical curve. If $C \cdot C < 0$, then C is finely approximated by a unique complex curve $\mathcal{C} \subset \mathcal{P}$.*

4   Obstructions coming from the Adjunction Formula

The adjunction formula for a non-singular curve $\mathcal{C}$ in a non-singular compact complex surface $\mathcal{X}$ reduces to (see [Sha94])

$$g(\mathcal{C}) = \frac{K_{\mathcal{X}} \cdot \mathcal{C} + \mathcal{C}^2 + 2}{2},$$

where $g(\mathcal{C})$ is the genus of $\mathcal{C}$, and $K_{\mathcal{X}}$ is the canonical class of $\mathcal{X}$. If $\mathcal{C}$ is singular but reduced, the right-hand side of the above formula defines the arithmetic genus of the curve and we denote it by $g_a(\mathcal{C})$. As before, denote by $g(\mathcal{C})$ the geometric genus of the curve $\mathcal{C}$, *i.e.,* the genus of its normalisation. If $\mathcal{C}$ is an irreducible curve, we have $0 \le g(\mathcal{C}) \le g_a(\mathcal{C})$. We interpret this bound on the level of the tropical curve in order to prove Theorem 4.1, which appeared in a simplified version in Theorem 1.3.

Beforehand, we need to introduce some more notation. Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and Δ be a primitive N-simplex giving a degree one compactification of $\mathcal{P}$. Recall that if the arrangement $\mathcal{A}$ corresponding to $\mathcal{P} \subset (\mathbb{C}^*)^N$ contains a point $\mathbf{p}_I$, then the fan tropical plane $P = \text{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ contains a ray in the corresponding direction $u_I = \sum_{i \in I} u_i$, where $u_0, \dots, u_N$ are the outgoing primitive normal vectors to the simplex Δ. Given a fan tropical curve $C \subset P$, let w_I denote the weight of the edge of C in the direction u_I, with the convention that $w_I = 0$ if C does not contain a ray in this direction. Note that this definition depends on the direction of u_I; therefore, it depends on the choice of the simplex Δ giving a degree one compactification when a choice exists.

Theorem 4.1   *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and $C \subset \text{Trop}(\mathcal{P})$ a fan tropical curve. If C is finely approximable by a complex curve $\mathcal{C} \subset \mathcal{P}$, then for a primitive N-simplex $\Delta \subset \mathbb{R}^N$ giving a degree one compactification of $\mathcal{P}$ we have*

$$2g(\mathcal{C}) \le C^2 + (N-2)\deg_{\Delta}(C) - \sum_{e_i \in \text{Edge}(C)} w_{e_i} - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I| - 2)w_I + 2,$$

with equality if and only if $\mathcal{C}$ is non-singular.

Recall that for a uniform plane, there is a unique degree one compactification. Therefore the simplex Δ is unique and the degree d used in the statement of Theorem 1.3 is well-defined. Moreover, all of the points of the corresponding arrangement are pairs i, j. This accounts for the simplified version of the above theorem stated in the introduction.

Proof   Let $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{C}}$ be the closures of $\mathcal{P}$ and $\mathcal{C}$ in the compatible compactification $\widetilde{\mathcal{X}}(\Delta)$ of $(\mathbb{C}^*)^N$ described in Example 3.7. Recall that $\widetilde{\mathcal{P}}$ is a blow up of $\mathbb{C}P^2$, let $\pi \colon \widetilde{\mathcal{P}} \to \mathbb{C}P^2$ denote the contraction map. The boundary $\partial \widetilde{\mathcal{P}} = \widetilde{\mathcal{P}} \backslash \mathcal{P}$ is a collection of non-singular divisors consisting of the proper transforms of the $N + 1$ lines in $\overline{\mathcal{P}} \setminus \mathcal{P}$ along with all exceptional divisors. Given $\mathbf{p}_I \in \mathbf{p}(\mathcal{A})$, we denote by $\mathcal{E}_I$ the proper transform in $\widetilde{\mathcal{P}}$ of the exceptional divisor of the blowup of $\mathbb{C}P^2$ at the point $\mathbf{p}_I$, by $\partial \widetilde{P}$ the sum of all divisors in $\widetilde{\mathcal{P}} \backslash \mathcal{P}$, and by $\mathcal{L}$ the divisor class of a line in $\mathbb{C}P^2$. Note that the divisors $\mathcal{E}_I$ are contained in the support of $\partial \widetilde{P}$. We will prove in Lemma 4.3

that the canonical class of $\widetilde{\mathcal{P}}$ can be written in the following way:

$$K_{\widetilde{\mathcal{P}}} = (N-2)\pi^*\mathcal{L} - \partial\widetilde{\mathcal{P}} - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I| - 2)\mathcal{E}_I.$$

With $K_{\widetilde{\mathcal{P}}}$ written this way, we can calculate $K_{\widetilde{\mathcal{P}}} \cdot \mathcal{C}$ using just the tropical curve $C = \mathrm{Trop}(\mathcal{C})$. First, $\pi^*\mathcal{L} \cdot \mathcal{C} = \deg_\Delta(C)$. By definition of the weights of the edges of $\mathrm{Trop}(C)$ we have

$$\mathcal{E}_I \cdot \widetilde{\mathcal{C}} = w_I \quad \text{and} \quad \partial\mathcal{P} \cdot \widetilde{\mathcal{C}} = \sum_{e \in \mathrm{Edge}(C)} w_e.$$

Therefore,

$$K_{\widetilde{\mathcal{P}}} \cdot \mathcal{C} = (N-2)\deg_\Delta(C) - \sum_{e \in \mathrm{Edge}(C)} w_e - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I| - 2)w_I.$$

By Theorem 3.8 we have $\widetilde{\mathcal{C}}^2 = C^2$. Applying the adjunction formula for $\widetilde{\mathcal{C}} \subset \widetilde{\mathcal{P}}$ we obtain the claimed inequality. ∎

Corollary 4.2 *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and $C \subset \mathrm{Trop}(\mathcal{P})$ a fan tropical curve. If C is finely approximable by a complex curve $\mathcal{C} \subset \mathcal{P}$, then for a primitive N-simplex $\Delta \subset \mathbb{R}^N$ giving a degree one compactification of $\mathcal{P}$ we have*

$$C^2 + (N-2)\deg_\Delta(C) - \sum_{e \in \mathrm{Edge}(C)} w_e - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I| - 2)w_I + 2 \geq 0.$$

The following lemma completes the proof of Theorem 4.1.

Lemma 4.3 *Using the same notations as in the proof of Theorem 4.1, we have*

$$K_{\widetilde{\mathcal{P}}} = (N-2)\pi^*\mathcal{L} - \partial\widetilde{\mathcal{P}} - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I| - 2)\mathcal{E}_I.$$

Proof To see that the canonical class can be expressed as claimed we first start with

$$K_{\mathbb{C}P^2} = -3\mathcal{L} = -\sum_{i=0}^{N} \mathcal{L}_i + (N-2)\mathcal{L},$$

where the $\mathcal{L}_i$'s are the lines in $\overline{\mathcal{P}} \setminus \mathcal{P}$. If $\pi': \mathcal{P}' \to \mathbb{C}P^2$ is the blowup of $\mathbb{C}P^2$ at the point $\mathbf{p}_I$, the canonical classes are related as follows:

$$K_{\mathcal{P}'} = \pi'^*K_{\mathbb{C}P^2} + \mathcal{E}_I.$$

Then

$$K_{\mathcal{P}'} = (N-2)\pi'^*\mathcal{L} - \pi'^*\left(\sum_{i=0}^{N} \mathcal{L}_i\right) + \mathcal{E}_I = (N-2)\pi'^*\mathcal{L} - \sum_{i=0}^{N} \widetilde{\mathcal{L}}_i - |I|\mathcal{E}_I + \mathcal{E}_I,$$

where $\widetilde{\mathcal{L}}_i$ is the proper transform of $\mathcal{L}_i$. Moreover, $\partial\mathcal{P}' = \sum_{i=0}^{N} \widetilde{\mathcal{L}}_i + \mathcal{E}_I$, so

$$K_{\mathcal{P}'} = (N-2)\pi'^*\mathcal{L} - \partial\mathcal{P}' - (|I| - 2)\mathcal{E}_I.$$

Blowing up further at points above $\mathbf{p}_I$ that are the intersection of two boundary divisors, the exceptional divisor is again a boundary divisor of the new surface. Continuing the process at each $\mathbf{p}_I$ to obtain $\widetilde{\mathcal{P}}$, we have

$$K_{\widetilde{\mathcal{P}}} = (N-2)\pi^*\mathcal{L} - \partial\widetilde{\mathcal{P}} - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I| - 2)\mathcal{E}_I,$$

which completes the proof. ■

5 Obstructions coming from Intersections with the Hessian

Consider a plane algebraic curve $\mathcal{C}$ in $\mathbb{C}P^2$ given by the homogeneous equation $F(z, w, u) = 0$. The Hessian of the polynomial $F(z, w, u)$, denoted by $\mathrm{Hess}_F(z, w, u)$, is the homogeneous polynomial defined as

$$\mathrm{Hess}_F(z, w, u) = \det \begin{pmatrix} \frac{\partial^2 F}{\partial^2 z} & \frac{\partial^2 F}{\partial z \partial w} & \frac{\partial^2 F}{\partial z \partial u} \\ \frac{\partial^2 F}{\partial z \partial w} & \frac{\partial^2 F}{\partial^2 w} & \frac{\partial^2 F}{\partial w \partial u} \\ \frac{\partial^2 F}{\partial z \partial u} & \frac{\partial^2 F}{\partial w \partial u} & \frac{\partial^2 F}{\partial^2 u} \end{pmatrix}.$$

If Hess_F is not the null polynomial, it defines a curve $\mathrm{Hess}_{\mathcal{C}}$ called the *Hessian* of $\mathcal{C}$. The polynomial $\mathrm{Hess}_{\mathcal{C}}$ of course depends on the chosen coordinate system on $\mathbb{C}P^2$; however, the curve $\mathrm{Hess}_{\mathcal{C}}$ does not; *i.e.,* the curve $\mathrm{Hess}_{\mathcal{C}}$ is invariant under projective change of coordinates in $\mathbb{C}P^2$. Note that if $\mathcal{C}$ has degree d, then $\mathrm{Hess}_{\mathcal{C}}$ has degree $3(d-2)$ and intersects $\mathcal{C}$ in finitely many points if $\mathcal{C}$ is reduced and does not contain a line as a component. Intersecting a curve with its Hessian detects singularities and inflection points of the curve. In particular, if $\mathcal{L}$ is a line and $p \in \mathcal{C}$ are such that $(\mathcal{C} . \mathcal{L})_p = m$, then $(\mathcal{C} . \mathrm{Hess}_{\mathcal{C}})_p \geq m - 2$.

Before we present the obstructions we fix some further notation. Given $\mathcal{C}$ an algebraic curve in affine space $\mathbb{C}^2$ defined by a polynomial $\sum a_{i,j} z^i w^j$, we denote its Newton polygon by $\Delta(\mathcal{C}) = \mathrm{Conv}\{(i, j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$, and we define

$$\Gamma(\mathcal{C}) = \mathrm{Conv}(\Delta(\mathcal{C}) \cup \{(0,0)\}) \quad \text{and} \quad \Gamma^c(\mathcal{C}) = \Gamma(\mathcal{C}) \setminus \Delta(\mathcal{C}).$$

Once a coordinate system is fixed in $\mathbb{C}^2$, the equation of an algebraic curve is defined up to a non-zero multiplicative constant. In particular the polygons $\Delta(\mathcal{C})$, $\Gamma(\mathcal{C})$, and $\Gamma^c(\mathcal{C})$ do not depend on the particular choice of the defining polynomial. The latter definition translates literally to tropical curves in $\mathbb{T}^2$. If C is the tropicalisation of a projective plane curve $\mathcal{C}$ in the coordinates (z, w), then we have

$$\Delta(\mathcal{C}) = \Delta(C), \quad \Gamma(\mathcal{C}) = \Gamma(C), \quad \text{and} \quad \Gamma^c(\mathcal{C}) = \Gamma^c(C).$$

Finally, given a polygon Δ in $\mathbb{R}^2$ we will denote its lattice area by $A(\Delta)$, *i.e.,* twice its euclidean area.

Lemma 5.1 *Let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}P^2$, and let us fix a coordinate system on $\mathbb{C}P^2$. If $\mathcal{C}$ is reduced and does not contain any line as a component, then*

$$(\mathcal{C} . \mathrm{Hess}_{\mathcal{C}})_{[0:0:1]} \geq 3A(\Gamma^c(\mathcal{C})) + r_0(\mathcal{C}) - 2v_0(\mathcal{C}) - 2h_0(\mathcal{C}).$$

where

- $r_0(\mathcal{C}) = \mathrm{Card}(e \cap \mathbb{Z}^2) - 1$ *if there exists an edge e of $\Gamma^c(\mathcal{C})$ of slope -1, and $r_0(\mathcal{C}) = 0$ otherwise;*
- $v_0(\mathcal{C}) = \mathrm{Card}(\Gamma^c(\mathcal{C}) \cap (\{0\} \times \mathbb{Z})) - 1$;
- $h_0(\mathcal{C}) = \mathrm{Card}(\Gamma^c(\mathcal{C}) \cap (\mathbb{Z} \times \{0\})) - 1$.

Proof The intersection multiplicity at $[0:0:1]$ of the curve $\mathcal{C}$ and its Hessian is bigger than the number of inflection points in $\mathbb{C}P^2$ of a curve with Newton polygon $\Gamma(\mathcal{C})$ minus the number of inflection points in $\mathbb{C}P^2 \setminus \{[0:0:1]\})$ of a curve with Newton polygon $\Delta(\mathcal{C})$. Hence the result follows from [BLdM12, Proposition 6.1]. ∎

Example 5.2 We will use Lemma 5.1 in the two following simple situations:

- if p is a non-degenerate node of a complex curve $\mathcal{C}$, then $(\mathcal{C} \,.\, \mathrm{Hess}_{\mathcal{C}})_p \geq 6$;
- if the curve $\mathcal{C}$ has a unique branch at a point p, then $(\mathcal{C} \,.\, \mathrm{Hess}_{\mathcal{C}})_p \geq 3M_p m_p - 2M_p - 2m_p$, where m_p is the multiplicity of C at p, and M is the maximal order of contact of a line with $\mathcal{C}$ at p (note that $M_p > m_p$ and that there exists a unique line $\mathcal{L}$ such that $(\mathcal{C} \,.\, \mathcal{L})_p > m_p$).

Consider a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ and a tropical morphism $f \colon C' \to \mathrm{Trop}(\mathcal{P})$; denote the image by $C = f(C')$. As usual, given a primitive N-simplex Δ giving a degree one compactification of $\mathcal{P}$, denote by $u_0, \ldots, u_N$ the outward primitive integer vectors normal to the faces of Δ. We also denote by $\mathcal{A}$ the line arrangement $\overline{\mathcal{P}} \setminus \mathcal{P}$. We define the three following subsets of $\mathrm{Edge}(C')$:

$$\mathrm{Bis}_I(C') = \{ e \in \mathrm{Edge}(C') \mid u_{f,e} = u_I \},$$
$$\mathrm{Bis}(C') = \bigcup_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} \mathrm{Bis}_I(C'),$$
$$K_w(C') = \left\{ e \in \mathrm{Edge}(C') \mid \exists i, \ u_{f,e} = u_i, \text{ and } w_{f,e} > 1 \right\},$$
$$K_1(C') = \left\{ e \in \mathrm{Edge}(C') \mid \exists i, \ u_{f,e} = u_i, \text{ and } w_{f,e} = 1 \right\}.$$

Note that the sets defined above are dependent on the choice of N-simplex Δ, used to obtain a degree one compactification of $\mathcal{P} \subset (\mathbb{C}^*)^N$, since they depend on the vectors u_i. Finally, we denote by $m_I(C)$ the multiplicity of the curve C at the point p_I, *i.e.*, its intersection multiplicity at p_I with the ray u_I. For a simple expression of this multiplicity, let $\mathrm{Edge}_I(C') = \{ e \in \mathrm{Edge}(C') \mid p_I \in \overline{f(e)} \}$. For an edge $e \in \mathrm{Edge}_I(C')$, the vector $u_{f,e}$ has a unique expression $p_e u_I + q_e u_k$ for some $k \in I$. Then the multiplicity at point p_I is given by,

$$m_I(C) = \sum_{e \in \mathrm{Edge}_I(C')} w_{f,e} p_e.$$

Again, the multiplicity $m_I(C)$ is dependent on the choice of N-simplex Δ, when a choice exists.

We can now state and prove the main result of this section.

Theorem 5.3 *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, and Δ a primitive N-simplex giving a degree one compactification of $\mathcal{P}$. Let $f \colon C' \to \mathrm{Trop}(\mathcal{P})$ be a tropical*
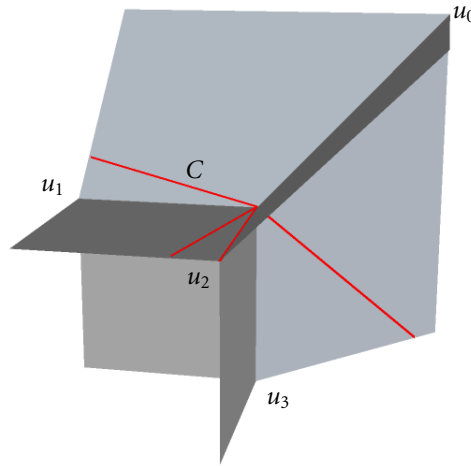
Figure 15: The 4-valent curve from Corollary 5.4.

morphism such that $C = f(C')$ has $\deg_\Delta(C) > 1$. If the morphism f is finely approximable in $\mathcal{P}$, then

$$3C^2 + 2(N-2)\deg_\Delta(C) - \sum_{\boldsymbol{p}_I \in \boldsymbol{p}(\mathcal{A})} 2(|I|-2)m_I(C)$$

$$- \sum_{e \in \mathrm{Bis}(C')(C') \cup K_w(C')} (3w_{f,e} - 2) - 2|K_1(C')| \geq 0.$$

Proof Suppose that $f\colon C' \to \mathrm{Trop}(\mathcal{P})$ is finely approximable by $\mathcal{F}\colon \mathcal{C} \to \mathcal{P}$. Since $\mathcal{F}(\mathcal{C})$ is irreducible, we will identify $\mathcal{C}$ and $\mathcal{F}(\mathcal{C})$ in $\mathcal{P}$. Consider $\overline{\mathcal{C}}$ the closure of $\mathcal{C}$ in $\overline{\mathcal{P}} = \mathbb{C}P^2$, and define $q_1, \ldots, q_s$ the points in $\overline{\mathcal{C}} \cap (\mathcal{A} \setminus \boldsymbol{p}(\mathcal{A}))$ for which the mulitplicity of intersection is at least 2. We denote by m_j this intersection multiplicity at the point q_j.

To lighten the notation slightly, we denote by $d = \deg_\Delta(C)$. Since the number of intersection points of $\overline{\mathcal{C}}$ with its Hessian in $\overline{\mathcal{P}} \setminus \{q_1, \ldots, q_s, \mathbf{p}_I \in \mathbf{p}(\mathcal{A})\}$ is non-negative and the Hessian is of degree $3(d-2)$, we have

(5.1) $$3d(d-2) - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (\overline{\mathcal{C}} . \mathrm{Hess}_{\overline{\mathcal{C}}})_{\mathbf{p}_I} - \sum_{j=1}^{s} (m_j - 2) \geq 0.$$

It is immediate from the definition of the weights of the tropicalisation of a morphism from Definition 2.10 that

$$\sum_{j=1}^{s} (m_j - 2) = \sum_{e \in K_w(C')} (w_{f,e} - 2).$$

It remains to estimate the quantities $(\overline{\mathcal{C}} . \mathrm{Hess}_{\overline{\mathcal{C}}})_{\mathbf{p}_I}$. As before, we denote $\partial\overline{\mathcal{P}} = \overline{\mathcal{P}} \setminus \mathcal{P}$, and we claim that

$$(5.2) \qquad (\overline{\mathcal{C}} . \mathrm{Hess}_{\overline{\mathcal{C}}})_{\mathbf{p}_I} \geq 3(\overline{C}^2)_{p_I} - 2(\overline{\mathcal{C}} . \partial\overline{\mathcal{P}})_{\mathbf{p}_I} - 2|\mathrm{Bis}_I(C')|$$
$$+ 3 \sum_{e \in \mathrm{Bis}_I(C')} w_{f,e} + 2(|I| - 2)m_I(C).$$

To prove inequality (5.2), we have to estimate the number of inflection points that are contained in the Milnor fiber F_I of $\overline{\mathcal{C}}$ at $\mathbf{p}_I$. Let us denote by $b_1, \ldots, b_k$ the local branches of $\overline{\mathcal{C}}$ at p_I. Note that these branches are in one to one correspondence with the edges of $\mathrm{Edge}_I(C')$. A small perturbation of F_I can be constructed as follows: in a small Milnor ball centered at $\mathbf{p}_I$, we translate each branch b_i such that they intersect transversally; then we replace each singular point by its Milnor fiber to obtain a surface Γ. Let us denote by m_i the multiplicity of b_i at $\mathbf{p}_I$, and by M_i the maximal order of contact of a line with b_i at $\mathbf{p}_I$. Note that if $e_i \in \mathrm{Bis}_I(C')$, then $(b_i . \partial\overline{\mathcal{P}}) = |I|m_i$, and that if $e_i \notin \mathrm{Bis}_I(C')$, then $(b_i . \partial\overline{\mathcal{P}}) = (|I| - 1)m_i + M_i$. In addition, note that if $e_i \in \mathrm{Bis}_I(C')$, then $m_i = w_{f,e_i}$ and $M_i \geq m_i + 1$. Combining both the second and first part of Example 5.2, the number of inflection points contained in Γ is at least,

$$6\sum_{i \neq j}(b_i . b_j)_{\mathbf{p}_I} + \sum_{e_i \notin \mathrm{Bis}_I(C')}(3M_im_i - 2M_i - 2m_i) + \sum_{e_i \in \mathrm{Bis}_I(C')}(3(m_i + 1)m_i - 4m_i - 2).$$

By Definition 3.6, we have that

$$\sum_{e_i \notin \mathrm{Bis}_I(C')} M_im_i + \sum_{e_i \in \mathrm{Bis}_I(C')} m_i^2 + 2\sum_{i \neq j}(b_i . b_j)_{\mathbf{p}_I} = (\overline{C}^2)_{p_I}.$$

Combining the above two lines we obtain inequality (5.2).

Hence from inequality we get (5.1) and (5.2) that

$$(5.3) \quad 3d(d - 2) - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})}\left(3(\overline{C}^2)_{p_I} - 2(\overline{\mathcal{C}} . \partial\overline{\mathcal{P}})_{\mathbf{p}_I} + 2(|I| - 2)m_I(C)\right)$$

$$+ 2|\mathrm{Bis}(C')| - 3\sum_{e \in \mathrm{Bis}(C')} w_{f,e} - \sum_{e \in K_w(C')}(w_{f,e} - 2) \geq 0$$

By Definition 3.6, we have

$$\sum_{0 \leq i < j \leq N}(\overline{C}^2)_{p_{i,j}} = d^2 - C^2.$$

Summing up all intersection multiplicities of $\overline{\mathcal{C}}$ with $\overline{\mathcal{P}} \setminus \mathcal{P}$, we get

$$\mathcal{C} . \partial\overline{\mathcal{P}} = (N + 1)d = \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})}(\overline{\mathcal{C}} . \partial\overline{\mathcal{P}})_{\mathbf{p}_I} + \sum_{e \in K_w(C')} w_{f,e} + |K_1(C')|.$$

Plugging all of the latter equalities into inequality (5.3) we obtain the desired inequality. ∎

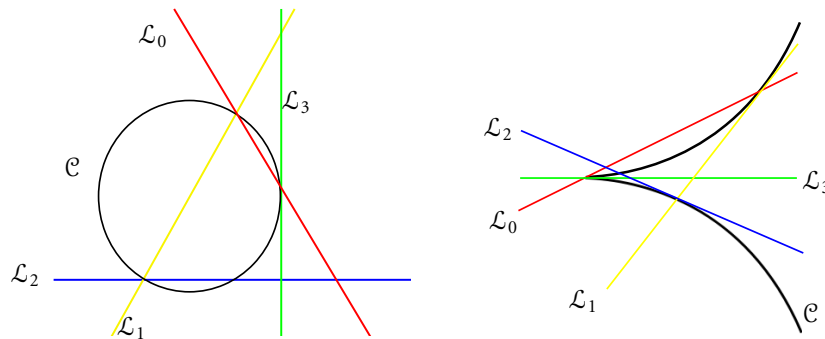As an application, we prove the following corollary.

Figure 16: The conic and the cuspidal cubic from Corollary 5.4 with their positions with respect to the 4 lines.

Corollary 5.4   *Let $\mathcal{P}$ be a uniform plane in $(\mathbb{C}^*)^3$, and denote by u_0, u_1, u_2, and u_3 the primitive integer directions of its four edges. Let C be a fan tropical curve in $\mathrm{Trop}(\mathcal{P})$ with four rays of weight one in the primitive integer directions*

$$u_1 + (d-1)u_2, \quad (d-1)u_1 + u_0, \quad du_3 + (d-1)u_0, \quad and \quad u_2,$$

(see Figure 15). Then C is approximable by a complex curve $\mathcal{C} \subset \mathcal{P}$ if and only if $d = 1, 2,$ or 3.

Proof   The tropical cycle induced by C is irreducible, hence if an approximation $\mathcal{C}$ exists, it must be an irreducible and reduced curve. Also, as $\mathcal{P}$ is uniform, there is a unique N-simplex Δ giving a degree one compactification of $\mathcal{P}$. Moreover, $\deg_\Delta(C) = d$.

If $d = 1$, the tropical curve C is the skeleton of the plane $\mathrm{Trop}(\mathcal{P})$ and it is approximated by a generic line $\mathcal{L} \subset \mathcal{P}$. For $d > 1$ the closure of the tropical curve $\overline{C} \subset \overline{\mathrm{Trop}(\mathcal{P})} \subset \mathbb{T}P^3$ contains the corner points $p_{i,j} \subset \overline{\mathrm{Trop}(\mathcal{P})}$ for $\{i, j\} = \{0, 1\}, \{0, 3\}, \{1, 2\}$. At these corner points we have the polygons

$$\Gamma_{i,j}^c = \mathrm{Conv}\{(0,0), (0,1), (d-1,0)\}, \text{ for } \{i, j\} = \{0, 1\} \text{ and } \{1, 2\}$$
$$\Gamma_{0,3}^c = \mathrm{Conv}\{(0,0), (0,d), (d-1,0)\}.$$

It is immediate that $\mathcal{C}$ exists if $d = 2$ or 3; the curves relative to the line arrangements are drawn in Figure 16. For $d = 3$ the curve has a cusp at $\mathbf{p}_{0,3}$ and simple tangencies to $\mathcal{L}_1$ and $\mathcal{L}_0$ at the points $\mathbf{p}_{1,2}$ and $\mathbf{p}_{0,1}$. For $d = 2$, the curve is a conic passing through the point $\mathbf{p}_{0,3}$ with a tangency to the line $\mathcal{L}_3$, and also passing through the points $\mathbf{p}_{0,1}$ and $\mathbf{p}_{1,2}$ having intersection multiplicity one with the respective lines at these points.

Let us now prohibit the remaining situations. An easy computation yields $C^2 = 2 - d$. Then, according to Theorem 5.3, if $\mathcal{C}$ exists then

$$3(2 - d) + 2d - 2 \geq 0$$

which reduces to $d \leq 4$. We are left to study by hand the case $d = 4$. Choose coordinates on $\overline{\mathcal{P}} = \mathbb{C}P^2$ such that $\mathcal{L}_1, \mathcal{L}_0, \mathcal{L}_3$ are coordinates axes. In this coordinate system, $\Delta(\mathcal{C}) = \text{Conv}\{(0,4), (0,3), (1,0)\}$. It is not hard to verify (see for example [BLdM12, Lemma 6.4]) that such a curve cannot have an inflection point on $\mathcal{L}_0 \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$ ∎

These curves will be revisited in Section 7 as Vigeland lines in a tropical surface.

6 Trivalent Fan Tropical Curves in Planes

In this section we classify all 3-valent tropical curves $C \subset \text{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ which are finely approximable in a given non-degenerate plane $\mathcal{P} \subset (\mathbb{C}^*)^N$. To this end we first focus on the case when $\mathcal{P}$ is a uniform hyperplane in $(\mathbb{C}^*)^3$.

6.1 Tropical Curves in $\mathbb{R}^3$ with $\text{aff}_C \leq 2$

Given a fan tropical curve $C \subset \mathbb{R}^N$, we denote by $\text{Aff}(C)$ its affine span in $\mathbb{R}^N$, and by aff_C the dimension of $\text{Aff}(C)$. The space $\text{Aff}(C)$ has a natural tropical structure since it is the tropicalisation of a binomial surface in $(\mathbb{C}^*)^N$.

In [BK12], T. Bogart and E. Katz gave some combinatorial obstructions to the approximation of a fan tropical curve C in a uniform tropical plane in $\mathbb{R}^3$ satisfying $\text{aff}_C \leq 2$. Combining their results with our own we obtain the complete classification of such approximable tropical curves.

Theorem 6.1 *Let $\mathcal{P}$ be a uniform plane in $(\mathbb{C}^*)^3$, and let us denote by u_0, u_1, u_2, and u_3 the four rays of $\text{Trop}(\mathcal{P})$. Let $C \subset \text{Trop}(\mathcal{P})$ be a reduced fan tropical curve in $\mathbb{R}^3$ with $\text{aff}_{f(C)} \leq 2$. Then the curve C is finely approximable in $\mathcal{P}$ if and only if one of the two following conditions holds:*

(i) *C is equal to the tropical stable intersection of $\text{Aff}(f(C))$ and $\text{Trop}(\mathcal{P})$ (see [RGST05]);*

(ii) *C has three edges e_1, e_2, and e_3 satisfying*

$$w_{e_1} u_{e_1} = u_i + d u_k, \quad w_{e_2} u_{e_2} = u_j + d u_l, \quad w_{e_3} u_{e_3} = (d-1)(u_i + u_j)$$

with $\{i, j, k, l\} = \{0, 1, 2, 3\}$. In particular, if $d = 1$ the fan tropical curve C is 2-valent with directions $u_i + u_j$, and $u_k + u_l$.

Moreover, in this case, $C = f(C_0)$ where C_0 is a fan tropical curve with $d + 1$ edges, and $f: C_0 \to \text{Trop}(\mathcal{P})$ is a tropical morphism finely approximable in $\mathcal{P}$ by a rational curve with $d + 1$ punctures. In particular, f maps all edges of C_0 to $\text{Trop}(\mathcal{P})$ with weight 1, and $(d - 1)$ edges of C_0 are mapped to e_3 by f.

The proof of Theorem 6.1 decomposes into several steps. Let us first recall two lemmas from [BK12].

Lemma 6.2 (Bogart–Katz, [BK12, Lemma 3.1]) *Let C be a tropical curve in $\mathbb{R}^3$ such that $\text{aff}_C \leq 2$ and which is approximable by a reduced and irreducible complex algebraic*

curve $\mathcal{C} \subset (\mathbb{C}^*)^3$. *Then there exists a reduced and irreducible binomial algebraic surface* $\mathcal{H} \subset (\mathbb{C}^*)^3$ *such that* $\mathrm{Trop}(\mathcal{H}) = \mathrm{Aff}(C)$ *and* $\mathcal{C} \subset \mathcal{H}$.

Lemma 6.3 (Bogart–Katz, [BK12, Lemmas 8.5–8,8])   *Let* $\mathcal{P} \subset (\mathbb{C}^*)^3$ *be a uniform plane, and let* $\mathcal{H} \subset (\mathbb{C}^*)^3$ *be a reduced and irreducible binomial surface. Then either the curve* $\mathcal{P} \cap \mathcal{H}$ *is non-singular or it has a unique singular point in* $(\mathbb{C}^*)^3$, *which is a node. Moreover, if* $\mathcal{P} \cap \mathcal{H}$ *has two irreducible components* $\mathcal{C}_1$ *and* $\mathcal{C}_2$, *then the embedded tropical curve* $\mathrm{Trop}(\mathcal{C}_1) \cup \mathrm{Trop}(\mathcal{C}_2)$ *is 4-valent, the two tropical curves* $\mathrm{Trop}(\mathcal{C}_1)$ *and* $\mathrm{Trop}(\mathcal{C}_2)$ *are at most 3-valent, and at least one of them is 2-valent.*

Lemma 6.3 immediately implies that a fan tropical curve $C \subset \mathbb{R}^3$ with $\mathrm{aff}_C \leq 2$ that is finely approximable in a uniform plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ and not equal to the tropical stable intersection of $\mathrm{Trop}(\mathcal{P})$ and $\mathrm{Aff}(C)$ must be either 2 or 3-valent.

Lemma 6.4   *Let* $\mathcal{P} \subset (\mathbb{C}^*)^3$ *be a uniform plane, and let* $\mathcal{H} \subset (\mathbb{C}^*)^3$ *be a reduced and irreducible binomial surface. We denote by* $\Delta_{\mathcal{P},\mathcal{H}}$ *the Newton polytope of the tropical surface* $\mathrm{Trop}(\mathcal{P}) \cup \mathrm{Trop}(\mathcal{H})$, *and by* $\mathcal{X}(\Delta_{\mathcal{P},\mathcal{H}})$ *the toric variety defined by* $\Delta_{\mathcal{P},\mathcal{H}}$. *Let* $\overline{\mathcal{P}}$ *and* $\overline{\mathcal{H}}$ *be respectively the closure of* $\mathcal{P}$ *and* $\mathcal{H}$ *in* $\mathcal{X}(\Delta_{\mathcal{P},\mathcal{H}})$, *and let* $\overline{\mathcal{C}} = \overline{\mathcal{P}} \cap \overline{\mathcal{H}}$. *Then the curve* $\overline{\mathcal{C}}$ *is reduced and* $\overline{\mathcal{C}}^2 = 0$ *in* $\overline{\mathcal{P}}$. *Moreover, if* $\overline{\mathcal{C}}$ *is reducible, then* $\overline{\mathcal{C}}$ *has exactly two irreducible components* $\overline{\mathcal{C}}_1$ *and* $\overline{\mathcal{C}}_2$, *and* $\overline{\mathcal{C}}_1^2 = \overline{\mathcal{C}}_2^2 = -1$ *in* $\overline{\mathcal{P}}$.

In particular, $\mathrm{Trop}(\mathcal{C}_1)^2 = \mathrm{Trop}(\mathcal{C}_2)^2 = -1$ *in* $\mathrm{Trop}(\mathcal{P})$.

Proof   We define $\mathcal{C} = \mathcal{P} \cap \mathcal{H} = \overline{\mathcal{C}} \cap (\mathbb{C}^*)^3$. According to Lemma 6.3, the curve $\mathcal{C}$ has at most one singular point, so it has to be reduced and cannot have more than two irreducible components. Hence the same is true for $\overline{\mathcal{C}}$. Since $\overline{\mathcal{H}}^2 = 0$, we also have $\overline{\mathcal{C}} = 0$ in $\overline{\mathcal{P}}$. Suppose that $\overline{\mathcal{C}}$ has two irreducible components, $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$. Since $\overline{\mathcal{H}}^2 = 0$, we have $\overline{\mathcal{C}}_1 . \overline{\mathcal{C}} = \overline{\mathcal{C}}_2 . \overline{\mathcal{C}} = 0$ which implies that

$$\overline{\mathcal{C}}_1^2 + \overline{\mathcal{C}}_1 . \overline{\mathcal{C}}_2 = \overline{\mathcal{C}}_2^2 + \overline{\mathcal{C}}_1 . \overline{\mathcal{C}}_2 = 0.$$

So we are left to show that $\overline{\mathcal{C}}_1 . \overline{\mathcal{C}}_2 = 1$. Since the curve $\overline{\mathcal{C}}$ is reducible, it follows from Lemma 6.3 that $\mathcal{C}$ has a unique singular point, which is a node, in $(\mathbb{C}^*)^3$. Hence the result will follow from the fact that $\overline{\mathcal{C}}$ intersects the boundary $\mathcal{X}(\Delta_{\mathcal{P},\mathcal{H}}) \setminus (\mathbb{C}^*)^3$ transversely at non-singular points of $\overline{\mathcal{C}}$.

To prove this last claim, we can assume that $\mathcal{H}$ is a subtorus of $(\mathbb{C}^*)^3$. In this case, there is a surjection $\phi \colon \mathrm{Hom}((\mathbb{C}^*)^3, \mathbb{C}^*) \otimes \mathbb{R} \to \mathrm{Hom}(\mathcal{H}, \mathbb{C}^*) \otimes \mathbb{R}$. Moreover, if $Q \in \mathrm{Hom}((\mathbb{C}^*)^3, \mathbb{C}^*)$ is an equation of $\mathcal{P}$ in $(\mathbb{C}^*)^3$, then $\phi(Q)$ is an equation of $\mathcal{C}$ in $\mathcal{H}$. In particular, the Newton polygon of $\phi(Q)$ is dual to the tropical curve $\mathrm{Trop}(\mathcal{C})$, seen as a tropical curve in $\mathrm{Trop}(\mathcal{H})$. According to Lemma 6.3, the tropical curve $\mathrm{Trop}(\mathcal{C})$ is 4-valent, so the Newton polygon of $\phi(Q)$ is a quadrangle. The polynomial Q has exactly four monomials and $\phi(Q)$ has no fewer monomials than Q, so we get that $\phi(Q)$ also has exactly four monomials. In particular, the only non-zero coefficients of $\phi(Q)$ are the vertices of its Newton polygon. This implies that $\overline{\mathcal{C}}$ intersects the boundary $\overline{\mathcal{H}} \setminus \mathcal{H}$ transversely at non-singular points of $\overline{\mathcal{C}}$.   ∎
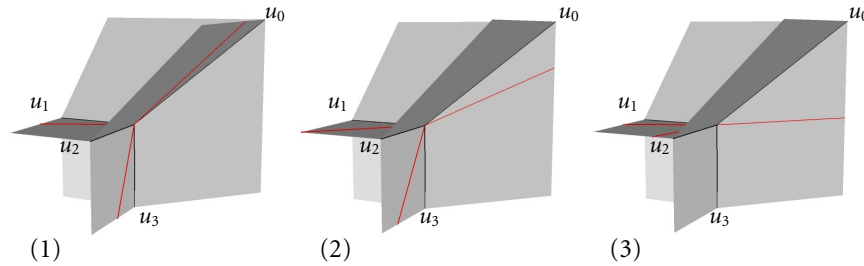
$$(1) \qquad\qquad (2) \qquad\qquad (3)$$

Figure 17: The three types of curves from Lemma 6.5.

To prove Theorem 6.1, we need to list all possible 3-valent fan tropical plane curves C with $\mathrm{aff}_C \leq 2$ and to compute C^2 for each of them. This is the content of the next lemma.

Lemma 6.5 *Let $\mathcal{P} \subset (\mathbb{C}^*)^3$ be a uniform plane, and let $C \subset \mathrm{Trop}(\mathcal{P})$ be an irreducible 3-valent fan tropical curve with $\mathrm{aff}_C \leq 2$. Let $u_0, u_1, u_2,$ and u_3 be the primitive integer directions of the edges of $\mathrm{Trop}(\mathcal{P})$, and let $e_1, e_2,$ and e_3 denote the edges of C. Then, up to the action of S_4 on the rays of $\mathrm{Trop}(\mathcal{P})$ and re-ordering of the edges of C, the curve C is one of the following types.*

(i) *There exists $0 \leq \alpha, \beta$ with $\gcd(d, \alpha, \beta) = 1$ and $\alpha + \beta \leq d$, such that*

$$w_{e_1} u_{e_1} = du_1 + \alpha u_2, \quad w_{e_2} u_{e_2} = \beta u_2 + du_3, \quad w_{e_3} u_{e_3} = (d - \alpha - \beta)u_2 + du_4,$$

see Figure 17(1). In this case the curve C is the tropical intersection of $\mathrm{Trop}(\mathcal{P})$ and $\mathrm{Aff}(C)$, and $C^2 = 0$.

(ii) *There exists $0 \leq \alpha, \beta \leq d$ with $\gcd(d, \alpha, \beta) = 1$ such that*

$$w_{e_1} u_{e_1} = du_1 + \alpha u_2, \quad w_{e_2} u_{e_2} = (d - \alpha)u_2 + (d - \beta)u_3, \quad w_{e_3} u_{e_3} = \beta u_3 + du_4,$$

see Figure 17(2). In this case, $C^2 = -\alpha\beta$.

(iii) *There exists $0 \leq \alpha < \beta \leq d$ with $\gcd(d, \alpha, \beta) = 1$ such that*

$$w_{e_1} u_{e_1} = \alpha u_1 + \beta u_2, \quad w_{e_2} u_{e_2} = (d - \alpha)u_1 + (d - \beta)u_2, \quad w_{e_3} u_{e_3} = du_3 + du_4,$$

see Figure 17(3). In this case, $C^2 = -d^2 + \beta d - \alpha\beta$.

Note that cases (i) and (ii) for $\alpha = \beta = 0$ (and consequently $d = 1$) coincide with the case (iii) for $\alpha = 0$ and $\beta = d = 1$. In addition, notice that when $d > 1$ and $\alpha, \beta > 0$, an approximation of a curve from case (i) must intersect the line arrangement $\overline{\mathcal{P}} \setminus \mathcal{P}$ in only three collinear points $\mathbf{p}_{1,2}, \mathbf{p}_{2,3}, \mathbf{p}_{2,4}$. Under the same conditions on $d, \alpha,$ and β, an approximation of a curve from case (ii) must intersect the arrangement in only the three points $\mathbf{p}_{1,2}, \mathbf{p}_{2,3}, \mathbf{p}_{3,4}$. Finally, an approximation of a curve from case (iii) may only intersect the arrangement in two points $\mathbf{p}_{1,2}, \mathbf{p}_{3,4}$, if we assume again the same restrictions on $d, \alpha,$ and β.

Proof The intersection numbers follow from a direct computation. In case (i), we have to prove in addition that the curve C is the tropical intersection of $\mathrm{Trop}(\mathcal{P})$ and $\mathrm{Aff}(C)$, which is non-trivial only for $\alpha \neq 0$ and $\beta \neq 0$. If C' denotes this

tropical intersection, it is clear that C and C' have the same underlying sets. Since C is irreducible, it remains to prove that C' is also irreducible.

Without loss of generality, we can assume that

$$u_0 = (1, 1, 1), \quad u_1 = (-1, 0, 0), \quad u_2 = (0, -1, 0), \quad u_3 = (0, 0, -1).$$

The surface Aff(C) is given by a classical linear equation of the form

$$ax + by + cz = 0 \quad \text{with} \quad \gcd(a, b, c) = 1.$$

Let us denote by $w_{1,2}$ (resp. $w_{2,3}$) the weight of the edge of C' lying in the convex cone spanned by u_1 and u_2 (resp. u_3 and u_2). A computation gives $w_1 = \gcd(a, b)$, and $w_2 = \gcd(b, c)$. Hence w_1 and w_2 are relatively prime and C' is irreducible, which implies $C' = C$. This completes the proof. ∎

Remark 6.6 Note that the same proof gives that the tropical stable intersection of any tropical surface of degree one in $\mathbb{R}^3$ made of an edge and 3 faces with any non-singular binomial tropical surface in $\mathbb{R}^3$ is always irreducible.

Completion of the proof of Theorem 6.1: It follows from Lemma 6.4 that if the tropical curve C is finely approximable in $\mathcal{P}$, then $C^2 = 0$ or -1 in Trop($\mathcal{P}$). Now, it follows from Lemma 6.5 that

- The case when $C^2 = 0$ corresponds to Theorem 6.1(i). The approximation of such a tropical curve follows from the fact that tropical stable intersections are always approximable (see [OP13, Theorem 5.3.3]).
- The case when $C^2 = -1$ corresponds to Theorem 6.1(ii). Up to automorphism of $\mathbb{C}P^2$ we are free to fix the 4 generic lines of the arrangement corresponding to $\mathcal{P}$ as we wish. Let $\mathcal{L}_2$ be the line at infinity and

$$\mathcal{L}_1 = \{y = 0\}, \quad \mathcal{L}_3 = \{x = 0\}, \quad \text{and} \quad \mathcal{L}_4 = \{x + y - 1 = 0\}.$$

The existence of an approximation in $\mathcal{P}$ of the tropical morphism $f \colon C_0 \to \text{Trop}(\mathcal{P})$ is equivalent to the existence of an irreducible complex algebraic curve $\mathcal{C}$ in $\mathbb{C}^2$ defined by the equation $yP(x) - 1 = 0$, where $P(x)$ is a complex polynomial of degree $d - 1$ with no multiple root, and such that $\mathcal{C}$ has order of contact d at $(0, 1)$ with the line $\mathcal{L}_4$. It is not hard to check that one has to have $P(x) = 1 + x + \cdots + x^{d-1}$. The curve $\mathcal{C}$ has the correct order of contact with each of the lines $\mathcal{L}_i$ of $\overline{\mathcal{P}} \backslash \mathcal{P}$ in order to tropicalise to $C \subset P$, and it is immediate that $\mathcal{C}$ is a rational curve with $d + 1$ punctures.

This completes the proof of the theorem. ∎

Notice that by Corollary 3.13, the above constructed complex curve $\mathcal{C}$ is the unique curve that approximates a tropical curve C from Theorem 6.1(ii), since $C^2 = -1$.

6.2 Classification of 3-valent Fan Tropical Curves in a Tropical Plane

The complete classification of fan tropical curves $C \subset \mathbb{R}^3$ with $\text{aff}_C \leq 2$ that are finely approximable in a non-degenerate plane in $(\mathbb{C}^*)^3$ can be extended to classify all 3-valent fan tropical curves finely approximable in a given plane of any codimension.

To this aim the next lemma determines whether a tropical curve $L \subset \text{Trop}(\mathcal{P})$ with $\deg_\Delta(L) = 1$ and any valency is approximable.

Lemma 6.7 *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and Δ a primitive N-simplex giving a degree one compactification of $\mathcal{P}$. Let $L \subset \text{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ be a k-valent fan tropical curve with $\deg_\Delta(L) = 1$. Then L is approximated by some line $\mathcal{L} \subset \mathcal{P}$ if and only if for some $0 \leq m \leq k$ the arrangement determined by $\mathcal{P}$ contains m collinear points $\boldsymbol{p}_{I_1}, \ldots, \boldsymbol{p}_{I_m}$ such that the sets $I_1, \ldots, I_m$ are disjoint and $I_1 \cup \cdots \cup I_m = \{0, \ldots, N\} \backslash J$, where $|J| = k - m$; moreover, the k rays of L are in the directions*

$$u_{I_1}, \ldots u_{I_m}, \quad and \quad \{u_i \mid i \in J\},$$

and $\mathcal{L}$ is the line passing through the m collinear points $\boldsymbol{p}_{I_i}$.

Proof Suppose a curve $\mathcal{L}$ approximates L. Since $\deg_\Delta(L) = 1$, the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ must be a line in the compactification of $\mathcal{P}$ to $\overline{\mathcal{P}} = \mathbb{C}P^2$ given by Δ. Therefore, $\overline{\mathcal{L}}$ intersects each line in the arrangement determined by $\mathcal{P}$ exactly once. Since L is k-valent and each edge is of weight one, the line $\overline{\mathcal{L}}$ may intersect the arrangement in only k points. These k intersection points induce a partition of the set of lines in the arrangement into k subsets. The subsets of size greater than one correspond to points $\mathbf{p}_{I_i}$ of the arrangement through which the line $\mathcal{L}$ passes. ∎

Before considering the general case we remark that not every tropical plane $P \subset \mathbb{R}^N$ contains trivalent fan tropical curves. In fact, in order for P to contain a trivalent curve there must exist three sets I_1, I_2, I_3 satisfying: $I_1 \cup I_2 \cup I_3 = \{0, \ldots, N\}$, and if $|I_i| > 1$, then p_{I_i} is a point of the corresponding line arrangement.

Given two line arrangements $\mathcal{A} \subset \mathcal{A}'$ in $\mathbb{C}P^2$, there is a natural inclusion of their respective planes $i \colon \mathcal{P}' \hookrightarrow \mathcal{P}$.

Lemma 6.8 *Let $\mathcal{P} \subset (\mathbb{C}^*)^3$ be a uniform plane, and denote the lines of the associated arrangement by $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$, and $\mathcal{L}_l$. In addition, let $\mathcal{C}_2 \subset \mathcal{P}$ be the degree two curve from Theorem 6.1(ii). Then the further fan tropical curves are finely approximated in the plane $\mathcal{P}'$ in following cases (see Figure 18).*

(i) *The plane $\mathcal{P}' \subset (\mathbb{C}^*)^4$ corresponds to the arrangement of five lines obtained by adding to the arrangement of $\mathcal{P}$ the unique line $\mathcal{L}_m$ that passes through the two points $\boldsymbol{p}_{i,k}, \boldsymbol{p}_{j,l}$; the three rays of $C \subset \text{Trop}(\mathcal{P}')$ are of weight one with primitive integer directions*

$$u_i + u_j, \quad u_i + 2u_k + u_m, \quad and \quad u_j + 2u_l + u_m.$$

(ii) *The plane $\mathcal{P}' \subset (\mathbb{C}^*)^4$ corresponds to the arrangement of five lines obtained by adding to the arrangement of $\mathcal{P}$ the unique line $\mathcal{L}_n$ that is tangent to $\mathcal{C}_2$ at the point $\boldsymbol{p}_{i,j}$; the three rays of $C_2 \subset \text{Trop}(\mathcal{P}')$ are of weight one with primitive integer directions*

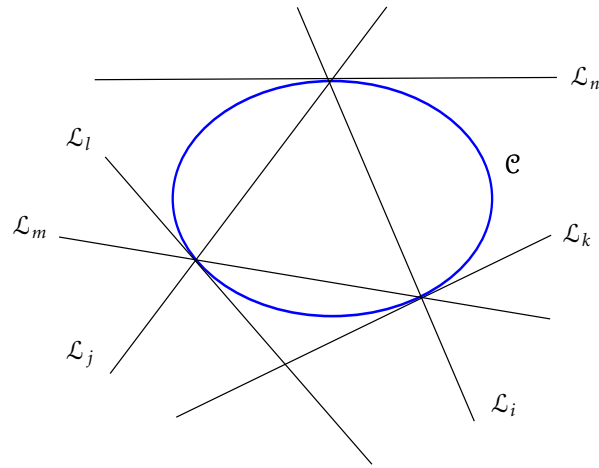$$u_i + u_j + 2u_n, \quad u_i + 2u_k, \quad and \quad u_j + 2u_l.$$

Figure 18: The curve $\mathcal{C}$ from Lemma 6.8 with respect to the lines indexed by i, j, k, l, m, n.

(iii)   *The plane $\mathcal{P}' \subset (\mathbb{C}^*)^5$ corresponds to the arrangement of six lines obtained from the arrangement of $\mathcal{P}$ by adding the lines $\mathcal{L}_m$ and $\mathcal{L}_n$ from parts* (i) *and* (ii)*; the three rays of $C \subset \mathrm{Trop}(\mathcal{P}')$ are of weight one and with primitive integer directions*

$$u_i + u_j + 2u_m, \quad u_i + 2u_k + u_n, \quad and \quad u_j + 2u_l + u_n.$$

Proof   In each of the three above cases the tropical curves are approximated by the curve $\mathcal{C} = \mathcal{C}_2 \cap \mathcal{P}'$. ∎

The curve Lemma 6.8(i) has already appeared in Example 3.4, in which it is remarked that there is a choice of compactification of $\mathcal{P}$ given by the simplex Δ' so that the closure of the curve $\mathcal{C}$ approximating C is a line. So in fact, case (i) above is also contained in Lemma 6.7.

Theorem 6.9   *Let $N \geq 3$, let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, and let $C \subset \mathrm{Trop}(\mathcal{P})$ be an irreducible 2 or 3-valent fan tropical curve. Then the curve C is finely approximable in $\mathcal{P}$ if and only if we have one of the following cases:*

(i)   *there exists a primitive N-simplex Δ giving a degree one compactification of $\mathcal{P}$ such that $\deg(C)_\Delta = 1$ and C and $\mathcal{P}$ satisfy Lemma 6.7;*

(ii)   *C and $\mathcal{P}$ satisfy one of the three situations described in Lemma 6.8;*

(iii) *the plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ is non-uniform and C is any irreducible trivalent fan tropical curve;*

(iv)   *the plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ is uniform and C is a trivalent curve from Theorem 6.1(ii) or Lemma 6.5(i).*

We remark that the trivalent lines with $N \geq 6$ in Theorem 6.9(i) and the curves of Lemma 6.8(ii) and (iii) are *exceptional*, in the sense that for a generic choice of

plane $\mathcal{P}$ that tropicalises to the fans in each of these cases (*i.e.*, whose line arrangement has the right intersection lattice), the corresponding tropical curve will not be approximable.

Proof All of the above tropical curves were shown to be approximable in the corresponding plane in Lemmas 6.7 and 6.8, Remark 6.6, and Theorem 6.1.

Let C be an irreducible 2 or 3-valent fan tropical curve that is finely approximable by a curve $\mathcal{C}$ in some plane $\mathcal{P}$. It remains to show that the pair $(\mathcal{P}, \mathcal{C})$ is one of those described in the theorem. According to Lemma 6.7, this is true if $\deg_\Delta(C) = 1$ for some primitive N-simplex Δ giving a degree one compactification of $\mathcal{P}$, so let us suppose that $\deg_\Delta(C) \geq 2$ for all such Δ.

Suppose first that the arrangement determined by $\mathcal{P}$ contains a uniform subarrangement $\mathcal{A}_0$ of 4 lines $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$, and $\mathcal{L}_l$ yielding a plane $\mathcal{P}_0$. Then there is a natural inclusion $\mathcal{P} \hookrightarrow \mathcal{P}_0$. If $\mathcal{C} \subset \mathcal{P}$ approximates C, let $\mathcal{C}_0 \subset \mathcal{P}_0$ be the closure of $\mathcal{C}$ in $\mathcal{P}_0$ and $C_0 \subset \mathrm{Trop}(\mathcal{P}_0)$ its tropicalisation. Since the plane $\mathcal{P}_0$ is uniform there is a unique 3-simplex Δ_0 giving a degree one compactification of $\mathcal{P}_0$, and the degree of C_0 is well-defined, denote it $\deg(C_0)$. Then we can find a degree one compactification of $\mathcal{P}$ given by a N-simplex Δ such that $\deg(C_0) = \deg_\Delta(C) \geq 2$. This implies by Theorem 6.1 that the curve C_0 is trivalent. Therefore, C_0 is either of type (i) from Lemma 6.5 or from Theorem 6.1(ii).

Suppose it is the former. Since $\deg(C_0) \geq 2$, up to relabeling the four lines in $\mathcal{A}_0$, it follows from Lemma 6.5 that $\mathcal{C}_0 \subset \mathcal{P}_0$ intersects this uniform arrangement in the three points $\mathbf{p}_{i,j}, \mathbf{p}_{i,k}, \mathbf{p}_{i,l}$. Let $\mathcal{L}$ be another line of the arrangement $\mathcal{A}$ determined by $\mathcal{P}$. Since C is trivalent, the line $\mathcal{L}$ must intersect $\mathcal{C}$ with multiplicity $\deg(C)$ at one of these three points, which is impossible according to Lemma 6.5.

If $C_0 \subset \mathrm{Trop}(\mathcal{P}_0)$ satisfies Theorem 6.1(ii), then $\mathcal{C}_0$ is the unique curve given in the proof of Theorem 6.1 (up to relabelling the four lines in $\mathcal{A}_0$). This curve intersects the arrangement $\mathcal{A}_0$ in the points $\mathbf{p}_{i,j}, \mathbf{p}_{i,k}$, and $\mathbf{p}_{j,l}$. Note that the only singular point of $\overline{\mathcal{C}} \subset \overline{\mathcal{P}}$ may be at the point $\mathbf{p}_{i,j}$ and that the tangent line to $\overline{\mathcal{C}}_0$ at the points $\mathbf{p}_{i,k}$ and $\mathbf{p}_{j,l}$ is already contained in the arrangement $\mathcal{A}_0$. Therefore, any other line passing through $\mathbf{p}_{i,k}$ or $\mathbf{p}_{j,l}$ has intersection multiplicity 1 with $\overline{\mathcal{C}}_0$ at this point. Let $\mathcal{L}$ be a line in $\mathcal{A} \setminus \mathcal{A}_0$. Since the tropical curve C is trivalent, we have one of the following two situations:

(a) The line $\mathcal{L}$ passes through the points $\mathbf{p}_{i,k}, \mathbf{p}_{j,l}$, and the sum of the intersection multiplicities of $\overline{\mathcal{C}}$ and $\mathcal{L}$ at these two points is equal $\deg(C)$. Since the intersection multiplicity of $\overline{\mathcal{C}}$ and $\mathcal{L}$ is one at these points, this is possible only if $\deg_\Delta(C) = 2$.

(b) The line $\mathcal{L}$ passes through the point $\mathbf{p}_{i,j}, \mathbf{p}_{i,k}$, or $\mathbf{p}_{j,l}$, and intersects $\overline{\mathcal{C}}$ with multiplicity $\deg(C)$ at this point. Since the intersection multiplicity of $\overline{\mathcal{C}}$ and $\mathcal{L}$ is 1 at $\mathbf{p}_{i,k}$ and $\mathbf{p}_{j,l}$, the line $\mathcal{L}$ necessarily passes through $\mathbf{p}_{i,j}$, which is an ordinary point of multiplicity $d - 1$ of $\overline{\mathcal{C}}$. Since C is 3-valent, the line $\mathcal{L}$ must have the same intersection multiplicity with all local branches of $\overline{\mathcal{C}}$ at $\mathbf{p}_{i,j}$, which is possible only if $\deg_\Delta(C) = 2$.

Hence if we do not have cases (i) or (iv) from the statement of the theorem, we have case (b).

If the arrangement $\mathcal{A}$ does not contain a uniform subarrangement of four lines, then according to Lemma 6.10, all but one line of $\mathcal{A}$ must belong to the same pencil. Then $\Delta(\mathcal{P})$ is $N-1$ dimensional and we must choose an N-simplex Δ containing $\Delta(\mathcal{P})$ and giving a degree one compactification of $\mathcal{P}$. The arrangement $\mathcal{A}$ contains a subarrangement $\mathcal{A}_0'$ of four lines, three of which belong to the same pencil, and defining a plane $\mathcal{P}_0' \subset (\mathbb{C}^*)^3$. As previously, if $\mathcal{C} \subset \mathcal{P}$ approximates C, let $\mathcal{C}_0' \subset \mathcal{P}_0'$ be the closure of $\mathcal{C}$ in $\mathcal{P}_0'$ and $C_0' \subset \text{Trop}(\mathcal{P}_0')$ its tropicalisation. From the N-simplex Δ yielding a degree one compactification of $\mathcal{P}$ we can obtain a 3-simplex Δ_0' compactifying $\mathcal{P}_0'$ so that $\deg_{\Delta_0'}(C_0') = \deg_\Delta(C) \geq 2$. Since C is a trivalent curve and $\deg_\Delta(C_0') \geq 2$, the curve C_0' must also be trivalent. Hence according to Remark 6.6, C_0' is the tropical stable intersection of $\text{Trop}(\mathcal{P}_0')$ and $\text{Aff}(C)$. If $\mathcal{C}_0'$ does not pass through the triple point of $\mathcal{A}_0'$, then since $\text{Trop}(\mathcal{C})$ is trivalent, we must have $\mathcal{A} = \mathcal{A}_0'$. If $\mathcal{C}_0'$ passes through the triple point of $\mathcal{A}_0'$, then there exist $|I|-2$ lines of $\mathcal{A}$ such that each branch of $\mathcal{C}$ at $\mathbf{p}_I$ has order of contact $\deg_\Delta(C)$ with these lines. This implies that $|I| = 3$, which completes the proof. ∎

Lemma 6.10 *If $\mathcal{A}$ is a line arrangement not containing a uniform subarrangement of 4 lines, then all but one of the lines in $\mathcal{A}$ are contained in the same pencil.*

Proof By assumption, not all lines of $\mathcal{A}$ belong to the same pencil, so there is a subarrangement of three lines $\mathcal{L}_i$, $\mathcal{L}_j$, and $\mathcal{L}_k$, that is uniform. Every other line in $\mathcal{A}$ must belong to the pencil determined by a pair of lines in this subarrangement, otherwise there would be four lines forming a uniform subarrangement. If two of the additional lines indexed by l, m belong to different pencils given by say $\mathbf{p}_{i,j}$ and $\mathbf{p}_{i,k}$, then the subarrangement given by j, k, l, m is uniform, and we obtain a contradiction. ∎

Proof of Theorem 1.4 Point (i) is a consequence of Theorem 6.1 and Lemma 6.5. Point (ii) is contained in Theorem 6.9. ∎

7 Application to Tropical Lines in Tropical Surfaces

In the space of tropical surfaces of degree d in $\mathbb{T}P^3$, there exists an open subset of surfaces that contain (potentially infinitely many) tropical lines. In this section we prove that a generic non-singular tropical surface S in $\mathbb{R}^3$ of degree three contains finitely many tropical lines L such that the pair (S, L) is approximable. In addition, a generic non-singular tropical surface S of degree greater than 3 contains no tropical lines L such that the pair (S, L) is approximable. Therefore, if we restrict to approximable lines, the tropical situation is analogous to the classical algebro-geometric one.

Examples of generic non-singular tropical surfaces in $\mathbb{R}^3$ of any degree containing infinitely many lines were first constructed by Vigeland in [Vig09]. In [Vig], Vigeland later classified by combinatorial type all tropical lines in generic non-singular tropical surfaces. In this entire section, we denote by Δ_d the simplex

$$\Delta_d = \text{Conv}\{(0,0,0), (d,0,0), (0,d,0), (0,0,d)\},$$
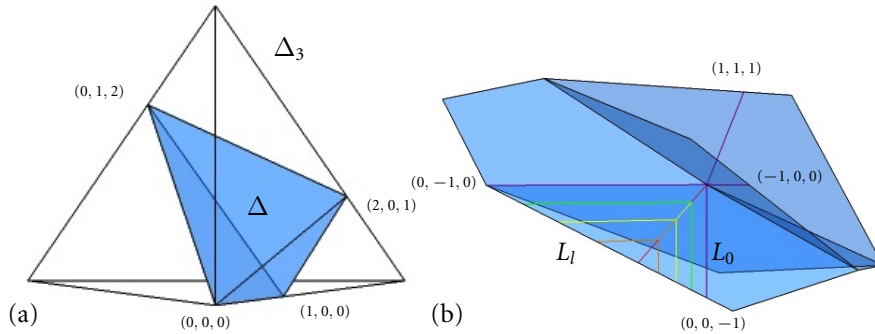
and its facets by $F_1, \ldots, F_4$.

Figure 19: (a) A pathological 3-simplex Δ from Theorem 7.2 drawn inside Δ_3. (b) The vertex of the surface dual to Δ and the infinite family of tropical lines. By Theorem 7.2 only L_0 is approximable in S.

7.1 1-parametric Families of Lines

A simplex $\Delta \subset \Delta_d$ with vertices in $\mathbb{Z}^3$ is said to be *d-pathological* if Δ is primitive and if Δ has one edge in $F_i \cap F_j$, one edge in F_k, and one edge in F_l for $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Theorem 7.1 (Vigeland, [Vig, Theorem 5.11]) *Let S be a generic non-singular tropical surface of degree $d \geq 3$ in $\mathbb{R}^3$ with Newton polytope Δ_d. If S contains a 1-parameter family of tropical lines, then there exists a vertex p of S, dual to a d-pathological simplex, such that any line in the family is contained in the fan $p + \mathrm{Star}_p(S)$.*

Conversely, any compact tropical surface S in $\mathbb{R}^3$ containing a d-pathological simplex in its dual subdivision contains infinitely many tropical lines.

Examples of tropical surfaces with infinitely many lines from [Vig09] were constructed using the *d*-pathological simplex with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, d-1)$, and $(d-1, 0, 1)$. It is easy to see that any *d*-pathological simplex defines a tropical fan containing a one-parameter family of tropical lines. This family consists of a unique 4-valent line, and the remaining lines have two 3-valent vertices. Here we prove that among all of these families, there is only a single line that is approximable. Before giving the rigourous statement, lets us first describe in detail tropical fans with Newton polygon a *d*-pathological simplex.

Let Δ be such a *d*-pathological simplex with $d \geq 3$. Without loss of generality, we can suppose that Δ has one edge in $F_1 \cap F_2$. Hence either Δ has one edge in $F_3 \cap F_4$, or Δ has one edge in F_3 and the last one in F_4. The first case is impossible, since Δ would not be primitive. Hence, up to permutation of the coordinates, the vertices of Δ are

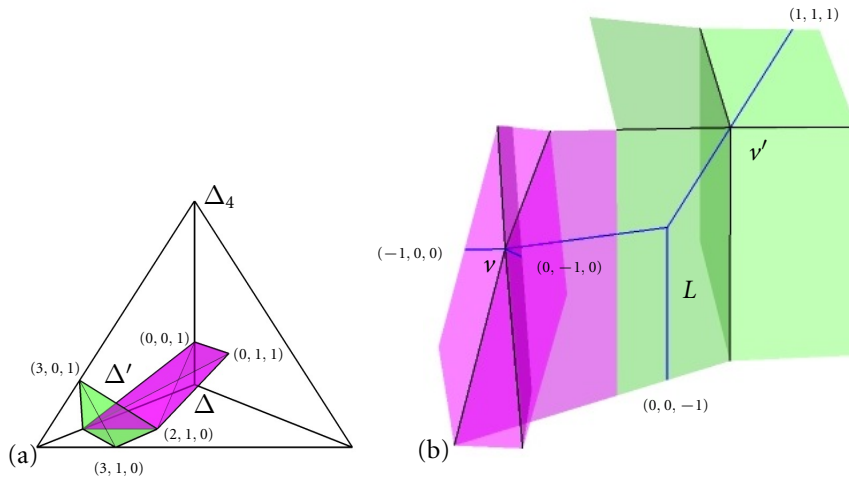$$(0, 0, 0), \quad (1, 0, 0), \quad (0, d - \alpha, \alpha), \quad (d - \beta - \gamma, \beta, \gamma)$$

Figure 20: (a) A pair (Δ, Δ') of type I in Δ_4.   (b) The corresponding vertices v, v' along with the isolated line L.

with the conditions that

$$0 < \alpha < d, \quad 0 < \beta + \gamma < d, \quad \begin{vmatrix} 1 & 0 & d - \beta - \gamma \\ 0 & d - \alpha & \beta \\ 0 & \alpha & \gamma \end{vmatrix} = \gamma(d - \alpha) - \alpha\beta = 1,$$

to ensure that Δ is primitive. See Figure 19 for an example.

Let S be a tropical surface in $\mathbb{R}^3$ with Newton polygon Δ. Without loss of generality, we can assume that the vertex of S is the origin. By computation we get that S has four rays with the four following primitive outgoing directions

$$u_0 = (0, \alpha, \alpha - d), \quad u_1 = (0, -\gamma, \beta),$$
$$u_2 = \big(-1, -\alpha(d - \beta - \gamma), (d - \alpha)(d - \beta - \gamma)\big),$$
$$u_3 = \big(1, \gamma + \alpha(d - \beta - \gamma - 1), -\beta - (d - \alpha)(d - \beta - \gamma - 1)\big).$$

The tropical surface S contains the following one parameter family of tropical lines $(L_l)_{l \in \mathbb{R}_{\geq 0}}$: the tropical line L_l has one vertex V_1 at $(0, 0, 0)$ adjacent to three rays with outgoing directions

$$U_1 = (0, -1, -1), \quad U_2 = (-1, 0, 0), \quad \text{and} \quad U_3 = (1, 1, 1),$$

and another vertex V_2 at $(0, -l, -l)$ adjacent to three rays with outgoing directions

$$-U_1, \quad U_4 = (0, -1, 0), \quad \text{and} \quad U_5 = (0, 0, -1).$$

If $l = 0$, then L_0 is a tropical line with one 4-valent vertex. The tropical line L_l is indeed in S since we have (see Figure 19)

$$U_1 = (\beta + \gamma)u_0 + du_1, \qquad U_2 = (d - \beta - \gamma)u_0 + u_2,$$
$$U_3 = (d - 1)u_2 + du_3, \qquad U_4 = \beta u_0 + (d - \alpha)u_1, \qquad U_5 = \gamma u_0 + \alpha u_1.$$

Theorem 7.2 *Let $S \subset (\mathbb{C}^*)^3$ be an algebraic surface with Newton polytope a pathological d-simplex Δ. The tropical line $\mathrm{Star}_{(0,0,0)}(L_l) \subset S$ is approximable by a complex algebraic line $\mathcal{L}_l \subset S$ if and only if $l = 0$ and S has Newton polytope*

$$\mathrm{Conv}\big\{(0,0,0),(1,0,0),(0,1,2),(2,0,1)\big\}.$$

In the case of 3-valent tropical lines, two instances of Theorem 7.2 were already known, namely the cases

$$(\alpha, \beta, \gamma) = (d-1, 0, 1) \qquad ([\text{BK12}]) \quad \text{and}$$
$$(\alpha, \beta, \gamma) = (1, d-2, 1) \qquad ([\text{Sha13}]; \text{see also Example } 3.11).$$

Proof The case $l > 0$ follows from initial degeneration and the classification given in Theorem 6.1. Suppose now that $l = 0$. We have that $C^2 = -(\beta + \gamma)(d-1) + 1$, which implies that

$$d + C^2 - \sum_{e \in \mathrm{Edge}(L_l)} w_e + 2 = -(d-1)(\beta + \gamma - 1).$$

Hence, if $\beta + \gamma > 1$, then the result follows from Theorem 1.3. If $\beta + \gamma = 1$, then since $\gamma(d - \alpha) - \alpha\beta = 1$ we deduce that $\beta = 0$ and $\gamma = 1$, and so $\alpha = d - 1$, and these are the Vigeland lines. The result now follows from Corollary 5.4. ∎

7.2 Isolated Lines

A pair (Δ, Δ') of simplices Δ and Δ' contained in Δ_d and with vertices in $\mathbb{Z}^3$ is said to be *d-pathological* if

(a) Δ and Δ' are primitive and intersect along a common edge e;
(b) Δ has 2 edges distinct from e and contained in the faces F_i and F_j of Δ_d;
(c) one of the two situations occurs:
   - (c1) the edge e is contained in F_k, and the opposite edge of Δ' is contained in F_l; in this case we say that the pair (Δ, Δ') is of type *I* (see Figure 20);
   - (c2) the polytope Δ' has a face F containing e and intersecting F_k and F_l; in this case we say that the pair (Δ, Δ') is of type *II* (see Figure 21);
(d) the set $\{i, j, k, l\}$ is equal to the set $\{1, 2, 3, 4\}$.

Isolated lines on a generic non-singular tropical surface of degree $d \geq 4$ in $\mathbb{R}^3$ have also been classified by combinatorial type by Vigeland.

Theorem 7.3 (Vigeland, [Vig, Theorem 5.11]) *Let S be a generic non-singular tropical surface of degree $d \geq 4$ in $\mathbb{R}^3$ with Newton polytope Δ_d. If S contains an isolated tropical line L, then S contains two vertices v and v', respectively dual to the simplices Δ and Δ', such that the pair (Δ, Δ') is d-pathological, and such that L has one vertex at v and*

(i) *passes through v' if (Δ, Δ') is of type I;*
(ii) *has another vertex on the edge of S dual to F if (Δ, Δ') is of type II.*

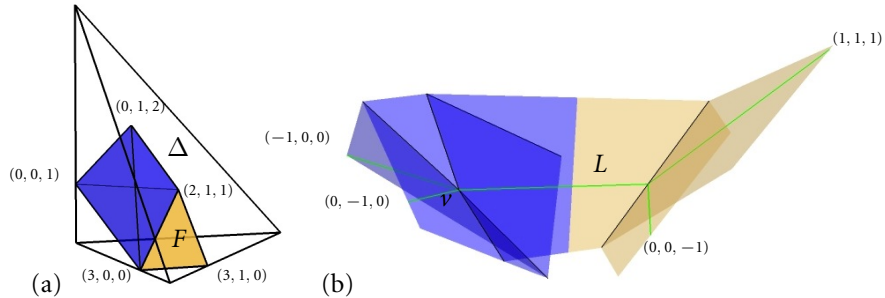According to the next theorem, none of the tropical lines of Theorem 7.3 is approximable in S.

Figure 21: (a) A pathological pair (Δ, Δ') of type *II* in Δ_4.   (b) The corresponding vertex and edge of the dual surface along with the isolated line *L*.

Theorem 7.4   *Let S be a generic non-singular tropical surface of degree $d \geq 4$ in $\mathbb{R}^3$ with Newton polytope Δ_d, and let $(\mathcal{S}_t)$ be a one-parameter family of algebraic surfaces in $(\mathbb{C}^*)^3$ with Newton polytope Δ_d such that*

$$\lim_{t \to +\infty} \mathrm{Log}_t(\mathcal{S}_t) = S.$$

Suppose that the tropical surface S contains a tropical line L. Then there does not exist a 1-parametric family of lines $\mathcal{L}_t \subset \mathcal{S}_t$ such that

$$\lim_{t \to +\infty} \mathrm{Log}_t(\mathcal{L}_t) = L.$$

Proof   Suppose that such a one-parameter family of lines $(\mathcal{L}_t)$ exists. It follows from Theorem 7.2 that the tropical line *L* is isolated. Hence the surface *S* contains two vertices v and v', respectively dual to the simplices Δ and Δ', such the pair (Δ, Δ') is *d*-pathological, and the line *L* is as described in Theorem 7.3. By initial degeneration, the family $(\mathcal{S}_t, \mathcal{L}_t)$ produces an approximation of the pair $(\mathrm{Star}_v(S), \mathrm{Star}_v(L))$ by a constant family. Let us denote by u_0, u_1, u_2, and u_3 the primitive integer directions of the rays of $\mathrm{Star}_v(S)$. Since *L* is isolated the fan tropical curve $\mathrm{Star}_v(L)$ cannot be equal to the tropical stable intersection of $\mathrm{Aff}(\mathrm{Star}_v(L))$ and $\mathrm{Star}_v(S)$. Moreover, *L* only has edges of weight 1, so according to Theorem 6.1 the curve $\mathrm{Star}_v(L)$ has degree two in $\mathrm{Star}_v(S)$. Without loss of generality, we can assume that the 4 rays of $\mathrm{Star}_v(S)$ satisfy one of the two cases:

Case 1: $2u_0 + u_3 = (-1, 0, 0)$, $2u_1 + u_2 = (0, -1, 0)$, and $u_2 + u_3 = (1, 1, 0)$,
Case 2: $2u_0 + u_1 = (-1, 0, 0)$, $u_1 + u_3 = (0, -1, 0)$, and $2u_2 + u_3 = (1, 1, 0)$.

See Figure 22 for the dual polytope Δ in each case.

   Case 1 corresponds to the situation when the two edges of Δ contained in the faces F_i, F_j are opposite edges of Δ and Case 2 is when these two edges are adjacent; see Figure 22.
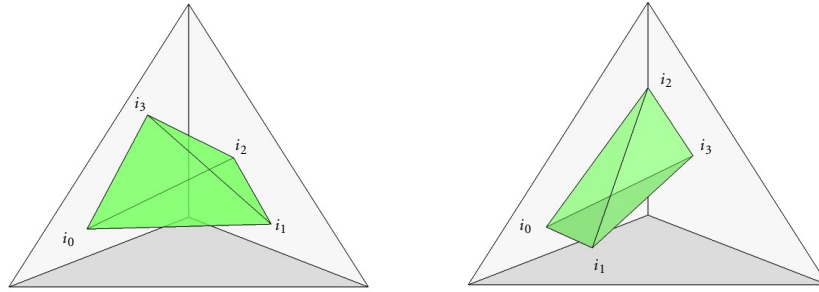
Figure 22: The two types of polytopes $\Delta \subset \Delta_d$ dual to the vertex v yielding the lines $\mathrm{Star}_v(L)$. Case 1 is on the left and Case 2 on the right.

In Case 1 we can set

$$u_0 = (a, b, c), \qquad\qquad u_1 = (-a - 1, -b - 1, -c),$$
$$u_2 = (2a + 2, 2b + 1, 2c), \qquad u_3 = (-2a - 1, -2b, -2c).$$

Since Δ is primitive, any three vectors among u_0, u_1, u_2, and u_3 form a basis for the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$, hence

$$\pm 1 = \begin{vmatrix} a & -a - 1 & -2a - 1 \\ b & -b - 1 & -2b \\ c & -c & -2c \end{vmatrix},$$

which gives $c = \pm 1$.

Denote by i_0, i_1, i_2, and i_3 the vertices of Δ in such a way that i_j is dual to the region of $\mathbb{R}^3 \setminus \mathrm{Star}_v(S)$ that contains the vector $-u_j$. The edge e of Δ is dual to the face of $\mathrm{Star}_v(S)$ generated by u_2 and u_3, so that $e = [i_0; i_1]$. Then we have

$$i_0 = (\alpha_0, 0, \gamma_0), \quad i_1 = (0, \beta_1, \gamma_1), \quad i_2 = (0, \beta_2, \gamma_2), \quad \text{and} \quad i_3 = (\alpha_3, 0, \gamma_3).$$

The edge e is orthogonal to u_2 and u_3, so it has direction $(-2c, 2c, 2a - 2b + 1)$. Since $c = \pm 1$, the direction of e is $(-2, 2, \pm(2a - 2b + 1))$. From this we deduce that

$$i_0 = (2, 0, \gamma_0) \quad \text{and} \quad i_1 = (0, 2, \gamma_0 \pm (2a - 2b + 1)).$$

Now we have to distinguish the two cases depending on the type of the pair (Δ, Δ').

If (Δ, Δ') is of type *I*, then the edge e is contained in a face of Δ_d, which is neither $\{x = 0\}$ nor $\{y = 0\}$. Moreover, a, b are integers so $2b - 2a - 1 \neq 0$ for any choice of a, b. Therefore e cannot be contained in the face $\{z = 0\}$, and neither can it be contained in $\{x + y + z = d\}$ by symmetry.

If (Δ, Δ') is of type *II*, then up to a change of coordinates we have

$$i_0 = (2, 0, 0), \quad i_1 = (0, 2, d - 2), \quad i_2 = (0, \beta_2, \gamma_2), \quad \text{and} \quad i_3 = (\alpha_3, 0, \gamma_3).$$

Moreover, the third vertex of the face F of Δ' has coordinates $(\alpha, d - \alpha, 0)$. Let us denote by (β, γ, δ) the coordinates of the fourth vertex of Δ'. Since the polytope Δ'

is primitive, we must have

$$\pm 1 = \begin{vmatrix} -2 & \alpha - 2 & \beta - 2 \\ 2 & d - \alpha & \gamma \\ d - 2 & 0 & \delta \end{vmatrix} = (d-2)C,$$

where $C \in \mathbb{Z}$ is some constant. Therefore, we obtain $d = 3$, so no isolated lines of this type exist when $d \geq 4$.

Now consider Case 2. Here we can set

$$u_0 = (a, b, c), \qquad\qquad u_1 = (-2a - 1, -2b, -2c),$$
$$u_2 = (-a, -b + 1, -c), \qquad u_3 = (2a + 1, 2b - 1, 2c).$$

By a calculation of the determinant of the vectors u_0, u_1, u_2 we find again that $c = \pm 1$. Again the edge e is orthogonal to both u_2, u_3 so that it has direction $(c, -c, b - a - 1)$. Since $c = \pm 1$, this becomes $(1, -1, \pm(b - a - 1))$.

In this case the vertices of the polytope Δ have coordinates:

$$i_0 = (\alpha_0, 0, \gamma_1), \quad i_1 = (\alpha_1, \beta_1, \gamma_1), \quad i_2 = (0, 0, \gamma_2), \quad i_3 = (0, \beta_3, \gamma_3).$$

The edge $[i_0; i_2]$ is orthogonal to both u_1, u_3 by tropical duality. Hence this edge has direction $(\pm 2, 0, 1 + 2a)$, so that $\alpha_0 = 2$.

Suppose the pair (Δ, Δ') is of type *I*. By a change of coordinates we can assume that the edge e lies in the face $\{z = 0\}$. Hence $\gamma_0 = \gamma_1 = 0$, and the edge e must have direction $(1, -1, 0)$. Therefore, $i_0 = (2, 0, 0)$ and $i_1 = (1, 1, 0)$. Now, Δ' has vertices $i_0, i_1, (\alpha, \beta, d - \alpha - \beta)$, and $(\gamma, \delta, d - \gamma - \delta)$. The polytope Δ' is primitive, so

$$\pm 1 = \begin{vmatrix} -1 & \alpha - 2 & \gamma - 2 \\ 1 & \beta & \delta \\ 0 & d - \alpha - \beta & d - \gamma - \delta \end{vmatrix} = (d-2)C,$$

where $C \in \mathbb{Z}$ is some constant. This implies $d = 3$, so again there are no isolated lines of this type for $d \geq 4$.

Finally, if (Δ, Δ') is of type *II*, then i_0 is in $\{y = z = 0\}$ and i_1 is in $\{x + y + z = d\}$, so we have $i_0 = (2, 0, 0)$ and $i_1 = (1, 1, d - 2)$. Now Δ' has two other vertices $(\alpha, d - \alpha, 0)$ and (β, γ, δ), and since Δ' is primitive, we have

$$\pm 1 = \begin{vmatrix} -1 & \alpha - 2 & \beta - 2 \\ 1 & d - \alpha & \gamma \\ d - 2 & 0 & \delta \end{vmatrix} = (d-2)C,$$

where $C \in \mathbb{Z}$ is some constant. Once again we obtain $d = 3$, and there are no isolated lines of this type for $d \geq 4$. This completes the proof. ∎

Proof of Theorem 1.8 According to [Vig, Theorem 5.6], such a generic tropical surface contains finitely many isolated lines and finitely many 1-parametric families of tropical lines. Now the result follows from initial degeneration, Theorems 7.1–7.4. ∎

7.3 Singular Tropical Lines

To conclude this paper, let us point out a strange phenomenon in tropical geometry that we were not aware of before starting this investigation. Let S be a generic non-singular tropical surface of degree $d \geq 3$ in $\mathbb{T}P^3$ that has a vertex dual to a d-pathological simplex with $(\alpha, \beta, \gamma) \neq (d-1, 0, 1)$ and $(1, 1, 0)$. Then any tropical line L in the corresponding 1-parameter family is singular when considered as a tropical curve in S. Indeed the self-intersection of such a line is $-(\beta + \gamma)(d-1) + 1$ in S, which is far from being equal to $2 - d$, which is the self-intersection of a complex algebraic line in a complex algebraic surface of degree d in $\mathbb{C}P^3$. In particular, this means that none of these tropical lines satisfy the adjunction formula in S; *i.e.,* they are singular.

It is possible, however, to give a sufficient condition for a tropical curve in a non-singular tropical surface to behave as a non-singular curve. If a tropical curve C in a non-singular compact tropical surface X can be locally given by a fan curve of degree one in a tropical plane, then it is shown in [Sha, Section 3.4.6] that the curve satisfies the tropical adjunction formula, where the genus of the curve given by the first Betti number of C. However, this condition is not necessary for a curve to satisfy the adjunction formula.

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