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ISOTROPIC IMMERSIONS

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0. Introduction. Recently, Ferus ([5], [6]) classified connected Riemannian manifolds with parallel second fundamental form in a real space form of constant curvature $\tilde{c} \geqq 0$. In this paper we may restrict our attention to isotropic submanifolds with parallel second fundamental form in the Euclidean sphere $S^m(k)$ of constant curvature k. Due to Ferus, we find that an isotropic submanifold with parallel second fundamental form in S^m is locally congruent to one of compact symmetric spaces of rank one and the immersion is locally equivalent to the second or the first standard immersion according as M is a sphere or not. In Section 2, we characterize the first standard immersion of a complex projective space into a sphere in terms of isotropic immersions. We have

THEOREM 1. *Let M be a real $2n(\geqq 4)$-dimensional connected Kaehler manifold of constant holomorphic sectional curvature and $\tilde{M}$ be a $(2n + p)$-dimensional real space form of curvature $\tilde{c} > 0$. If $p \leqq n^2$ and M is an isotropic submanifold of $\tilde{M}$, then $p = n^2 - 1$ or $p = n^2$. Moreover, M is one of the following*:

(i) *M is locally congruent to a complex projective space which is immersed in $\tilde{M}$ through the first standard minimal immersion.*

(ii) *M is locally congruent to a complex projective space which is immersed in some totally umbilical hypersphere of $\tilde{M}$ through the first standard minimal immersion.*

In Section 3, we classify submanifolds all of whose geodesics are circles in a complex projective space with codimension 2.

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1. Preliminaries. A Riemannian manifold of constant curvature is called a *real space form*. Let M be an n-dimensional submanifold of $\tilde{M}^{n+p}$ with metric g. We denote by ∇ and $\tilde{\nabla}$ the covariant differentiations on M and $\tilde{M}$, respectively. Then the second fundamental form B of the immersion is defined by

$$B(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y,$$

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where X and Y are vector fields tangent to M. We call

$$\mathfrak{H} = (1/n)(\mathrm{tr}\ B)$$

the *mean curvature vector* of M in $\tilde{M}$. The *mean curvature* H of M in $\tilde{M}$ is the length of $\mathfrak{H}$. If $\mathfrak{H}$ is identically zero, the submanifold M is said to be *minimal*. The submanifold M is *totally umbilic* provided that

$$B(X, Y) = g(X, Y)\mathfrak{H}$$

for all vector fields X and Y on M. In particular, if B vanishes identically, M is said to be a *totally geodesic* submanifold of $\tilde{M}$. For a vector field ξ normal to M, we write

$$\tilde{\nabla}_X\xi = -A_\xi X + D_X\xi,$$

where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. the normal) component of $\tilde{\nabla}_X\xi$. A normal vector field ξ is said to be *parallel* if $D_X\xi = 0$ for each vector field X tangent to M. The submanifold M of $\tilde{M}$ is called an *extrinsic sphere* if it is totally umbilic and has parallel mean curvature vector. We recall the notion of circles in a Riemannian manifold $\tilde{M}$. A curve $x(t)$ of $\tilde{M}$ parametrized by arc length t is called a *circle*, if there exists a field of unit vectors Y_t along the curve which satisfies, together with the unit vectors $X_t = \dot{x}(t)$, the differential equations:

$$\tilde{\nabla}_t X_t = kY_t \quad \text{and} \quad \tilde{\nabla}_t Y_t = -kX_t,$$

where k is a positive constant and $\tilde{\nabla}_t$ denotes the covariant differentiation $\tilde{\nabla}$ with respect to X_t. We call M a *circular geodesic* submanifold of $\tilde{M}$ provided that every geodesic in M is a circle in $\tilde{M}$. We define the covariant differentiation $\bar{\nabla}$ of the second fundamental form B with respect to the connection in (tangent bundle) + (normal bundle) as follows:

$$(\bar{\nabla}_X B)(Y, Z) = D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

The second fundamental form B is said to be *parallel* if

$$(\bar{\nabla}_X B)(Y, Z) = 0$$

for all tangent vector fields X, Y and Z on M. Let $\xi_1, \dots, \xi_p$ be an orthonormal basis of the normal bundle $T^\perp(M)$ and A_α be the second fundamental form with respect to ξ_α:

$$g(A_\alpha X, Y) = g(B(X, Y), \xi_\alpha).$$

$\|B\|$ is the length of the second fundamental form B of the immersion so that

$$\|B\|^2 = \sum_{\alpha=1}^{p} \mathrm{tr}\ A_\alpha^2.$$

The manifold M is said to be a (λ-)*isotropic* submanifold of $\tilde{M}$ provided that $\|B(X, X)\|$ is equal to a constant $(= \lambda)$ for all unit tangent vectors X at each point. Let R and $\widetilde{R}$ be the curvature tensors of M and $\widetilde{M}$, respectively. For later use, we write Gauss and Codazzi equations respectively:

$$(1.1)\quad g(R(X, Y)Z, W) = g(\widetilde{R}(X, Y)Z, W) + g(B(X, W), B(Y, Z)) - g(B(X, Z), B(Y, W)),$$

$$(1.2)\quad \{\widetilde{R}(X, Y)Z\}^{\perp} = (\overline{\nabla}_X B)(Y, Z) - (\overline{\nabla}_Y B)(X, Z),$$

where X, Y, Z and W are vector fields tangent to M and $\{*\}^{\perp} \Rightarrow \{*\}^{\perp}$ means the normal component of $\{*\} \Rightarrow \{*\}$.

Here we prepare without proof the following two theorems in order to prove Theorem 1. For orthonormal vectors $X, Y \in T_x(M)$, we denote by $K(X, Y)$ (resp. $\widetilde{K}(X, Y)$) the sectional curvature of the plane spanned by X and Y for M (resp. for $\widetilde{M}$) and we put

$$\Delta_{XY} = K(X, Y) - \widetilde{K}(X, Y).$$

We call Δ the discriminant at $x \in M$.

THEOREM 1.1 ([**10**]). *Let M^n be a $\lambda(> 0)$-isotropic submanifold in a Riemannian manifold $\widetilde{M}^{n+p}$. Assume that the discriminant Δ at $x \in M$ is constant. Then the following inequalities hold at x:*

$$-((n + 2)/2(n - 1))\lambda^2 \leqq \Delta \leqq \lambda^2.$$

Let N_x^1 be the first normal space at x of the above immersion, that is, the vector space spanned by all vectors $B(X, Y)$. Then we have

(1) $\Delta = \lambda^2 \Leftrightarrow M$ *is umbilic at* $x \Leftrightarrow \dim N_x^1 = 1$,

(2) $\Delta = -((n + 2)/2(n - 1))\lambda^2 \Leftrightarrow M$ *is minimal at* x $\Leftrightarrow \dim N_x^1 = n(n + 1)/2 - 1$,

(3) $-((n + 2)/2(n - 1))\lambda^2 < \Delta < \lambda^2 \Leftrightarrow \dim N_x^1 = n(n + 1)/2.$

THEOREM 1.2 ([**6**]). *Let M be an n-dimensional connected Kaehler manifold with complex structure J which is isometrically immersed into a real space form $\widetilde{M}^{2n+p}(\widetilde{c})$. Suppose that*

$$B(JX, JY) = B(X, Y) \quad \textit{for all } X, Y.$$

Then the second fundamental form of the immersion is parallel.

Let M be a Riemannian submanifold of a Kaehler manifold $\widetilde{M}$ with complex structure J. The submanifold M is called a *Kaehler* submanifold (resp. a *totally real* submanifold) of $\widetilde{M}$ if each tangent space of M is mapped into the tangent space of M (resp. the normal space of M) by the complex structure J. A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form.*

2. Proof of theorem 1. Let M be a real $2n$-dimensional Kaehler manifold with complex structure J of constant holomorphic sectional curvature $4c$. Then the equation of Gauss (1.1) is reduced to (2.1):

$$\begin{aligned}(2.1)\quad & g(B(X, Y), B(Z, W)) - g(B(Z, Y), B(X, W)) \\ &= c(g(X, Y)g(Z, W) - g(Z, Y)g(X, W) \\ &+ g(JX, Y)g(JZ, W) - g(JZ, Y)g(JX, W) \\ &+ 2g(Z, JX)g(JY, W)) - \tilde{c}(g(X, Y)g(Z, W) \\ &- g(Z, Y)g(X, W)).\end{aligned}$$

By assumption, all normal curvature vectors $B(X, X)$ at x have the same length, say, λ. Namely we have

$$g(B(X, X), B(X, X)) = \lambda^2 g(X, X)g(X, X),$$

which is equivalent to

$$\begin{aligned}(2.2)\quad & g(B(X, Y), B(Z, W)) + g(B(X, Z), B(Y, W)) \\ &+ g(B(X, W), B(Y, Z)) = \lambda^2(g(X, Y)g(Z, W) \\ &+ g(X, Z)g(Y, W) + g(X, W)g(Y, Z))\end{aligned}$$

for all vectors X, Y, Z and W tangent to M.

On the other hand, exchanging X and Y in (2.1), we see

$$\begin{aligned}(2.3)\quad & g(B(Y, X), B(Z, W)) - g(B(Z, X), B(Y, W)) \\ &= c(g(Y, X)g(Z, W) - g(Z, X)g(Y, W) \\ &+ g(JY, X)g(JZ, W) - g(JZ, X)g(JY, W) \\ &+ 2g(Z, JY)g(JX, W)) - \tilde{c}(g(Y, X)g(Z, W) \\ &- g(Z, X)g(Y, W)).\end{aligned}$$

Summing up (2.1), (2.2) and (2.3), we obtain

$$\begin{aligned}(2.4)\quad & g(B(X, Y), B(Z, W)) \\ &= (\lambda^2 + 2(c - \tilde{c}))/3 \cdot g(X, Y)g(Z, W) \\ &+ (\lambda^2 - (c - \tilde{c}))/3 \cdot (g(X, W)g(Y, Z) \\ &+ g(X, Z)g(Y, W)) + c(g(JX, Z)g(JY, W) \\ &+ g(JY, Z)g(JX, W)).\end{aligned}$$

Here we put

$$\Sigma = \{x \in M:\lambda(x) > 0\}.$$

Since a Kaehler manifold cannot be immersed in a real space form as a totally geodesic submanifold, we see that the set Σ is an open dense subset of M. In the following, we study at a fixed point x of Σ. Now we investigate the first normal space of M by using (2.4). We choose a local field of the orthonormal frame

$$e_1, \dots, e_n, e_{n+1} = Je_1, \dots, e_{2n} = Je_n$$

around x. Since the curvature tensor R of M is a nice form, we immediately find

$$g(R(e_i, e_j)e_j, e_i) = c \quad \text{for } 1 \leqq i \neq j \leqq n.$$

So we may apply Theorem 1.1 to the linear subspace of $T_x(M)$, which is generated by $\{e_1, \dots, e_n\}$. First we consider the case (1) of Theorem 1.1, that is,

$$\lambda^2 = c - \tilde{c}.$$

From (2.2), we have

$$(2.5) \quad 2g(B(e_i, e_j), B(e_i, e_j)) + g(B(e_i, e_i), B(e_j, e_j)) = (c - \tilde{c})(2\delta_{ij}\delta_{ij} + 1),$$

where i and j run over the range $\{1, 2, \dots, 2n\}$.

Hence the equation (2.5) yields

$$(2.6) \quad 2\|B\|^2 + 4n^2H^2 = 4n(n + 1)(c - \tilde{c}),$$

where $\|B\|$ is the length of the second fundamental form B and H is the mean curvature of M.

On the other hand the Gauss equation (2.1) shows

$$(2.7) \quad -\|B\|^2 + 4n^2H^2 = 4n(n + 1)c - 2n(2n - 1)\tilde{c}.$$

As an immediate consequence of (2.6) and (2.7), we get

$$\|B\|^2 = -2n\tilde{c} < 0,$$

which is a contradiction. So we find that $\lambda^2 \neq c - \tilde{c}$ and either the case (2) or the case (3) of Theorem 1.1 must happen at x. Moreover a straightforward calculation, by virtue of (2.4), yields the following orthogonal relations:

$$(2.8) \quad g(B(e_i, Je_j), B(e_k, Je_l)) = (\lambda^2 - (c - \tilde{c}))/3 \cdot \delta_{ik}\delta_{jl}$$

for $1 \leqq i < j \leqq n$ and $1 \leqq k < l \leqq n$;

$$(2.9) \quad g(B(e_i, e_j), B(e_k, Je_l)) = 0$$

for $1 \leqq i \leqq j \leqq n$ and $1 \leqq k < l \leqq n$.

Then, in consideration of Theorem 1.1, (2.8) and (2.9), we see that the codimension

$$p \geqq \frac{n(n+1)}{2} - 1 + \frac{n(n-1)}{2} = n^2 - 1$$

at a fixed point x. Here we take n vectors

$$B(e_i, Je_i) \quad (i = 1, 2, \dots, n).$$

A similar calculation shows the following orthogonal relations:

(2.10) $g(B(e_i, Je_i), B(e_j, Je_j)) = (\lambda^2 - (4c - \tilde{c}))/3 \cdot \delta_{ij}\delta_{ij}$

for $i, j = 1, 2, \dots, n$;

(2.11) $g(B(e_i, e_j), B(e_k, Je_k)) = 0$

for $1 \leqq i \leqq j \leqq n$ and $1 \leqq k \leqq n$;

(2.12) $g(B(e_i, Je_j), B(e_k, Je_k)) = 0$

for $1 \leqq i < j \leqq n$ and $1 \leqq k \leqq n$.

Now suppose that

$$B(e_i, Je_i) \neq 0 \quad (i = 1, 2, \dots, n),$$

that is,

$$\lambda^2 \neq 4c - \tilde{c}.$$

Then, in view of (2.10), (2.11) and (2.12), we find that the codimension

$$p \geqq (n^2 - 1) + n,$$

which contradicts the assumption $p \leqq n^2$. And hence we have

(2.13) $\lambda^2 = 4c - \tilde{c}$.

Substituting (2.13) into the right-hand side of (2.4), we obtain

(2.14) $g(B(X, Y), B(Z, W))$

$$= (2c - \tilde{c})g(X, Y)g(Z, W) + c(g(X, W)g(Y, Z) + g(X, Z)g(Y, W) + g(JX, Z)g(JY, W) + g(JY, Z)g(JX, W)).$$

The equation (2.14) shows the following:

(2.15) $g(B(X, Y), B(X, Y)) = g(B(JX, JY), B(JX, JY))$

$$= (3c - \tilde{c})(g(X, Y))^2 + c(\|X\|^2\|Y\|^2 - (g(JX, Y))^2).$$

(2.16) $g(B(X, Y), B(JX, JY))$

$$= (3c - \tilde{c})(g(X, Y))^2 + c(\|X\|^2\|Y\|^2 - (g(JX, Y))^2).$$

Thus, in consideration of (2.15) and (2.16), we see

$$B(JX, JY) = B(X, Y) \quad \text{for all } X, Y.$$

And hence, from Theorem 1.2 we find that the second fundamental form of our immersion is parallel on the open dense subset Σ of M so that $\bar{\nabla} B \equiv 0$ at each point of M. Therefore, due to the classification of parallel submanifolds, we obtain our conclusion (for details, see [**6**]).

3. Flat torus in a complex projective space. Let $P_n(\mathbf{C})$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. Now we construct a flat manifold T^n in $P_n(\mathbf{C})$ as follows: We consider

$$M^{n+1} = S^1(1/\sqrt{n+1}) \times \ldots \times S^1(1/\sqrt{n+1}) \quad \text{in } S^{2n+1}(1),$$

where $S^1(1/\sqrt{n+1})$ is a circle with radius $1/\sqrt{n+1}$.

Making use of this manifold M^{n+1}, we get a fibration

$$S^1 \to M^{n+1} \to T^n$$

which is compatible with the Hopf fibration

$$S^1 \to S^{2n+1} \to P_n(\mathbf{C})$$

(for details, see [**15**]).

The submanifold T^n thus obtained has various beautiful properties. In fact, for each n, we can show that T^n is a totally real minimal submanifold with parallel second fundamental form in $P_n(\mathbf{C})$ (cf. [**1**], [**4**]). In our paper, we will pay particular attention to the case $n = 2$. The purpose of this section is to provide some characterizations of the flat torus T^2 in $P_2(\mathbf{C})$. We have

THEOREM 2. *Let M be an n-dimensional real space form of constant curvature c. If M is a totally real isotropic submanifold of $P_n(\mathbf{C})$, then M is totally geodesic ($c = 1$) or $n = 2$ and M is locally congruent to* $T^2 \;\; (c = 0) \Rightarrow T^2 \;\; (c = 0)$.

THEOREM 3. *Let M be a $(2n - 2)$-dimensional submanifold of $P_n(\mathbf{C})$. If every geodesic in M is a circle in $P_n(\mathbf{C})$, then $n = 2$ and M is locally congruent to T^2 or $Q_1(\mathbf{C})$.*

The other model space $Q_1(\mathbf{C})$ of Theorem 3 is defined as

$$Q_1(\mathbf{C}) = \{ (z_0, z_1, z_2) \in P_2(\mathbf{C}) : {z_0}^2 + {z_1}^2 + {z_2}^2 = 0 \},$$

where $\{z_0, z_1, z_2\}$ is a homogeneous coordinate system of $P_2(\mathbf{C})$.

Remark 3.1. We note that the following problem is still open: Classify all circular geodesic submanifolds in a complex space form. When the ambient manifold is a real space form, the above problem was solved by Sakamoto ([**13**]).

Here we prepare three lemmas for Theorem 3.

LEMMA 3.1. *Let M be a submanifold in a Riemannian manifold $\widetilde{M}$. Then the following two conditions are equivalent.*

(i) *The submanifold M is nonzero (λ-)isotropic and the second fundamental form B of M in $\widetilde{M}$ satisfies $(\overline{\nabla}_X B)(X, X) = 0$ for all vector fields X tangent to M.*

(ii) *M is a circular geodesic submanifold of $\widetilde{M}$.*

Proof. (i) $\Rightarrow$ (ii): Let $x(t)$ be any geodesic with unit tangent vectors X_t. Put

$$\lambda(t) = \|B(X_t, X_t)\|.$$

Then λ is constant along $x(t)$, because, by the second assumption of (i), we have

$$\begin{aligned} X_t(\lambda^2) &= X_t \cdot g(B(X_t, X_t), B(X_t, X_t)) \\ &= 2g((\overline{\nabla}_t B)(X_t, X_t), B(X_t, X_t)) = 0. \end{aligned}$$

(This, together with the first assumption of (i), yields that the immersion is constant isotropic.)

Here we put

$$Y_t = (1/\lambda(t)) \cdot B(X_t, X_t).$$

It follows from the Gauss formula that

$$\widetilde{\nabla}_t X_t = B(X_t, X_t) = \lambda Y_t.$$

Moreover we get

$$\widetilde{\nabla}_t Y_t = (1/\lambda)\widetilde{\nabla}_t(B(X_t, X_t)).$$

By virtue of the Weingarten formula we see

$$\widetilde{\nabla}_t Y_t = (1/\lambda)(-A_{B(X_t,X_t)}X_t + D_t(B(X_t, X_t))).$$

Since M is an isotropic submanifold, we easily find that

$$A_{B(X_t,X_t)}X_t = \lambda^2 X_t$$

(see [**10**], Lemma 1) and again by the second assumption we get

$$D_t(B(X_t, X_t)) = 0.$$

Thus we find that every geodesic in M is a circle in $\widetilde{M}$.

(ii) $\Rightarrow$ (i): Let $x(t)$ be any geodesic on M with unit tangent vectors X_t. By the Gauss formula we have

$$\widetilde{\nabla}_t X_t = \nabla_t X_t + B(X_t, X_t) = B(X_t, X_t).$$

So we obtain

(1) $\widetilde{\nabla}_t^2 X_t = -A_{B(X_t, X_t)} X_t + D_t(B(X_t, X_t))$.

On the other hand, since $x(t)$ is a circle in $\widetilde{M}$ by assumption, there exists a field of unit tangent vectors Y_t along $x(t)$ and $k > 0$ such that

$$\widetilde{\nabla}_t X_t = kY_t \quad \text{and} \quad \widetilde{\nabla}_t Y_t = -kX_t,$$

thus

(2) $\widetilde{\nabla}_t^2 X_t = -k^2 X_t$.

From (1) and (2) we find

(3) $A_{B(X_t, X_t)} X_t = k^2 X_t$

and

(4) $D_t(B(X_t, X_t)) = 0$.

Since $x(t)$ is a geodesic in M,

$$(\overline{\nabla}_t B)(X_t, X_t) = 0$$

by virtue of (4). And hence, we have

$$(\overline{\nabla}_X B)(X, X) = 0$$

for all tangent vector fields X to M. From (3) it follows that for any unit tangent vector X to M there exists a certain constant $k > 0$ such that

$$A_{B(X,X)} X = k^2 X.$$

If Y is a tangent vector perpendicular to X, then we see

$$g(A_{B(X,X)} X, Y) = 0$$

so that

$$g(B(X, X), B(X, Y)) = 0$$

whenever $g(X, Y) = 0$. This condition implies that the immersion is (nonzero) isotropic (see [**10**], Lemma 1).

LEMMA 3.2. *Let M be a submanifold of a complex space form $\widetilde{M}(\widetilde{c})$ of constant holomorphic sectional curvature $\widetilde{c}$ with complex structure J. Then the following two conditions are equivalent.*

(A) $(\overline{\nabla}_X B)(X, X) = 0$

for all vector fields X tangent to M.

(B) $(\overline{\nabla}_X B)(Y, Z) = (\widetilde{c}/4)\{g(X, JY)JZ + g(X, JZ)JY\}^{\perp}$

for all vector fields X, Y and Z tangent to M, where $\{*\}^{\perp} \Rightarrow \{*\}^{\perp}$ means the normal component of $\{*\} \Rightarrow \{*\}$.

Proof. Now we may easily see that

$$(\overline{\nabla}_X B)(X, X) = 0$$

is equivalent to

(1) $(\bar{\nabla}_X B)(Y, Z) + (\bar{\nabla}_Y B)(Z, X) + (\bar{\nabla}_Z B)(X, Y) = 0.$

On the other hand, since the curvature tensor $\tilde{R}$ of $\tilde{M}(\tilde{c})$ is a nice form, from (1.2) we have

(2) $$(\bar{\nabla}_X B)(Y, Z) - (\bar{\nabla}_Y B)(X, Z) = (\tilde{c}/4)\{g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\}^{\perp}.$$

Exchanging Y and Z in (2), we get

(3) $$(\bar{\nabla}_X B)(Z, Y) - (\bar{\nabla}_Z B)(X, Y) = (\tilde{c}/4)\{g(JZ, Y)JX - g(JX, Y)JZ + 2g(X, JZ)JY\}^{\perp}.$$

Summing up (1), (2) and (3), we obtain (B).

The converse is trivial.

Remark 3.2. It is well known that $(\bar{\nabla}_X B)(Y, Z)$ is symmetric for X, Y and Z in case that the ambient manifold $\tilde{M}$ is a real space form. Then we easily see that the condition (A) of Lemma 3.2 is equivalent to $\bar{\nabla} B \equiv 0$. On the other hand, when the ambient manifold $\tilde{M}$ is a complex space form, this is not true. In fact, Chen and Vanhecke ([3]) proved that any small geodesic hypersphere in a complex space form satisfies the condition (A). However the second fundamental form of any geodesic hypersphere in a complex space form is not parallel (cf. [14]).

Finally we prepare the following lemma without proof.

LEMMA 3.3 ([7]). *Let M be an m-dimensional isotropic submanifold with codimension p of a Riemannian manifold $\tilde{M}$. If $m \geqq 3$ and $p < \max\{m/2, 3\}$, then the immersion is totally umbilic.*

4. Proof of theorem 2. First of all we note the following: Naitoh ([8]) classified totally real isotropic submanifolds with parallel second fundamental form in $P_n(\mathbf{C})$. So, by virtue of his classification, we have only to show that the second fundamental form of the immersion is parallel in order to prove Theorem 2.

Now we rewrite (1.1) and (1.2). Since the submanifold M is a real space form which is immersed in $P_n(\mathbf{C})$ as a totally real submanifold, the equations (1.1) and (1.2) are reduced to (4.1) and (4.2) respectively:

(4.1) $$g(B(X, Y), B(Z, W)) - g(B(Z, Y), B(X, W)) = (c - 1)(g(X, Y)g(Z, W) - g(Z, Y)g(X, W)),$$

(4.2) $$(\bar{\nabla}_X B)(Y, Z) = (\bar{\nabla}_Y B)(X, Z).$$

Now, by the assumption that the immersion is isotropic, all normal curvature vectors at x have the same length, say, λ. Namely, we have

$$g(B(X, X), B(X, X)) = \lambda^2 g(X, X)g(X, X).$$

Here the same calculation as in the proof of Theorem 1 (see (2.1), (2.2), (2.3) and (2.4)) yields

$$\begin{aligned}(4.3)\quad & g(B(X, Y), B(Z, W)) \\ & = (c - 1)/3 \cdot (2g(X, Y)g(Z, W) - g(Z, Y)g(X, W) \\ & - g(Z, X)g(Y, W)) \\ & + \lambda^2/3 \cdot (g(X, Y)g(Z, W) + g(X, Z)g(Y, W) \\ & + g(X, W)g(Y, Z)).\end{aligned}$$

Now we consider (at a fixed point x of M) two cases $\lambda = 0$ and $\lambda \neq 0$. In view of Theorem 1.1, we see that the latter case $\lambda \neq 0$ is divided into three cases. But the case (3) of Theorem 1.1 does not occur, since $n(n + 1)/2 > n$ ($=$ codim M). Hence we have only to investigate the following three cases:

(I) $\lambda = 0$,

(II) $\lambda^2 = c - 1$,

(III) $\lambda^2 = -(2(n - 1)/(n + 2)) \cdot (c - 1)$.

Since λ is a continuous function on the submanifold of M, we see that λ is constant on the connected component U of x. And hence, in the following we consider the above three cases on U.

Case (I). We immediately find that the immersion is totally geodesic so that the submanifold is locally congruent to $P^n(\mathbf{R})$ which is the real part of $P_n(\mathbf{C})$.

Case (II). This case means that the immersion is totally umbilic. However [**2**] asserts that the Case (II) does not occur.

Case (III). Since λ is constant on U, differentiating (4.3) with respect to any tangent vector field T, we have the following:

$$(4.4)\quad g((\bar{\nabla}_T B)(X, Y), B(Z, W)) = -g(B(X, Y), (\bar{\nabla}_T B)(Z, W)).$$

By using (4.4) and the Codazzi equation (4.2) repeatedly, we get

$$\begin{aligned}(4.5)\quad & g((\bar{\nabla}_T B)(X, Y), B(Z, W)) \\ & = -g(B(X, Y), (\bar{\nabla}_Z B)(T, W)) \\ & = g((\bar{\nabla}_X B)(Z, Y), B(T, W)) \\ & = -g(B(Z, Y), (\bar{\nabla}_W B)(X, T)) \\ & = g((\bar{\nabla}_Y B)(Z, W), B(X, T)) \\ & = -g(B(Z, W), (\bar{\nabla}_T B)(X, Y)).\end{aligned}$$

So we have

(4.6) $g((\bar{\nabla}_T B)(X, Y), B(Z, W)) = 0.$

We remark that the above calculation is due to Nakagawa-Itoh ([**9**]). On the other hand, it follows from

$$n(n + 1)/2 - 1 \leqq n \quad (= \text{codim } M)$$

that

$$n = 2 \text{ and codim } M = n(n + 1)/2 - 1,$$

which yields that the first normal space spans the normal space. This, combined with (4.6), shows that the second fundamental form of the immersion is parallel. Thus, in consideration of [**8**], we see that M is locally congruent to T^2 (for details, see [**8**]).

5. Proof of theorem 3. First of all we note the following: Naitoh ([**8**]) classified circular geodesic totally real submanifolds in $P_n(\mathbf{C})$ and Nomizu ([**11**]) classified circular geodesic Kaehler submanifolds in $P_n(\mathbf{C})$. Due to their works, we find that the model space T^2 (resp. $Q_1(\mathbf{C})$) is a circular geodesic totally real (resp. a circular geodesic Kaehler) submanifold in $P_2(\mathbf{C})$.

So, in order to prove Theorem 3, we have only to show that the submanifold M is a totally real or a Kaehler submanifold.

Now we shall consider the case of $n \geqq 3$. As an immediate consequence of Lemma 3.1 and Lemma 3.3, we see that M^{2n-2} is a totally umbilic submanifold of $P_n(\mathbf{C})$. However Chen-Ogiue's work [**2**] asserts that this case does not happen.

And hence, in the following we shall solve our problem by using Lemma 3.1 and Lemma 3.2 in the case of $n = 2$. Our aim here is to find that the immersion is minimal. Let Σ be the set of all non-umbilic points of M^2. We remark that Σ is a dense subset of M^2, since there exists no totally umbilic (not totally geodesic) surface in $P_2(\mathbf{C})$ (cf. [**2**]). Let x be a point of Σ and e_1, e_2 be an orthonormal basis in $T_x(M)$. Since x is a non-umbilic point and the immersion is (λ-)isotropic, we have

(5.1) $B(e_1, e_2) \neq 0$ at x.

Moreover, again by the fact that the immersion is isotropic, we see (cf. [**10**])

(5.2) $g(B(e_1, e_1), B(e_1, e_2)) = g(B(e_2, e_2), B(e_1, e_2)) = 0.$

Therefore, in consideration of the assumption that codim $M = 2$ and the equations (5.1), (5.2), we have only to consider the following two cases:

(I) $B(e_1, e_1) = B(e_2, e_2),$

(II) $B(e_1, e_1) = -B(e_2, e_2).$

On the other hand, from (2.2) we have

(5.3) $g(B(e_1, e_1), B(e_2, e_2)) + 2g(B(e_1, e_2), B(e_1, e_2)) = \lambda^2.$

The equation (5.3), together with (5.1), tells us that the Case (I) does not happen at a non-umbilic point x, from which we see that the immersion is minimal on the open dense subset Σ of M^2 so that our surface M^2 is minimal in $P_2(\mathbf{C})$. We fix an arbitrary point p of M^2. Let X, Y be orthonormal vector fields on a neighborhood U of p such that $\nabla X = \nabla Y = 0$ at p. Since the immersion is λ-isotropic, we see

(5.4) $g(B(X, X), B(X, X)) = \lambda^2 \quad (\neq 0) \quad$ on U.

We easily find that (5.4) is equivalent to (cf. [**10**])

(5.5) $g(B(X, X), B(X, Y)) = 0 \quad$ on U.

From (2.2), we get

(5.6) $g(B(X, X), B(Y, Y)) + 2g(B(X, Y), B(X, Y)) = \lambda^2 \quad$ on U.

The equation (5.6), combined with the fact that the immersion is minimal, shows

(5.7) $g(B(X, Y), B(X, Y)) = \lambda^2 \quad (\neq 0) \quad$ on U.

These equations (5.4), (5.5) and (5.7), combined with the assumption that codim $M = 2$, imply that the vectors $B(X, X)$, $B(X, Y)$ span the normal space at every point on U. On the other hand, since λ is constant (see the proof of Lemma 3.1), from (5.4) we get

$$Y \cdot g(B(X, X), B(X, X)) = 0,$$

that is,

(5.8) $g((\overline{\nabla}_Y B)(X, X), B(X, X)) = 0 \quad$ at p.

Hence, by virtue of Lemma 3.2, from (5.8) we find

(5.9) $g(Y, JX)g(JX, B(X, X)) = 0 \quad$ at p.

Similarly, from (5.7) we have

(5.10) $g(X, JY)g(JX, B(X, Y)) = 0 \quad$ at p.

Now assume that the tangent space $T_p(M)$ is not a totally real plane. Since dim $M = 2$, this assumption implies

(5.11) $g(X, JY) \neq 0 \quad$ at p.

And hence, from (5.9), (5.10) and (5.11) we get

$$g(JX, B(X, X)) = 0 \quad \text{and}$$
$$g(JX, B(X, Y)) = 0 \quad \text{at } p.$$

Then, due to the above discussion, we see that $T_p(M)$ is a holomorphic plane. Namely we find that our surface M^2 is a totally real submanifold or a Kaehler submanifold of $P_2(\mathbf{C})$. Therefore we conclude that M^2 is locally congruent to T^2 or $Q_1(\mathbf{C})$ (for details, see [**8**], [**11**]).

6. Another characterization of extrinsic spheres. Nomizu and Yano ([**12**]) proved that a submanifold M^n of a Riemannian manifold $\widetilde{M}^{n+p}$ is an extrinsic sphere if and only if every circle in M^n is a circle in $\widetilde{M}^{n+p}$. The purpose of this section is to provide another characterization of an extrinsic sphere $M^n (n \geqq 3)$ in an arbitrary manifold $\widetilde{M}^{n+p}$ in the case of $p < \max\{n/2, 3\}$. We have the following

THEOREM 4. *Let M^n be a Riemannian submanifold all of whose geodesics are circles in an ambient manifold $\widetilde{M}^{n+p}$. If $n \geqq 3$ and $p < \max\{n/2, 3\}$, then the submanifold M^n is an extrinsic sphere of $\widetilde{M}^{n+p}$.*

Proof. It follows from Lemma 3.1 and Lemma 3.3 that the immersion is totally umbilic, that is,

$$B(X, Y) = g(X, Y)\mathfrak{H} \quad \text{and} \quad (\overline{\nabla}_X B)(X, X) = 0$$

for all tangent vector fields X and Y on M^n. Namely we find that the submanifold M^n is totally umbilic and has parallel mean curvature vector so that M^n is an extrinsic sphere of $\widetilde{M}^{n+p}$.

Remark. As an immediate consequence of Lemma 3.1, we obtain the following fact: Let M be an extrinsic sphere of a Riemannian manifold $\widetilde{M}$. If the immersion is not totally geodesic, then M is a circular geodesic submanifold of $\widetilde{M}$.

However, the converse is not true. Motivated by Theorem 4, we consider a circular geodesic submanifold $M^n (n \geqq 3)$ in $\widetilde{M}^{n+p}$ with codimension $p = \max\{n/2, 3\}$. Of course the following submanifolds are not extrinsic spheres.

Example 1. $f:P_2(\mathbf{C}) \to S^7$, where the immersion f is called the first standard minimal immersion (for details, see [**13**]).

Example 2. $g:S^1 \times S^2 \to P_3(\mathbf{C})$, where the immersion g is constructed by Naitoh (for details, see [**8**]).

REFERENCES

1. B. Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc. *193* (1974), 257-266.

2. ——— *Two theorems on Kaehler manifolds*, Michigan Math. J. *21* (1974), 225-229.

3. B. Y. Chen and L. Vanhecke, *Differential geometry of geodesic spheres*, J. Reine Angew. Math. *325* (1981), 28-67.

4. N. Egiri, *Totally real minimal immersions of n-dimensional real space forms into n-dimensional complex space forms*, Proc. Amer. Math. Soc. *84* (1982), 243-246.

5. D. Ferus, *Immersions with parallel second fundamental form*, Math. Z. *140* (1974), 87-93.
6. ——— *Symmetric submanifolds of Euclidean space*, Math. Ann. *247* (1980), 81-93.
7. N. Kleinjohann and R. Walter, *Nonnegativity of the curvature operator and isotropy for isometric immersions*, Math. Z. *181* (1982), 129-142.
8. H. Naitoh, *Isotropic submanifolds with parallel second fundamental form in $P^m(c)$*, Osaka J. Math. *18* (1981), 427-464.
9. H. Nakagawa and T. Itoh, *On isotropic immersions of space forms into a sphere*, Proc. of Japan-United States Seminar on minimal submanifolds, including Geodesics (1978).
10. B. O'Neill, *Isotropic and Kaehler immersions*, Can. J. Math. *17* (1965), 905-915.
11. K. Nomizu, *A characterization of the Veronese varieties*, Nagoya Math. J. *60* (1976), 181-188.
12. K. Nomizu and K. Yano, *On circles and spheres in Riemannian geometry*, Math. Ann. *210* (1974), 163-170.
13. K. Sakamoto, *Planar geodesic immersions*, Tohoku Math. J. *29* (1977), 25-56.
14. Y. Tashiro and S. Tachibana, *On Fubinian and C-Fubinian manifolds*, Kodai Math. Sem. Rep. *15* (1963), 176-183.
15. K. Yano, M. Kon and I. Ishihara, *Anti-invariant submanifolds with flat normal connection*, J. Diff. Geom. *13* (1978), 577-588.

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