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ON THE ERROR ESTIMATE FOR CUBATURE ON WIENER SPACE

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Abstract It was pointed out by Crisan and Ghazali that the error estimate for the cubature on Wiener space algorithm developed by Lyons and Victoir requires an additional assumption on the drift. In this paper we demonstrate that it is straightforward to adopt the analysis of Kusuoka to obtain a general estimate without an additional assumptions on the drift. In the process we slightly sharpen the bounds derived by Kusuoka.

Keywords: cubature; stochastic differential equations; signature

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1. Introduction

In pricing and hedging financial derivatives as well as in assessing the risk inherent in complex systems we often have to find approximations to expectations of functionals of solutions to stochastic differential equations (SDEs). We consider a Stratonovich stochastic differential equation

$$\mathrm{d}\xi_{t,x} = V_0(\xi_{t,x})\,\mathrm{d}t + \sum_{i=1}^{d} V_i(\xi_{t,x}) \circ \mathrm{d}B_t^i, \quad \xi_{0,x} = x, \tag{1.1}$$

defined by a family of smooth vector fields V_i and driven by Brownian motion. It is well known that computing $P_{T-t}f := E(f(\xi_{T-t,x}))$ corresponds to solving a parabolic partial differential equation (PDE). The cubature on Wiener space method developed by Lyons and Victoir in [**13**], following Kusuoka [**5**] (in the following also referred to as the KLV method), is a high-order particle method for approximating the weak solution of stochastic differential equations in Stratonovich form. To obtain high-order error bounds, the test functions are assumed to be Lipschitz, and the vector fields defining the SDE satisfy Kusuoka's UFG condition (see [**6**]), which is a weaker assumption than the usual uniform Hörmander condition.

High-order particle methods have since been shown to be highly effective in practice (see, for example, [**14**,**15**]) and further extensions and applications of cubature on Wiener space have been developed by various authors. Applications include the nonlinear filtering problem [**2**], stochastic backward differential equations [**3**, **4**], calculating Greeks by cubature methods [**17**] and extending the KLV method by adding recombination [**10**]. It was pointed out in [**2**] that the analysis of the error bounds in [**13**] requires an additional assumption on the drift (see Definition 2.5) and thus the question of whether this additional assumption is necessary to derive high-order error bounds was raised.

We first give a brief introduction to cubature on Wiener space and outline how the need for an additional assumption on the drift arises in [**13**]. Then, based on [**9**], we demonstrate carefully how the analysis in [**7**] can be adopted to derive similar bounds for cubature on Wiener space. We show, for the KLV method based on a cubature measure of degree m over a k step partition $\mathcal{D}$, that the error $E_{\mathcal{D}}$ can be bounded by

$$E_{\mathcal{D}} := \sup_{x\in\mathbb{R}^N} |E_P f(\xi_{x,s}) - E_Q f(\xi_{x,s})| \leqslant C\biggl(\sum_{j=m+1}^{2m} s^{j/2}\|f\|_{V,j} + s^{(m+1)/2}\|\nabla f\|_\infty\biggr)$$

for any $s \in (0,1]$. Note that these bounds do not contain any higher-order derivatives in the direction of the drift V_0 and, although our proof contains many elements of the analysis of a version of Kusuoka's algorithm carried out in [**7**], we obtain slightly sharper error bounds in the process involving $2m$ instead of m^{m+1} derivatives. For suitable families of partitions (first considered in [**5**]) the error bounds immediately lead to convergence of order $(m-1)/2$ in the number of time steps in the partition. Finally, we clarify the relation of the KLV method to the version of Kusuoka's algorithm analysed in [**7**].

2. Cubature measures

Let $C_{\mathrm{b}}^\infty(\mathbb{R}^N,\mathbb{R}^N)$ denote the smooth bounded $\mathbb{R}^N$-valued functions whose derivatives of all orders are bounded. Then $V_i = (V_i^1,\dots,V_i^N) \in C_{\mathrm{b}}^\infty(\mathbb{R}^N,\mathbb{R}^N)$, $0 \leqslant i \leqslant d$, may be identified with smooth vector fields on $\mathbb{R}^N$. We denote by $C_0^0([0,T],\mathbb{R}^d)$ the continuous $\mathbb{R}^d$-valued paths starting at the origin and by $C_{0,\mathrm{bv}}^0([0,T],\mathbb{R}^d)$ the set of those paths that have in addition bounded variation. Let $B = (B_t^1,\dots,B_t^d)$ be a Brownian motion and let $B_t^0(t) = t$. Let $\xi_{t,x}$, $t \in [0,T]$, $x \in \mathbb{R}^N$, be a version of the solution of the Stratonovich SDE (1.1) that coincides with the pathwise solution on continuous paths of bounded variation (recall that the set of bounded variation paths have zero Wiener measure). We define the Itô functional $\Phi_{T,x}\colon C_0^0([0,T],\mathbb{R}^d) \to \mathbb{R}^N$ by

$$\Phi_{T,x}(\omega) = \xi_{T,x}(\omega).$$

The particular choice for the version of the SDE solution when defining $\xi_{t,x}$ implies that the Itô functional for a bounded variation path ω coincides with the usual ODE solution of (1.1) along the path ω.

Define the set of all multi-indices A by $A = \bigcup_{k=0}^\infty \{0,\dots,d\}^k$ and let $\alpha = (\alpha_1,\dots,\alpha_k) \in A$ be a multi-index. Furthermore, we define a degree on a multi-index α by $\|\alpha\| = k + \mathrm{card}\{j\colon \alpha_j = 0\}$ and

$$A(j) = \{\alpha \in A\colon \|\alpha\| \leqslant j\}.$$

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Let $A_1 = A \setminus \{\emptyset, (0)\}$ and $A_1(j) = \{\alpha \in A_1 \colon \|\alpha\| \leqslant j\}$. Following [**7**], we inductively define a family of vector fields indexed by A by taking

$$V_{[\emptyset]} = 0, \quad V_{[i]} = V_i, \quad 0 \leqslant i \leqslant d,$$
$$V_{[(\alpha_1,\dots,\alpha_k,i)]} = [V_{[\alpha]}, V_i], \quad 0 \leqslant i \leqslant d,\ \alpha \in A.$$

Moreover, let $V_\alpha = V_{\alpha_1} \cdots V_{\alpha_k}$, where the composition is taken in the sense of differential operators. Finally, we define a family of semi-norms on the space of functions $C_{\mathrm{b}}^\infty(\mathbb{R}^N)$:

$$\|f\|_{V,k} = \sum_{j=1}^{k} \sum_{\substack{\alpha_1,\dots,\alpha_j \in A_1, \\ \|\alpha_1\|+\cdots+\|\alpha_j\|=k}} \|V_{[\alpha_1]} \cdots V_{[\alpha_j]} f\|_\infty.$$

It is important to note that these semi-norms contain no derivatives in the direction of V_0. For $V \in C_{\mathrm{b}}^\infty(\mathbb{R}^N; \mathbb{R}^N)$ we define the flow $\operatorname{Exp}(tV)(x)$ to be the solution of the autonomous ODE

$$\dot{X}(t,x) = V(X(t,x)), \quad t > 0, \qquad X(0,x) = x \in \mathbb{R}^N.$$

A cubature measure on a finite-dimensional measure space is a discrete positive measure that integrates polynomials up to a certain (finite) degree correctly (i.e. as under Wiener measure). Together with the Taylor approximation for error estimation, cubature is a classical and efficient approach to the numerical integration of sufficiently smooth functions. For the Wiener space setting [**13**], a cubature measure is a discrete measure supported on paths of bounded variation and the role of polynomials is taken by the analogous Wiener functionals (iterated Stratonovich integrals).

Definition 2.1. For fixed $T > 0$ we say that a discrete measure Q_T assigning positive weights $\lambda_1, \dots, \lambda_n$ to paths

$$\omega_j \in C^0_{0,\mathrm{bv}}([0,T], \mathbb{R}^d), \quad \text{with } \omega_j^0 = t,\ j = 1, \dots, n,$$

is a cubature measure of degree m if, for all $(i_1, \dots, i_k) \in A(m)$,

$$E\biggl(\int_{0<t_1<\cdots<t_k<T} \circ\, \mathrm{d}B^{i_1}_{t_1} \cdots \circ\, \mathrm{d}B^{i_k}_{t_k}\biggr) = \sum_{j=1}^{n} \lambda_j \int_{0<t_1<\cdots<t_k<T} \mathrm{d}\omega_j^{i_1}(t_1) \cdots \mathrm{d}\omega_j^{i_k}(t_k), \quad (2.1)$$

where the expectation is taken under Wiener measure.

By the scaling property of Brownian motion any cubature measure Q_T may be obtained from Q_1 by letting $\omega^j_{T,i}(t) = \sqrt{T}\omega^j_i(t/T)$, $j = 1, \dots, d$, and keeping the weights of Q_1.

Taylor expansions play a crucial role in the estimation of the error when we replace the original (Wiener) measure by a cubature measure. On Wiener space the bounds for sufficiently smooth functions are obtained by considering stochastic Taylor expansion. The following proposition is a sharpened version of Proposition 2.1 in [**13**].

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Lemma 2.2. *Let $f \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$, $m \in \mathbb{N}$. Then, for every $t > 0$,*

$$f(\xi_{t,x}) = \sum_{(\alpha_1,\dots,\alpha_k)\in A(m)} V_{\alpha_1}\cdots V_{\alpha_k} f(x) \int_{0<t_1<\cdots<t_k<t} \circ\, \mathrm{d}B_{t_1}^{\alpha_1}\cdots \circ\, \mathrm{d}B_{t_k}^{\alpha_k} + R_m(t,x,f). \tag{2.2}$$

And the remainder process $R_m(t,x,f)$ satisfies

$$\sup_{x\in\mathbb{R}^N} \sqrt{E(R_m(t,x,f)^2)} \leqslant C \sum_{j=m+1}^{m+2} t^{j/2} \sup_{(\alpha_1,\dots,\alpha_i)\in A(j)\setminus A(j-1)} \|V_{\alpha_1}\cdots V_{\alpha_i} f\|_\infty,$$

where C is a constant depending only on d and m.

Proof. By induction one can prove that the remainder $R_m(t,x,f)$ of the Stratonovich stochastic Taylor expansion is given by

$$R_m(t,x,f) = \sum_{\substack{(\alpha_2,\dots,\alpha_k)\in A(m)\\ (\alpha_1,\dots,\alpha_k)\notin A(m)}} \int_{0<t_1<\cdots<t_k<t} V_{\alpha_1}\cdots V_{\alpha_k} f(\xi_{t_1,x}) \circ \mathrm{d}B_{t_1}^{\alpha_1}\cdots \circ \mathrm{d}B_{t_k}^{\alpha_k}.$$

The proposition follows from an elementary calculation using the Itô formula (see [**9**] for details). □

The following lemma is the analogue of Proposition 2.2 for the cubature measures Q_T, and its proof may be found in [**13**].

Lemma 2.3. *Let $R_m(T,x,f)$ be the process defined in (2.2). Then we have*

$$\sup_{x\in\mathbb{R}^N} E_{Q_T}|R_m(T,x,f)| \leqslant C(d,m,Q_1) \sum_{j=m+1}^{m+2} T^{j/2} \sup_{(\alpha_1,\dots,\alpha_i)\in A(j)\setminus A(j-1)} \|V_{\alpha_1}\cdots V_{\alpha_i} f\|_\infty,$$

where C is a constant depending only on d, m and the length of the bounded variation paths in the support of the cubature measure Q_1.

The constant in Lemma 2.3 can in fact be made explicit (see [**2**, Example 4]). The expectations of the Taylor approximation $f(\xi_{t,x}) - R_m(s,x,f)$ defined in (2.2) under Wiener and cubature measures coincide by definition of the cubature measure. Hence, one may apply the triangle inequality to Lemmas 2.2 and 2.3 and deduce that

$$\sup_{x\in\mathbb{R}^N} |E(f(\xi_{t,x})) - E_{Q_T}(f(\xi_{t,x}))| \leqslant C \sum_{j=m+1}^{m+2} s^{j/2} \sup_{(\alpha_1,\dots,\alpha_i)\in A(j)\setminus A(j-1)} \|V_{\alpha_1}\cdots V_{\alpha_i} f\|_\infty. \tag{2.3}$$

In general, the right-hand side of the inequality in (2.3) is not sufficient to directly obtain a good error bound for the approximation of the expectation; in particular, if f is only assumed to be Lipschitz, the estimate appears useless. Therefore, instead of approximating

$$P_T f(x) := E(f(\xi_{T,x}))$$

in one step, we consider a partition $\mathcal{D}$ of the interval $[0,T]$,

$$t_0 = 0 < t_1 < \cdots < t_k = T$$

with $s_j = t_j - t_{j-1}$, and solve the problem over each of the smaller subintervals by applying the cubature method recursively. If τ and τ' are two path segments, we denote their concatenation by $\tau \otimes \tau'$. For the approximation we consider all possible concatenations of cubature paths over the subintervals, i.e. all paths of the form $\omega_{s_1,i_1} \otimes \cdots \otimes \omega_{s_k,i_k}$. We define a corresponding probability measure ν as

$$\nu = \sum_{i_1,\dots,i_k=1}^{n} \lambda_{i_1} \cdots \lambda_{i_k} \delta_{\omega_{s_1,i_1} \otimes \cdots \otimes \omega_{s_k,i_k}}.$$

The iterated cubature method may be interpreted as a Markov operator and, hence the error of the approximation of $P_T f$ by $E_\nu(f(\xi_{T,x}))$ is bounded above by the sum of the errors of the approximations over the subintervals. The error over each subinterval can in turn be bounded by applying (2.3) to $P_{T-t_i} f$ instead of f and exploiting the regularity of $P_{T-t_i} f$. The following result is a corollary to [**6**, **8**]; for a detailed proof see [**2**].

Corollary 2.4. *Suppose the family of vector fields V_i, $0 \leqslant i \leqslant d$, satisfy the UFG condition. If $f \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$ and $\alpha_1, \dots, \alpha_j \in A_1$, then*

$$\|V_{[\alpha_1]} \cdots V_{[\alpha_j]} P_s f\|_\infty \leqslant \frac{C s^{1/2}}{s^{(\|\alpha_1\| + \cdots + \|\alpha_j\|)/2}} \|\nabla f\|_\infty \tag{2.4}$$

for all $s \in (0,1]$, where C is a constant independent of s and f.

As the regularity estimates in the previous corollary do not hold in the V_0 direction but the Taylor-based estimates used to obtain (2.3) require a higher derivative in the V_0 direction, it was pointed out in [**2**] that the analysis in [**13**] requires an additional assumption on the drift. We state this as follows.

Definition 2.5 (V0 condition). A family of vector fields V_i, $0 \leqslant i \leqslant d$, satisfies the V0 condition if

$$V_0 = \sum_{\beta \in A_1(2)} u_\beta V_{[\beta]}$$

for some $u_\beta \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$.

The following theorem, from [**13**], is the main error estimate for the iterated cubature method.

Theorem 2.6. *Suppose that the vector fields satisfy the UFG and V0 conditions. Then*

$$\sup_{x \in \mathbb{R}^N} |P_T f(x) - E_\nu(f(\xi_{T,x}))| \leqslant C(T) \|\nabla f\|_\infty \left(s_k^{1/2} + \sum_{j=m}^{m+1} \sum_{i=1}^{k-1} \frac{s_i^{(j+1)/2}}{(T-t_i)^{j/2}} \right).$$

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As an immediate corollary one obtains (see [**13**, Example 14]) high-order convergence of the KLV method for suitable partitions of $[0, T]$.

Corollary 2.7. *Consider the family of partitions given by*

$$t_j = T\left(1 - \left(1 - \frac{j}{k}\right)^\gamma\right),$$

and let v_k denote the corresponding iterated cubature measures. Suppose the vector fields satisfy the UFG and V0 conditions. Then

$$\sup_{x \in \mathbb{R}^N} |P_T f(x) - E_{\nu_k}(f(\xi_{T,x}))| \leqslant C k^{-(m-1)/2} \|\nabla f\|_\infty,$$

where C is a constant independent of k and f.

In the rest of the paper we will derive similar bounds for the KLV method that do not require the additional V0 assumption on the drift.

3. Algebraic preliminaries: the free Lie algebra and the signature

In the following we adopt the notation of Lyons and Victoir [**13**]. Given a Banach space W, we define the tensor algebra of non-commutative polynomials over W by

$$T(W) := \bigoplus_{i=0}^{\infty} W^{\otimes i}.$$

Define $T^{(j)}(W)$ to be the quotient of $T(W)$ by the ideal $\bigoplus_{i=j+1}^{\infty} W^{\otimes i}$. We identify $T^{(j)}(W)$ with the subspace

$$T^{(j)}(W) = \bigoplus_{i=0}^{j} W^{\otimes i}.$$

In the following we will not distinguish between the algebras of non-commutative polynomials and series, as we always work with their truncations. Let $\varepsilon_0, \dots, \varepsilon_d$ be a fixed orthonormal basis for $\mathbb{R} \oplus \mathbb{R}^d$. Let $T(\mathbb{R}, \mathbb{R}^d)$ denote the tensor algebra of polynomials over $\mathbb{R} \oplus \mathbb{R}^d$ endowed with a grading that assigns degree 2 to ε_0 and degree 1 to the remaining generators (see [**13**] for the details of the definition).

Let $\lambda \in R$, $a = (a_0, a_1, \dots)$, $b = (b_0, b_1, \dots) \in T(\mathbb{R}, \mathbb{R}^d)$. Define a homogeneous scaling operation by

$$\langle \lambda, a \rangle := (a_0, \lambda a_1, \dots, \lambda^i a_i, \dots),$$

and the exponential and logarithm on $T(\mathbb{R}, \mathbb{R}^d)$ using the usual power series. Let π_j denote the natural projection of $T(\mathbb{R}, \mathbb{R}^d)$ onto the subspace $T^{(j)}(\mathbb{R}, \mathbb{R}^d)$.

We define a Lie bracket on $T(\mathbb{R}, \mathbb{R}^d)$ by $[a, b] = a \otimes b - b \otimes a$. Let $\mathcal{L}$ denote the free Lie algebra generated by $\mathbb{R} \oplus \mathbb{R}^d$ (see [**16**]). Then $\mathcal{L}$ is the space of linear combinations of finite sequences of Lie brackets of elements in $W = R \oplus \mathbb{R}^d$, i.e.

$$W \oplus [W, W] \oplus [W, [W, W]] \oplus \cdots .$$

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We call an element u of $\pi_j(\mathcal{L})$ a Lie polynomial of degree j and an infinite sequence of Lie brackets a Lie series. Note that $\pi_j(\mathcal{L}) \subseteq T^{(j)}(\mathbb{R},\mathbb{R}^d)$.

Words of the form $\varepsilon_\alpha := \varepsilon_{\alpha_1} \otimes \cdots \otimes \varepsilon_{\alpha_k}$, $\alpha \in A \cup \{\emptyset\}$, form a basis for $T(\mathbb{R},\mathbb{R}^d)$ (note that $\varepsilon_\emptyset := 1$). Following Kusuoka [**7**], for

$$w_i = \sum_{\alpha \in A} w_{i\alpha}\varepsilon_\alpha \in T(\mathbb{R},\mathbb{R}^d), \quad i = 1, 2,$$

we define an inner product and a norm $\|\cdot\|_2$ on $T(\mathbb{R},\mathbb{R}^d)$ by

$$(w_1, w_2) = \sum_{\alpha \in A \cup \{\emptyset\}} w_{1\alpha} w_{2\alpha}, \quad \|w_1\|_2 = (w_1, w_1)^{1/2}. \tag{3.1}$$

Note that, restricted to $T^{(j)}(\mathbb{R},\mathbb{R}^d)$, all norms are equivalent, as $T^{(j)}(\mathbb{R},\mathbb{R}^d)$ is finite dimensional when regarded as a vector space.

The map sending ε_i to V_i, $i = 0, \dots, d$, extends to a unique linear map on W and by the universality property of the tensor algebra extends to a unique homomorphism Γ from $T(\mathbb{R},\mathbb{R}^d)$ into the differential operators on $\mathbb{R}^N$. The restriction of Γ to $\mathcal{L}$ is a Lie map from $\mathcal{L}$ into the smooth vector fields on $\mathbb{R}^N$.

Finally, we collect a number of simple algebraic facts.

Lemma 3.1. *Let $w \in \mathcal{L}$. Then*

(i) *the homogeneous scaling $\langle t, \cdot\rangle$ commutes with* exp *and* log*,*

(ii) $\pi_m \log(\pi_m w) = \pi_m \log(w)$ *and* $\pi_m \exp(\pi_m w) = \pi_m \exp(w)$*,*

(iii) *Γ restricted to $\pi_m\mathcal{L}$ is a linear map of finite-dimensional vector spaces and hence commutes with expectations on $\pi_m\mathcal{L}$.*

Proof. Part (i) is obvious from the definition of log and exp as power series. Part (ii) follows from the fact that, for $a, b \in T(\mathbb{R},\mathbb{R}^d)$, $\pi_m(\pi_m(a)\pi_m(b)) = \pi_m(ab)$. □

For a path $\phi \in C^0_{0,\mathrm{bv}}([0,T],\mathbb{R}^d)$ with $\phi^0(t) = t$, $s, t \in [0,T]$, we define its signature (also known as the Chen series) $S_{s,t}\colon C^0_{0,\mathrm{bv}}([0,T],\mathbb{R}^d) \to T(\mathbb{R},\mathbb{R}^d)$ by

$$S_{s,t}(\phi) = \sum_{k=0}^{\infty} \int_{s<t_1<\cdots<t_k<t} \mathrm{d}\phi(t_1) \otimes \cdots \otimes \mathrm{d}\phi(t_k), \tag{3.2}$$

where the summation is to be interpreted as a direct sum. Using Stratonovich iterated integrals, we may define $S_{s,t}(\circ B)$, the random Stratonovich signature of a Brownian motion (under Wiener measure).

With these definitions in mind, we can restate condition (2.1) in the definition of a cubature measure as

$$E(\pi_m(S_{0,1}(\circ B))) = \sum_{j=1}^{n} \lambda_j \pi_m(S_{0,1}(\omega_j)). \tag{3.3}$$

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Chen's theorem (see, for example, [**13**]) tells us that $L_i := \pi_m(\log(S_{0,1}(\omega_i)))$ is a Lie polynomial. The measure $Q_{\mathcal{L}} = \sum_{j=1}^n \lambda_j \delta_{L_j}$ satisfies

$$E(\pi_m(S_{0,1}(\circ B))) = E_{Q_{\mathcal{L}}(\mathrm{d}L)}(\pi_m \exp(L)). \tag{3.4}$$

Conversely, for any Lie polynomials L_i there exist continuous bounded variation paths ω_i with log-signature L_i. Moreover, if $Q_{\mathcal{L}}$ satisfies (3.4), Q will satisfy (3.3), so the identities (3.3) and (3.4) are equivalent. The proof of Chen's theorem can be extended to show that $\log(S_{s,t}(\circ B))$ is a (random) Lie series (see, for example, [**11**]). Such arguments can be used to obtain small-time asymptotics of the solution of Stratonovich SDEs (see, for example, [**1**]).

Motivated by this discussion, and following [**13**], we make the following equivalent definition for a cubature measure on Wiener space.

Definition 3.2. Let $m \in \mathbb{N}$ and $Q_{\mathcal{L}} = \sum_{j=1}^n \lambda_j \delta_{L_j}$ with $\lambda_i > 0$ and $L_i \in \pi_m(\mathcal{L})$ for $i = 1, \dots, n$. We say $Q_{\mathcal{L}}$ is a cubature measure on Wiener space if and only if

$$E(S_{0,1}^{(m)}(\circ B)) = E_{Q_{\mathcal{L}}(\mathrm{d}L)} \pi_m \exp(L).$$

In the following we will sometimes, where no confusion arises, drop the reference to the integration variable L and write $E_{Q_{\mathcal{L}}}$ in place of $E_{Q_{\mathcal{L}}(\mathrm{d}L)}$. A cubature measure over a general time interval $[0, T]$ may be obtained from $Q_{\mathcal{L}}$ by homogeneously rescaling the Lie polynomial in its support and leaving the weights unchanged. We have

$$E(S_{0,T}^{(m)}(\circ B)) = E_{Q_{\mathcal{L}}(\mathrm{d}L)} \pi_m \exp(\langle \sqrt{T}, L \rangle).$$

4. Error estimate for the cubature approximation

In this section we derive our main error estimate and demonstrate that $P_T f$ can be approximated to high order by a cubature measure, and the bounds on the error do not involve any derivative in the V_0 direction (just its Lie brackets).

Theorem 4.1. *Let P denote the Wiener measure, and let Q denote a degree-m cubature measure supported on paths of bounded variation. Then*

$$\sup_{x \in \mathbb{R}^N} |E_P f(\xi_{x,s}) - E_Q f(\xi_{x,s})| \leqslant C \biggl(\sum_{j=m+1}^{2m} s^{j/2} \|f\|_{V,j} + s^{(m+1)/2} \|\nabla f\|_\infty \biggr)$$

for any $s \in (0, 1]$, $f \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$. The constant C depends on d, m, Q_1,

$$\sup_{\alpha \in A(m+2) \setminus A(m)} \|V_\alpha \operatorname{Id}(\cdot)\|_\infty$$

and

$$E_P \|\pi_k(\log S(\circ B))\|_2^k, \quad E_{Q_1} \|\pi_k(\log S(\circ B))\|_2^k, \quad 1 \leqslant k \leqslant 2m.$$

As an immediate consequence, by substituting [**13**, Proposition 3.2] by Theorem 4.1, we obtain the following error estimate for the KLV method that preserves its higher-order convergence.

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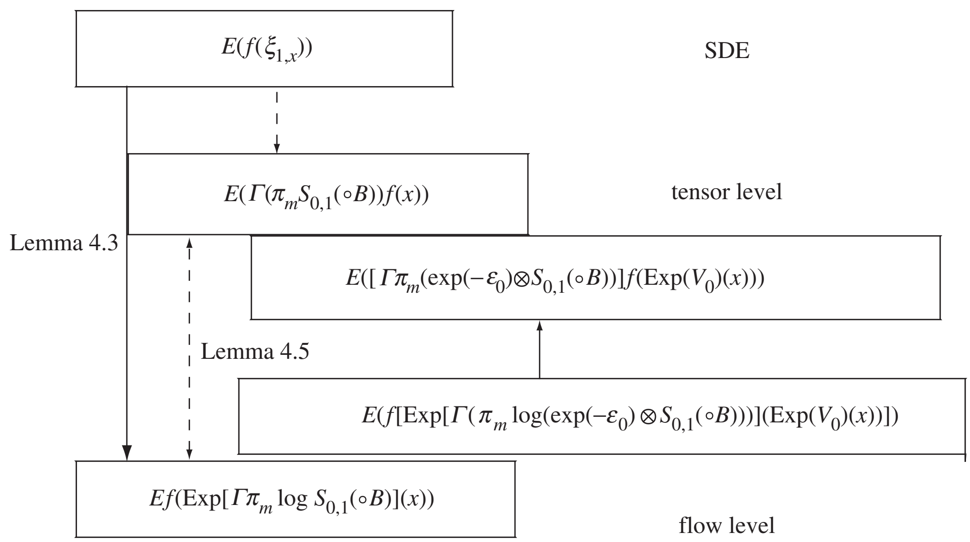

Figure 1. Structure of the approximations.

Corollary 4.2. *With the notation of Corollary 2.7, suppose the vector fields satisfy the UFG condition. Then*

$$\sup_{x\in\mathbb{R}^N} |P_T f(x) - E_\nu(f(\xi_{T,x}))| \leqslant C(T)\|\nabla f\|_\infty \left(s_k^{1/2} + \sum_{i=1}^{k-1}\left(\sum_{j=m+1}^{2m} \frac{s_i^{j/2}}{(T-t_i)^{(j-1)/2}} + s_i^{(m+1)/2} \right)\right)$$

and

$$\sup_{x\in\mathbb{R}^N} |P_T f(x) - E_{\nu_k}(f(\xi_{T,x}))| \leqslant Ck^{-(m-1)/2}\|\nabla f\|_\infty.$$

To prove the theorem we will adopt the analysis of Kusuoka [**7**] to the cubature in the Wiener space setting. In the process we sharpen the estimates slightly, allowing us to obtain a bound with at most $2m$ derivatives instead of m^{m+1} (cf. [**7**, Lemma 18]). Recall that Id is the identity function on $\mathbb{R}^N$ defined by $\mathrm{Id}(x) = x$.

Before going into technical details we give an interpretation of the ideas developed in [**7**], summarized in Figure 1. A stochastic Taylor expansion of $f(\xi_{x,s})$ can be written as $\Gamma(\pi_m(S_{0,1}(\circ B)))f(x)$, i.e. the differential operator obtained from the truncated signature under the map Γ acting on f at x. As the signature takes values in the tensor algebra, we may call the Taylor approximation the tensor level (approximation). It follows immediately from the definition of a degree-m cubature measure on Wiener space that the expectations of the degree-m Taylor approximation under P and Q are identical. Although the actual cubature step (exchanging the measures P and Q) has to take place at the tensor level, we cannot do it directly, as the error bounds would involve higher derivatives in the direction of V_0, which we have set out to avoid. Instead, we follow [**7**]

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and observe that the signature may be written as the exponential of the log signature, and by interchanging exp and Γ we obtain a new approximation at the level of the flow by $f(\mathrm{Exp}[\Gamma(\pi_m \log S_{0,1}(\circ B))](x))$. Lemma 4.5 formalizes this statement and allows us to move between the tensor algebra and the flow level. Crucially, at the level of flows, it suffices to approximate $\xi_{t,x}$ by $\hat{\xi}_{t,x}$ in the L^1 norm, as the bound for the approximation of $f(\hat{\xi}_{t,x})$ is only increased by a factor of $\|\nabla f\|_\infty$.

A key observation Kusuoka exploits is that if the Lie polynomial defining the flow does not involve an ε_0 component, the error bound for moving between flow and tensor level does not involve higher V_0 derivatives. By using a splitting argument at the level of flows (Lemma 4.4), Kusuoka can replace the log-signature by a Baker–Campbell–Hausdorff-style term that does not involve ε_0. This allows him to move to the tensor level and complete the approximation without using higher V_0 derivatives.

To apply Kusuoka's argument to cubature on Wiener space we will go through this approximation process (the solid lines in Figure 1) for both the Wiener measure and the cubature measure. By using the defining cubature identity (3.3), we will then be able to see that the approximations at the end of each chain agree and obtain the desired bound.

The following two lemmas may be found in [**7**, Corollaries 15 and 17] and we will state them without proof.

Lemma 4.3. *Let $m \geqslant 1$, then there exists $C > 0$ such that*

$$|E_P(f(\xi_{s,x})) - E_P\{f(\mathrm{Exp}[\Gamma\pi_m\langle\sqrt{s}, \log S_{0,1}(\circ B)\rangle](x))\}| \leqslant Cs^{(m+1)/2}\|\nabla f\|_\infty$$

for any $x \in \mathbb{R}^N$, $s \in (0,1]$ and $f \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$.

The second lemma is the splitting argument at the level of flows mentioned in the previous discussion.

Lemma 4.4. *Let $L^{(i)}$, $i = 1, 2$, denote two $\mathcal{L}$-valued random variables with*

$$E[\|\pi_k(L^{(i)})\|_2^k] < \infty \quad \text{for any } k \geqslant 1.$$

Then, for any $m \geqslant 1$ and $p \in [1,\infty)$, there exists $C > 0$ such that

$$\begin{aligned}\|\mathrm{Exp}(\Gamma\pi_m\langle\sqrt{s}, L^{(1)}\rangle)&(\mathrm{Exp}[\Gamma\pi_m\langle\sqrt{s}, L^{(2)}\rangle](x)) \\ &- \mathrm{Exp}[\Gamma(\pi_m \log(\exp\langle\sqrt{s}, L^{(2)}\rangle \otimes \exp\langle\sqrt{s}, L^{(1)}\rangle))](x)\|_{L^p} \leqslant Cs^{(m+1)/2}\end{aligned}$$

for all $s \in (0,1]$ and $x \in \mathbb{R}^N$.

Note that

$$\mathrm{Exp}(\Gamma\pi_m\langle\sqrt{s}, L^{(1)}\rangle)(\mathrm{Exp}[\Gamma\pi_m\langle\sqrt{s}, L^{(2)}\rangle](x))$$

is the composition of $\mathrm{Exp}(\cdot)$ functions.

The following lemma bounds the difference between flow and tensor approximation. It improves on [**7**] by considering a different truncation of the Taylor approximation.

Lemma 4.5. *Let $m \geqslant 2$ and suppose that $w \in \pi_m(\mathcal{L})$. Then, for any $f \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$, we have*

$$\sup_{x\in\mathbb{R}^N} |f(\mathrm{Exp}[\Gamma(w)](x)) - (\Gamma[\pi_m \exp(w)]f)(x)| \leqslant \sum_{j=2}^{m+1} \bigg\| \sum_{\substack{i_2+\cdots+i_j\leqslant m,\\ i_1+\cdots+i_j>m}} \frac{1}{(j-1)!}\Gamma(w_{i_1}\otimes\cdots\otimes w_{i_j})f\bigg\|_{\infty}.$$

If in addition w satisfies $(w,\varepsilon_0)=0$, there exists $C(w)>0$ independent of s and f such that

$$\sup_{x\in\mathbb{R}^N} |f(\mathrm{Exp}[\Gamma\langle\sqrt{s},w\rangle](x)) - (\Gamma\pi_m \exp(\langle\sqrt{s},w\rangle)f)(x)| \leqslant C(w)\sum_{j=m+1}^{2m} s^{j/2}\|f\|_{V,j}$$

for any $s\in(0,1]$.

Proof. Let $w=\sum_{i=1}^m w_i$, such that each $w_i\in(\pi_i-\pi_{i-1})(\mathcal{L})$ (i.e. each w_i is a homogeneous Lie polynomial of degree i). We first proceed as in [**7**, Proposition 9] by noting that $\Gamma(w)$ is a smooth vector field defined on all of $\mathbb{R}^N$. Thus, for any smooth f, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{Exp}(t\Gamma(w))(x)) = ((\Gamma w)f)(\mathrm{Exp}(t\Gamma(w))(x)).$$

Using this identity iteratively to expand $f(\mathrm{Exp}(t\Gamma(w))(x))$ in a Taylor expansion, one sees that

$$\begin{aligned} f(\mathrm{Exp}(t\Gamma(w))(x)) &- \sum_{j=0}^{m}\frac{t^j}{j!}\bigg(\Gamma\bigg(\sum_{i_1+\cdots+i_j=0}^{m} w_{i_1}\otimes\cdots\otimes w_{i_j}\bigg)\bigg)f(x) \\ &= \sum_{j=2}^{m+1}\sum_{\substack{i_2+\cdots+i_j\leqslant m,\\ i_1+\cdots+i_j>m}} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!}[\Gamma(w_{i_1}\otimes\cdots\otimes w_{i_j})f](\mathrm{Exp}[s\Gamma(w)](x))\,\mathrm{d}s. \end{aligned}$$

Setting $t=1$, we deduce that

$$\bigg| f(\mathrm{Exp}(\Gamma(w))(x)) - \sum_{j=0}^{m}\bigg(\Gamma\bigg(\sum_{i_1+\cdots+i_j=0}^{m}\frac{1}{j!} w_{i_1}\otimes\cdots\otimes w_{i_j}\bigg)\bigg)f(x)\bigg| \leqslant \sum_{j=2}^{m+1} \bigg\| \sum_{\substack{i_2+\cdots+i_j\leqslant m,\\ i_1+\cdots+i_j>m}} \frac{1}{(j-1)!}\Gamma(w_{i_1}\otimes\cdots\otimes w_{i_j})f\bigg\|_{\infty}.$$

Noting that

$$\pi_m\exp(w) = \sum_{j=0}^{m}\sum_{i_1+\cdots+i_j=0}^{m}\frac{1}{j!}w_{i_1}\otimes\cdots\otimes w_{i_j}$$

yields the first claim. The second follows by considering $\langle\sqrt{s},w\rangle$ in place of w and noting that under the assumption $(w,\varepsilon_0)=0$ the vector field V_0 does not appear on its own in the composition of the differential operators on the right-hand side of the last inequality. □

The last lemma is obtained by combining arguments from [**13**] and [**7**].

Lemma 4.6. *Let $t\in(0,1]$ and let Q_t be a cubature measure for Wiener space. Then, for any $x\in\mathbb{R}^N$,*

$$|E_{Q_t}(f(\xi_{t,x}))-E_{Q_1}\{f(\mathrm{Exp}[\Gamma\pi_m\langle\sqrt{t},\log S_{0,1}(\circ B)\rangle](x))\}|\leqslant Ct^{(m+1)/2}\|\nabla f\|_\infty$$

for all $f\in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$, where C is a constant independent of t and f.

Proof. Let $g\in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$. By Lemma 2.3, we may write

$$E_{Q_t}|g(\xi_{t,x})-\Gamma(\pi_m\exp(\log S_{0,t}(\circ B)))g(x)|\leqslant C\sum_{j=m+1}^{m+2}t^{j/2}\sup_{\alpha\in A(j)\setminus A(j-1)}\|V_\alpha g\|_\infty. \tag{4.1}$$

Letting $w=\pi_m\log S_{0,t}(\circ B)$ and applying Lemma 4.5, we see that

$$E_{Q_t}|g(\mathrm{Exp}[\Gamma w](x))-\Gamma(\pi_m\exp(w))g(x)|\leqslant C\sum_{j=m}^{2m}t^{j/2}\sup_{\alpha\in A(j)\setminus A(j-1)}\|V_\alpha g\|_\infty. \tag{4.2}$$

Combining (4.1) and (4.2) with g, the identity function, we see that

$$E_{Q_t}|\xi_{t,x}-\mathrm{Exp}[\Gamma\pi_m\log S_{0,t}(\circ B)](x)|\leqslant C\sum_{j=m}^{2m}t^{j/2}\sup_{\alpha\in A(j)\setminus A(j-1)}\|V_\alpha\,\mathrm{Id}\,\|_\infty$$

and the lemma follows. □

We are now ready to prove Theorem 4.1. Our proof is modelled on [**7**, Lemma 18]. We shall go through a sequence of approximations for the expectation of $f(\xi_{s,x})$ under the Wiener measure and the cubature measure. Finally, we shall show that the approximations at each end agree.

Proof of Theorem 4.1. Let $\mu_{1,s}=P$ be the Wiener measure on paths parametrized over $[0,s]$ and $\mu_{2,s}=Q_s$. From Lemmas 4.3 and 4.6 we see that, for $i=1,2$,

$$\sup_{x\in\mathbb{R}^N}|E_{\mu_{i,s}}(f(\xi_{s,x}))-E_{\mu_{i,1}}\{f(\mathrm{Exp}[\Gamma\pi_m\langle\sqrt{s},\log S_{0,1}(\circ B)\rangle](x))\}|\leqslant Cs^{(m+1)/2}\|\nabla f\|_\infty. \tag{4.3}$$

Let

$$L^{(1)}=\pi_m\log S_{0,1}^{(m)}(\circ B)\quad\text{and}\quad L^{(2)}=-\varepsilon_0.$$

It is well known (see, for example, [**11**]) that the log-signature of Brownian motion is a Lie series with probability 1. Also, we have $E[\|\pi_k(\log S_{0,1}(\circ B))\|_2^k]<\infty$. In fact, using the techniques of rough paths, a similar statement can be obtained at the level of paths.

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For example, in [**12**] Lyons and Sidorova compute the radius of convergence for the log signature.

Hence, Lemma 4.4 implies that, for $i = 1, 2$,

$$\begin{aligned} &|E_{\mu_{i,1}} f(\operatorname{Exp}(\Gamma\pi_m\langle\sqrt{s}, L^{(1)}\rangle)(\operatorname{Exp}[\Gamma\pi_m\langle\sqrt{s}, L^{(2)}\rangle](z))) \\ &\quad - E_{\mu_{i,1}} f(\operatorname{Exp}[\Gamma\pi_m \log(\exp\langle\sqrt{s}, L^{(2)}\rangle \otimes \exp\langle\sqrt{s}, L^{(1)}\rangle)](z))| \leqslant C s^{(m+1)/2} \|\nabla f\|_\infty. \end{aligned} \tag{4.4}$$

Writing

$$x = \operatorname{Exp}(\Gamma\langle\sqrt{s}, -\varepsilon_0\rangle)(z),$$

the inequality (4.4) becomes

$$\begin{aligned} &|E_{\mu_{i,1}} f(\operatorname{Exp}(\Gamma\pi_m\langle\sqrt{s}, L^{(1)}\rangle)(x)) \\ &\quad - E_{\mu_{i,1}} f(\operatorname{Exp}[\Gamma\pi_m \log(\exp\langle\sqrt{s}, L^{(2)}\rangle \otimes \exp\langle\sqrt{s}, L^{(1)}\rangle)](\operatorname{Exp}(\Gamma\langle\sqrt{s}, \varepsilon_0\rangle)(x)))| \\ &\qquad \leqslant C s^{(m+1)/2} \|\nabla f\|_\infty. \end{aligned} \tag{4.5}$$

Thus, combining inequalities (4.3) and (4.5) and using the triangle inequality, we see that

$$\begin{aligned} \sup_{x\in\mathbb{R}^N} &|E_{\mu_{i,s}} f(\xi_{s,x}) - E_{\mu_{i,1}} f(\operatorname{Exp}[\Gamma\pi_m \log(\exp(\langle\sqrt{s}, -\varepsilon_0\rangle) \\ &\qquad \otimes \langle\sqrt{s}, S_{0,1}(\circ B)\rangle)](\operatorname{Exp}(\Gamma\langle\sqrt{s}, \varepsilon_0\rangle)(x)))| \\ &\qquad \leqslant C s^{(m+1)/2} \|\nabla f\|_\infty. \end{aligned} \tag{4.6}$$

It follows from the Baker–Campbell–Hausdorff formula that

$$\pi_m \log(\exp(\langle\sqrt{s}, -\varepsilon_0\rangle) \otimes \langle\sqrt{s}, S_{0,1}(\circ B)\rangle) \tag{4.7}$$

has no ε_0 component, i.e.

$$(\pi_m \log(\exp(\langle\sqrt{s}, -\varepsilon_0\rangle) \otimes \langle\sqrt{s}, S_{0,1}(\circ B)\rangle), \varepsilon_0) = 0,$$

where the inner product is defined in (3.1).

Moreover, as the log signature of the Brownian motion is a Lie series with probability 1, (4.7) is a Lie polynomial.

Hence, we may apply Lemma 4.5 to inequality (4.6) and, once again using the triangle inequality, we obtain, for $i = 1, 2$,

$$\begin{aligned} \sup_{x\in\mathbb{R}^N} &|E_{\mu_{i,s}}(f(\xi_{s,x})) - E_{\mu_{i,1}}((\Gamma\pi_m \exp\langle\sqrt{s}, \pi_m \log(\exp(-\varepsilon_0) \otimes S_{0,1}(\circ B))\rangle f)(y))| \\ &\qquad \leqslant C\biggl(\sum_{j=m+1}^{2m} s^{j/2}\|f\|_{V,j} + s^{(m+1)/2}\|\nabla f\|_\infty\biggr), \end{aligned} \tag{4.8}$$

where $y = \mathrm{Exp}(\Gamma\langle\sqrt{s}, \varepsilon_0\rangle)(x) = z$. Note that in the previous step we have used the fact that the scaling operation $\langle s, \cdot\rangle$ commutes with log and exp. We have also used the fact that

$$\pi_m \exp\langle\sqrt{s}, \pi_m \log(\exp(-\varepsilon_0) \otimes S_{0,1}(\circ B))\rangle = \pi_m\langle\sqrt{s}, \exp(-\varepsilon_0) \otimes S_{0,1}(\circ B)\rangle.$$

Finally, using the cubature relation (3.3),

$$E_P(\pi_m S_{0,1}(\circ B)) = E_{Q_1}(\pi_m S_{0,1}(\circ B)),$$

and noting that the multiplication by a deterministic tensor can be taken out of the expectation, we have

$$E_P(\pi_m(\exp(-\varepsilon_0) \otimes S_{0,1}(\circ B))) = E_{Q_1}(\pi_m(\exp(-\varepsilon_0) \otimes S_{0,1}(\circ B))).$$

Hence, it follows that

$$E_P[(\Gamma\pi_m\langle\sqrt{s}, \exp(-\varepsilon_0) \otimes S_{0,1}(\circ B)\rangle f)(z)] = E_{Q_1}[(\Gamma\pi_m\langle\sqrt{s}, \exp(-\varepsilon_0) \otimes S_{0,1}(\circ B)\rangle f)(z)].$$

Using this identity in (4.8), a final application of the triangle inequality completes the proof of the theorem. □

Remark 4.7. The truncated log signatures of the cubature paths of a degree-m cubature measure satisfy the definition of an m-$\mathcal{L}$-moment-similar random variable of Kusuoka [**7**] with respect to the truncated log signature of the Brownian motion. Conversely, for any finite such family we can find paths that satisfy a degree-m cubature formula. The approximation operator for $P_s f$ whose error bounds are analysed in [**7**] can be written as

$$E_{Q_{\mathcal{L}}} f(\mathrm{Exp}[\Gamma\langle\sqrt{s}, \pi_m L\rangle](x))$$

and it is clear from our discussion that the same bounds as in Theorem 4.1 can be obtained for this approximation.

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