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FINITE GROUPS WITH LARGE CENTRALIZERS

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It is known that a finite non-abelian group G has a proper centralizer of order $> |G|^{\frac{1}{3}}$ if, for example, $|G|$ is even and $|Z(G)|$ is odd, or whenever G is solvable. Often the exponent $\frac{1}{3}$ can be improved to $\frac{1}{2}$, for example when G is supersolvable, or metabelian, or $|G| = p^{\alpha} q^{\beta}$. Here we show more generally that this improvement is possible in many situations where G is factorizable into the product of two subgroups. In particular, much more evidence is presented to support the conjecture that some proper centralizer has order $> |G|^{\frac{1}{2}}$ whenever G is a finite non-abelian solvable group.

1. Introduction

In [2] the first author proved that every finite non-abelian solvable group G has a proper centralizer of order $|C_G(x)| > |G|^{\frac{1}{3}}$. Furthermore it was shown that the exponent $\frac{1}{3}$ can be improved to $\frac{1}{2}$

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when non-abelian G is either supersolvable, metabelian, a solvable A-group, or has order $p^{\alpha}q^{\beta}$, p, q distinct primes.

Let C denote the collection of all finite non-abelian groups G which contain a proper (large) centralizer of order $> |G|^{\frac{1}{2}}$. Let S denote the collection of all finite non-abelian solvable groups. In [2] the question was raised as to whether $S \subset C$. In this paper we generalize most of the results in [2] and give much more evidence that $S \subset C$. Along the way we also prove, for example, that every finite group containing a conjugacy class of prime-power cardinality (> 1) belongs to C.

Specifically, in the solvable case we prove: (Theorem 2) If non-abelian $G = NM$ where N and M are nilpotent subgroups of G, then $G \in C$. Thus if G' is nilpotent (> 1) then $G \in C$. (Theorem 7) If $G \in S$ and $|G| = \Pi\, p_i^{\alpha_i}$ (distinct primes p_i), with each $\alpha_i \leq 4$, then $G \in C$. Finally, a few more results of numerical type (some not presented here) have enabled us to prove (Theorem 10); every non-abelian group of odd order $< 10^6$ belongs to C; every non-abelian solvable group of even order $\leq 10^4$ is a member of C. The proof of the last theorem amy be obtained from the authors.

2. Factorizable Groups

THEOREM 1. *If G is a finite non-abelian group with the factorization $G = AB$, where A and B are nilpotent subgroups of G and $(|A|, |B|) = 1$, then $G \in C$.*

Proof. By Wielandt's theorem ([4], p. 680) G is solvable. If $Z = Z(G) = \{1\}$, then $G \in C$ by Theorem 1 of [2]. So $\{1\} < Z = A_1 \times B_1$ with $A_1 \leq A$ and $B_1 \leq B$. Let $|A_1| = a_1$, $|B_1| = b_1$, $|A| = a$ and $|B| = b$. Since $|G| = ab = \left(\frac{ab_1}{a_1}\right)\left(\frac{ba_1}{b_1}\right)$, either one of the latter factors is larger that $|G|^{\frac{1}{2}}$, or $|Z| \geqq |G|^{\frac{1}{2}}$ and $G \in C$. Suppose without loss

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of generality that $\frac{ab_1}{a_1} > |G|^{\frac{1}{2}}$. If G is nilpotent, then $G \in C$ by Corollary 1.1(b) of [2]. So assume G/Z is not nilpotent, and $a_1 \neq a$. Let $xA_1 \in Z(A/A_1)^{\#}$. Then Z/A_1, $A/A_1 \leq C_{G/A_1}(xA_1)$, and since they are of coprime orders it follows from Lemma 1 of [2] that

$$|C_G(x)| \geq |C_{G/A_1}(xA_1)| \geq [Z:A_1][A:A_1] = \frac{b_1 a}{a_1} > |G|^{\frac{1}{2}}.$$ Since $x \notin Z$, $G \in C$ and the proof is complete. □

THEOREM 2. *If G is a finite non-abelian group with the factorization $G = NM$, where N and M are nilpotent subgroups of G, then $G \in C$.*

Proof. By the theorem of Wielandt and Kegel ([4], p. 674) G is solvable. Our proof is by induction on $k = \min\{|\pi(N)|, |\pi(M)|\}$. Assume without loss of generality that $|\pi(M)| = k$. If $k = 0$, then G is nilpotent and $G \in C$. If $k = 1$, then M is a p-group. Let $M \leq P = \mathrm{Syl}_p(G)$. Then $G = NP$, with $N_p = N \cap P$, so $G = N_{p'}P$ and $G \in C$ by Theorem 1.

So assume that $k = n \geq 2$ and that Theorem 2 holds for all $k < n$. We may assume without loss of generality that $Z = Z(G) \leq N$ (otherwise replace N by the nilpotent subgroup NZ), and again by earlier results we may assume that $\{1\} < Z < N$. Hence for some prime p, $Z_p < N_p$, $p \mid |Z(N/Z_p)|$ and there exists an $x \in N_p - Z_p$ such that

$$|C_G(x)| \geq |C_N(x)| \geq |C_{N/Z_p}(xZ_p)| \geq [N:Z_p].$$

Since $n \geq 2$, there exists a prime $q \in \pi(M)$, $q \neq p$. If $M_q \leq Z(<N)$, then $G = NM_{q'}$, and $G \in C$ by induction. So we may assume that $M_q > M_q \cap Z$. Considering MZ/Z_q, we conclude (again using Lemma 1 of [2]) that there exists an element $y \in M_q - Z$ such that $|C_G(y)| \geq [MZ:Z_q]$. Thus

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$$|C(x)| \cdot |C(y)| \geq \frac{|N|}{|Z_p|} \cdot \frac{|M||Z|}{|M \cap Z||Z_q|} = \frac{|N||M|}{|N \cap M|} \cdot \frac{|Z|}{|Z_p||Z_q|} \geq |G|$$

since $p \neq q$. If $|C(x)||C(y)| > |G|$ or $|C(x)| \neq |C(y)|$ then $G \in C$, since $x, y \notin Z$. Otherwise $|C(x)| = |C(y)| = |G|^{\frac{1}{2}}$, whence $Z = Z_p \times Z_q$. Since $Z > \{1\}$, either $|\pi(Z)| = 1$ or $|\pi(Z)| = 2$. If $|\pi(Z)| = 1$ then, in view of $|\pi(N)| \geq |\pi(M)| = n \geq 2$ there exist $x_1 \in N - Z$ and $y_1 \in M - Z$ such that $C(x_1) \geq N$ and $C(y_1) \geq MZ$. Then we have

$$|C(x_1)|\ |C(y_1)| \geq |N|\ |MZ| \geq \frac{|N||M|}{|N \cap M|} \cdot |Z| > |G|\ ,$$

since $\{1\} < Z < N$. Since $x_1, y_1 \notin Z$, $G \in C$.

Finally, consider the case $|\pi(Z)| = 2$. If $\pi(N) = \pi(M) = \pi(Z)$, then $|\pi(G)| = 2$ and $G \in C$ by Theorem 1. If $\pi(N) \neq \pi(Z)$, then $|\pi(N)| \geq 2$ implies that there exists $x_2 \in N - Z$ such that $C(x_2) \geq N$. Thus

$$|C(x_2)|\ |C(y)| \geq |N| \cdot \frac{|M||Z|}{|M \cap Z||Z_q|} = \frac{|N||M|}{|N \cap M|} \cdot \frac{|Z|}{|Z_q|} > |G|$$

(since $|\pi(Z)| = 2$). Again $G \in C$ since $x_2, y \notin Z$. Otherwise, $\pi(M) \neq \pi(Z)$, and $|\pi(M)| \geq 2$ implies that there exists a $y_2 \in M - Z$ such that $C_G(y_2) \geq M\, Z$. Thus

$$|C_G(x)|\ |C_G(y_2)| \geq \frac{|N|}{|Z_p|} \cdot |MZ| = \frac{|N||M||Z|}{|Z_p||M \cap Z|} \geq \frac{|N||M|}{|M \cap N|} \cdot \frac{|Z|}{|Z_p|} > |G|.$$

Since $x, y_2 \notin Z$, $G \in C$ and the proof of Theorem 2 is complete. □

COROLLARY 2.1. *Suppose $G \in S$, and G contains a nilpotent, maximal subgroup M. Then $G \in C$.*

Proof. Every maximal subgroup of a solvable group has prime-power index. Thus (considering the prime-power factorization of $|G|$) we have $G = M\,P$ where $P \in \mathrm{Syl}_p(G)$ for some prime p. Since M is nilpotent Theorem 2 applies, and $G \in C$. □

COROLLARY 2.2. *Let G be a finite non-abelian group with G' nilpotent. Then $G \in C$.*

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Proof. Since G' is nilpotent, G is solvable and we know (see for example [4], p. 271) that $G = G'U$ for some nilpotent subgroup $U \leq G$. Again Theorem 2 applies, and $G \in C$. □

THEOREM 3. *Suppose the finite group G contains an element $g \in G - Z$ such that $G = C_G(g)N$ for some nilpotent subgroup N. Then $|C_G(x)| \geq |G|^{\frac{1}{2}}$ for some $x \in G - Z$.*

Proof. Since $g \notin Z$, $N > N \cap Z$. Let $M = C_G(g)$ and $\bar{y} \in Z(N/N \cap Z)^{\#}$. Then $y \in N - Z$ and

$$|C_G(g)| \cdot |C_G(y)| \geq |M||C_{N/N\cap Z}(\bar{y})| = \frac{|M||N|}{|N \cap Z|} \geq \frac{|M||N|}{|M \cap N|} = G .$$

Thus either $|C_G(g)| \geq |G|^{\frac{1}{2}}$ or $|C_G(y)| \geq |G|^{\frac{1}{2}}$. □

COROLLARY 3.1. *If $|\pi(N)| \leq 2$, N as in Theorem 3, then $G \in C$.*

Proof. If equality holds, in the proof of the theorem, then $|\pi(G)| \geq 2$ and we may apply Theorem 1. □

COROLLARY 3.2. *Suppose the finite group G contains a conjugacy class of cardinality $|[g]| = p^r > 1$, where p is a prime. Then $G \in C$.*

Proof. Let $P \in \mathrm{Syl}_p(G)$. Then consideration of the prime-power factorization of $|G|$ shows that $G = C_G(g)P$. The result follows from Corollary 3.1, since $g \notin Z(G)$. □

COROLLARY 3.3. *If $G \in S$, and $C_G(g)$ is a (proper) maximal subgroup of G, then $G \in C$.*

Proof. Every maximal subgroup of a solvable group has prime-power index in G. The conclusion now follows from Corollary 3.2. □

COROLLARY 3.4. *Suppose $G \in S$, and G has "abelian centralizers", that is $C_G(g)$ is abelian for all $g \in G - Z(G)$. Then $G \in C$.*

Proof. It follows from the work of R. Baer [1] on normal non-trivial partitions of finite groups that one of the following holds

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(see [5] or [6]):

(a) $G/Z \cong \mathrm{Sym}(4)$;

(b) G/Z is a Frobenius group, with $C_G(x)/Z$ an (abelian) Frobenius complement, for some $x \in G - Z$. If N/Z is the Frobenius kernel, then either N/Z is a p-group, or N is abelian;

(c) G/Z is a p-group;

(d) there is an $x \in G$ such that $C_G(x)$ is the subgroup generated by $Z(G)$ and all $g \in G$ such that $g^p \notin Z$. Here $[G: C_G(x)] = p$.

In case (a), $(G/Z)' = G'Z/Z \cong G'/G' \cap Z$ is abelian. By Theorem 1 of [2] we may assume that $G' \cap Z \neq \{1\}$, in which case $G' \cap Z = Z(G')$ by Lemma 2b of [2] . Thus $G'/Z(G')$ is abelian and G' is nilpotent (of class 2). The conclusion now follows by Corollary 2.2. In case (b) $(G/Z)/(N/Z) \cong G/N \cong C_G(x)/Z$ is abelian in which case $G' \leq N$ and again G' is nilpotent. In case (c) G is nilpotent, and the result follows. In case (d), $C_G(x)$ is a maximal subgroup and the conclusion follows from Corollary 3.3. □

LEMMA 4.1. *Let G be a finite group and $G = AB$, $A,B \leq G$ with $Z(A)$, $Z_2(B) \not\leq Z(G)$. Then there exists an element $x \in G - Z$ such that* $|C_G(x)| \geq |G|^{\frac{1}{2}}$.

Proof. Clearly $G = (AZ)B$. Let $a \in Z(A) - Z(G)$. If there exists an element $b \in Z(B) - Z(G)$, then $|C_G(a)|\,|C_G(b)| \geq |A|\,|B| \geq |G|$, and either $|C_G(a)| \geq |G|^{\frac{1}{2}}$ or $|C_G(b)| \geq |G|^{\frac{1}{2}}$. Otherwise $Z(B) = Z(G)$ so $Z(G) \cap B = Z(B)$. Let $c \in Z_2(B) - Z(G)$. Then $|C_G(a)| \cdot |C_G(c)| \geq |AZ| \cdot |C_B(c)| \geq |AZ| \cdot |C_{B/Z(B)}(cZ(B))| = \frac{|AZ|\cdot|B|}{|Z(B)|} = \frac{|AZ|\cdot|B|}{|Z(G)\cap B|} \geq \frac{|AZ|\cdot|B|}{|AZ\cap B|} = |G|$. So either $|C_G(a)| \geq |G|^{\frac{1}{2}}$ or $|C_G(c)| \geq |G|^{\frac{1}{2}}$. □

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LEMMA 4.2. *Suppose, in addition to the hypotheses of the above lemma, that* $|A| \nmid |B|$. *Then there exists an* $x \in G - Z$ *with* $|C_G(x)| > |G|^{\frac{1}{2}}$.

Proof. In the case that $|C_G(a)| \cdot |C_G(b)| \geq |A| \cdot |B| = |G|$ with $a \in Z(A)$, $b \in Z(B)$ and $a, b \notin Z(G)$, clearly now $|A| > |G|^{\frac{1}{2}}$ or $|B| > |G|^{\frac{1}{2}}$. In the case that $|C_G(a)| \cdot |C_G(c)| \geq \frac{|AZ| \cdot |B|}{|B \cap Z(G)|} \geq |G|$, with $Z(B) = Z(G)$ and $c \in Z_2(B) - Z(B)$, suppose $|C_G(a)| = |C_G(c)| = |G|^{\frac{1}{2}}$. Then also $|AZ| = \frac{|B|}{|B \cap Z(G)|}$, so $|B| = |A| \cdot [Z:Z \cap A] \, |B \cap Z|$ a contradiction. □

THEOREM 4. *Let* G *be a finite group,* $G = AB$ *for* $A, B \leq G$ *and* $(|A|, |B|) = 1$. *If* $Z_2(A), Z_2(B) \nleq Z(G)$, *then* $|C_G(x)| > |G|^{\frac{1}{2}}$ *for some* $x \in G - Z(G)$.

Proof. If either $Z(A) \nleq Z(G)$ or $Z(B) \nleq Z(G)$ we are done by the previous lemma. So we may assume that $Z(A) = Z(G) \cap A$ and $Z(B) = Z(G) \cap B$. Our hypotheses imply $Z(G) = (Z \cap A) \times (Z \cap B) = Z(A) \times Z(B)$. If $a \in Z_2(A) - Z(G)$ and $b \in Z_2(B) - Z(G)$, then $|C_G(a)| \geq |C_{G/Z(A)}(aZ(A))| \geq [AZ:Z(A)] = [A:Z(A)]\,[Z(G):Z(A)]$, and $|C_G(b)| \geq [B:Z(B)]\,[Z(G):Z(B)]$. Thus $|C_G(a)| \cdot |C_G(b)| \geq |A| \cdot |B| = |G|$. If $|C_G(a)| \neq |C_G(b)|$ we are done; if $|C_G(a)| = |C_G(b)| = |G|^{\frac{1}{2}}$ then $|A| \cdot |Z(B)|^2 = |B| \cdot |Z(A)|^2$ and $(|A|, |B|) = 1$ give $|Z| = |G|^{\frac{1}{2}}$, a contradiction. □

COROLLARY 4.1. *If* G *is a non-abelian group,* $G = G_p G_{p'}$ *and*

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$Z_2(G_{p'}) \nleq Z(G)$ for some prime p, then $|C_G(x)| > |G|^{\frac{1}{2}}$ for some $x \in G - Z(G)$. (In particular, the conclusion holds if G is solvable and $Z_2(G_{p'}) \nleq Z(G)$ for some prime p)

Proof. If $Z_2(G_p) \nleq Z(G)$ then the previous theorem gives the conclusion. If $Z_2(G_p) \leq Z(G)$, then $G_p \leq Z(G)$ and $Z_2(G_{p'}) \leq Z_2(G)$. Now $Z_2(G) = Z(G)$ would yield $G_{p'} \leq Z(G)$ and G abelian, so $Z_2(G) > Z(G)$ and the conclusion follows by Lemma 2(c) of [2]. □

THEOREM 5. *Let $G \in S - C$. Then the following properties hold:*

(a) if $N \trianglelefteq G$ and $N \nleq Z$, then $|N| > |G|^{\frac{1}{2}}$ and $N \cap Z = Z(N) \neq \{1\}$;

(b) for exactly one prime $p \,\big|\, |G|$, $F(G) = ZO_p > Z$ and $|O_p| > |G|^{\frac{1}{2}}$. Also $\{1\} < Z_p = Z(O_p) = Z(G_p)$.

If p is the special prime in (b), then

(c) $O_{p'} = Z_{p'}$, and $F(G) = O_{p'p}(G)$;

(d) O_p is non-abelian of class 2, $F' \leq 2$, and

$$|O_p : Z_p| \leq |G_p : Z_p| < |G|^{\frac{1}{2}} < |O_p| \ ;$$

(e) $[C_G(O_p/Z_p)]_{p'} = Z_{p'}$.

Proof. (a) Suppose $|N| \leq |G|^{\frac{1}{2}}$ and $x \in N - Z(G)$. Then $|[x]| \leq |N| - 1 < |G|^{\frac{1}{2}}$ so $|C_G(x)| > |G|^{\frac{1}{2}}$, contradicting $G \notin C$. Also, if $y \in Z(N) - Z(G)$, then $|C_G(y)| \geq |N| > |G|^{\frac{1}{2}}$, again a contradiction.

(b) Since G is solvable but not nilpotent, $Z(G) < F(G) < G$, so $O_p > Z_p$ for at least one prime p. If prime $q \neq p$ and $O_q > Z_q$ then $|O_p O_q| > |G|$ by (a), a contradiction. Thus for exactly one prime

p, $F = ZO_p > Z$ and $|O_p| > |G|^{\frac{1}{2}}$.

Also $\{1\} < Z(G_p) \leq Z(O_p) = O_p \cap Z(G) \leq Z_p \leq Z(G_p)$.

(c) Let $R = O_{p'}(G)$. Then $Z_{p'} \leq R \trianglelefteq G$.

By (a), if $R \nleq Z(G)$, then $|R| > |G|^{\frac{1}{2}}$ from which $|RO_p| > |G|$, a contradiction. Thus $O_{p'}(G) < Z(G)$ and so $O_{p'}(G) = Z_{p'}$. Clearly $F(G) = ZO_p \leq O_{p'p}(G)$. But $O_{p'}(G) < Z(G)$, so $O_{p'p}(G)$ is nilpotent and thus contained in $F(G)$, so (c) is proved.

(d) As $O_p > Z_p$ and $|O_p| > |G|^{\frac{1}{2}}$, O_p is non-abelian. By Exercise 3 p. 214 in [3], if the nilpotence class of O_p is ≥ 3, then O_p contains a characteristic abelian subgroup A, which is not contained in $Z(O_p)$, and hence is not contained in $Z(G)$. But then A char $O_p \trianglelefteq G$, $A \nleq Z(G)$. This contradicts (a). Thus class $(O_p) = 2$. Since $O_q \leq Z$ whenever $q \neq p$, we have class$(F) = 2$, so $F' \leq Z(F) \leq Z(G)$ (the latter follows from $|F| \geq |O_p| > |G|^{\frac{1}{2}}$). Finally, let $x \in Z_2(G_p) - Z(G_p) = Z_2(G_p) - Z_p$. Then, if $\bar{x} = xZ_p$

$|C_G(x)| \geq |C_{G_p}(x)| \geq |C_{G_p/Z_p}(\bar{x})| = [G_p:Z_p]$. Since $G \notin C$, $[G_p:Z_p] < |G|^{\frac{1}{2}}$.

(e) Clearly $C_G(O_p) \leq Z(G)$ (since $|O_p| > |G|^{\frac{1}{2}}$). If $y \in [C_G(O_p/Z_p)]_{p'}$, then $y \in C_G(O_p) \leq Z(G)$. The latter follows from Theorem 5.3.2, p. 178 of [3]. For suppose y is a p' element and y satisfies $x^{-1}y^{-1}xy \in Z_p$ for all $x \in O_p$ (that is $y \in C_G(O_p/Z_p)$). Then the group $\langle y \rangle$, acting by conjugation on O_p, is a p'-subgroup of $\mathrm{Aut}(O_p)$ which stabilizes the normal series $O_p \geq Z_p \geq \{1\}$ (Lemma 5.3.1 of [3]), and therefore Theorem 5.3.2 applies. Thus conjugation is the identity automorphism, that is $y \in C_G(O_p)$. □

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THEOREM 6. *Let* $G \in S - C$, *and let* p *be the unique prime satisfying* $p \mid [F(G):Z(G)]$. *If* $|Z_p| = p$ *then* G *satisfies the following properties:*

(i) $G_p = O_p$ *is extraspecial;*

(ii) $|G_p| = p^{2m+1} \geq p^5$;

(iii) $Z_2(G_{p'}) < Z(G)$, *and hence* $Z(G_{p'}) = Z(G)_{p'}$.

Proof. Property (iii) follows immediately from Corollary 4.1. By Theorem 5(b),(d) we have that $|O_p(G)| > |G|^{\frac{1}{2}}$ and O_p is non-abelian; hence $|O_p(G)| \geq p^3$. If either $G_p > O_p$ or O_p/Z_p is not elementary abelian, then either $[G_p:Z_p] \geq |O_p| > |G|^{\frac{1}{2}}$, contradicting Theorem 5(d), or $[O_p:Z_p] \geq |H|$ for some characteristic subgroup of H of $O_p \trianglelefteq G$ such that $Z_p = Z(O_p) < H < O_p$. From the latter, $H \nleq Z$, $H \trianglelefteq G$ and $|H| < |G|^{\frac{1}{2}}$ in contradiction to Theorem 5(a). We have thus proved (i), and $|G_p| = p^{2m+1} > |G|^{\frac{1}{2}}$. As for (ii), suppose $|G_p| = p^3$, so G_p/Z_p is elementary abelian of order p^2. By Theorem 5(c) $O_{p'}(G) = Z_{p'}(G)$, and by Theorem 6.3.4 of [3] $G_{p'}/Z_{p'}$ is faithfully represented on $O_p(G)/\Phi(O_p(G)) = G_p/Z_p$ regarded as a vector space over Z_p. Thus $H = G_{p'}/Z_{p'} \leq GL(2, p)$; in face $H \leq PGL(2, p)$ since (by iii) $Z_2(G_{p'}) \leq Z(G) \cap G_{p'} = Z_{p'} = Z(G_{p'})$ and $Z(H) = \{1\}$. If $p = 2$. then $PGL(2, 2) = PSL(2, 2)$ has no subgroups H of odd order with $Z(H) = \{1\}$, from which we get $H = \{1\}$ and $G \in C$. If $p > 2$, then $[PGL(2, p) : PSL(2, p)] = 2$. If $p = 3$, then $|PGL(2, 3)| = 24$ and has no $3'$- subgroups H with $Z(H) = \{1\}$, so $G \in C$. Thus suppose that $p \geq 5$. Here the only solvable p'- subgroups of $PSL(2, p)$ are, by the theorem of Dickson (see [7], Theorem 3.6.25, p. 412):

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(i) dihedral groups of order $p \pm 1$ and their subgroups;

(ii) $\mathrm{Alt}(4)$; (iii) $\mathrm{Sym}(4)$.

Since $Z(H) = \{1\}$, it follows (by [4], Theorem V 8.18(c), p. 506) that some element $xZ_{p'}$ of $H^{\#}$ fixes an element yZ_p of $(G_p/Z_p)^{\#}$ and hence x stabilizes the normal series $\{1\} < Z_p < \langle Z_p, y\rangle$. Thus (by [3], Theorem 5.3.2, p.178) x centralizes $\langle Z_p, y\rangle$ and so we have $|C_G(x)| \geq p^2\ |C_{G_{p'}}(x)|$, $x \notin Z$. If $H \cap PSL(2, p)$ is of type (i), then $|H| \leq 2(p + 1)$ and hence:

$|C_G(x)|^2 \geq p^4 \cdot 2^2 \cdot |Z_{p'}|^2 > p^3 \cdot 2(p + 1)\cdot|Z_{p'}| \geq |G_p|\ |G_{p'}| = |G|$,

yielding $G \in C$. Suppose that $H \cap PSL(2, p) = A_4$ or S_4. Since the Sylow 2-subgroups of A_4 and S_4 are not cyclic or generalized quaternion, it follows (by [4], Theorem V 8.18(a) p. 506) that some nontrivial 2-element of $H^{\#}$ fixes $yZ_p \in (G_p/Z_p)^{\#}$. So we may assume that x is a 2-element, $x \notin Z$. If $Z_{p'} = \{1\}$, then $|C_{G_{p'}}(x)| \geq 4$, and if $Z_{p'} \neq \{1\}$ then $|C_{G_{p'}}(x)| \geq 2\ |Z_{p'}|$. Thus in both cases, $|C_{G_{p'}}(x)|^2 \geq 8\ |Z_{p'}|$, and when $p \geq 7$ we obtain

$|C_G(x)|^2 \geq p^4\ |C_{G_{p'}}(x)|^2 \geq 8p^4\ |Z_{p'}| > 48p^3\ |Z_{p'}| \geq |H|\ p^3\ |Z_{p'}| = |G|$,

and $G \in C$. Whenever $|Z_{p'}| > 2$, then $|C_{G_{p'}}(x)|^2 \geq 12 \cdot |Z_{p'}|$ and $G \in C$ since $p > 3$. Finally, suppose that $|Z_{p'}| = 2$, $p = 5$. If $|H| = 24$ then $G \in C$, as above. By [7] Exercise 9, p. 418 $PGL(2, q)$ contains only solvable p'-subgroups of types (i) - (iii), and thus $|H| = 48$ is impossible. The proof of the theorem is now complete. □

THEOREM 7. *If* $G \in S$ *and* $|G| = \prod_{i=1}^{n} p_i^{\alpha_i}$ *where the* p_i *are distinct primes and* $\alpha_i \leq 4$ *for all* i*, then* $G \in C$.

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Proof. The proof is by induction on the number of prime factors, n. If $n = 1$, then G is nilpotent, so $G \in C$. So assume $n > 1$ and the theorem holds for smaller values of n. By Theorem 5(b) $O_p(G) > Z_p(G)$ for a unique prime $p \,\big|\, |G|$, say $O_{p_1} > Z_{p_1}$. If $|Z_{p_1}| = p_1$, then $G \in C$ by Theorem 6. If $|Z_{p_1}| \geq p_1^2$ then either all groups of order $\prod_{i=2}^{n} p_i^{\alpha_i}$ are abelian and, $G \in C$ by Theorem 1, or by induction there exists a subgroup $H \in C$, $|H| = \prod_{i=2}^{n} p_i^{\alpha_i}$. Hence for some $x \in H - Z(H)$ (so $x \notin Z(G)$) we obtain

$|C_G(x)| \geq |Z_{p_1}|\ |C_H(x)| > p_1^2 (\prod_{i=2}^{n} p_i^{\alpha_i})^{\frac{1}{2}} \geq |G|^{\frac{1}{2}}$, and again $G \in C$. □

THEOREM 8. *Let* $G \in S$, *and* $|G| = p^n qr$ *with* p, q, r *distinct primes. If* $Syl_p(G) \trianglelefteq G$, *then* $G \in C$.

Proof. If $(|Z(G)|, qr) > 1$, then G contains an abelian subgroup of order qr and $G \in C$ by Theorem 1. Otherwise, by Theorem 5(b), $Z(G) = Z(G_p) < O_p = G_p$. Thus $Z = Z(G_p) < Z_2(G_p) = Z_2$, so a subgroup H of order qr acts on Z_2/Z. Since H is non-abelian an element h, say of order r, fixes some $xZ \in (Z_2/Z)^{\#}$. Since $(r, p) = 1$, h centralizes x (using Theorem 5.3.2 of [3]). Thus $|C_G(x)| \geq r \cdot |C_{G_p}(x)| \geq r \cdot |C_{G_p/Z}(xZ)| \geq r \cdot [G_p:Z]$. Also,some element y, of order q, satisfies $|C_G(y)| \geq q \cdot |Z|$, $y \notin Z$. We obtain $|C_G(x)| \cdot |C_G(y)| \geq q \cdot r \cdot |G_p| = |G|$. Since $(q, r) = 1$ we must have $|C_G(x)| \neq |C_G(y)|$, so $G \in C$. □

COROLLARY 8.1. *Let* $G \in S$ *and* $|G| = p^n qr$, *where* p, q, r *are primes with* $p > q > r$. *Then* $G \in C$.

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Proof. Due to the ordering of the primes, it is clear that $\mathrm{Syl}_p(G) \trianglelefteq G$, and $G \in C$ by Theorem 8. □

COROLLARY 8.2. *Let* $G \in S$ *and* $|G| = p^n q r$, *where* p, q, r *are distinct primes and* $ord(p) \geq n - 1 \pmod{q}$. *Then* $G \in C$.

Proof. If $q \,\big|\, |Z(G)|$, then $G \in C$ by Theorem 1. If $\mathrm{Syl}_p(G) \trianglelefteq G$ then $G \in C$ by Theorem 8. So assume that $q \nmid |Z|$ and $O_p < G_p$. By Theorem 5(b) $Z_p > \{1\}$, so $|O_p/Z_p| \leq p^{n-2}$. Thus an element $x \notin Z$, x of order q, centralizes O_p/Z_p. But then $G \in C$ by Theorem 5(e). □

COROLLARY 8.3. *If* $G \in S$ *and* $|G| = p^5 q r$ *with* p, q, r *distinct primes, then* $G \in C$.

Proof. By Theorem 1 we may assume that $|Z(G)| \,\big|\, p^5$. Clearly $G \in C$ if $|Z| \geq p^3$. If $|Z| = p$ then $\mathrm{Syl}_p(G) \trianglelefteq G$ by Theorem 6, and thus $G \in C$ by Theorem 8. So suppose that $|Z| = p^2$. Since a subgroup of order qr is non-cyclic, we may assume without loss of generality that an element of order r centralizes an element $\bar{x}$ of the abelian group $O_p/Z_p = O_p/Z$, by the theory of Frobenius complements. Thus $|C_G(x)| \geq |C_{O_p/Z_p}(\bar{x})| \cdot r \geq p^3 \cdot r$. Also $|C_G(y)| \geq p^2 \cdot q$, for an element y of order q, and we obtain $|C_G(x)||C_G(y)| \geq |G|$. Since $x, y \notin Z(G)$ and $(p, q) = 1$ we have $G \in C$. □

THEOREM 9. *Suppose* $G \in S - C$ *and* $|G| = p^n m$, $(p, m) = 1$, $O_p > Z_p$.
(a) If every non-abelian solvable group of order m *is in* C, *then* $n \geq 5$.
(b) If p *is the minimal prime dividing* $|G|$, *then* $n \geq 7$.

Proof. (a) By Theorem 5(b), $Z_p = Z(O_p) \neq \{1\}$. If $|Z_p| = p$, then $n \geq 5$ by Theorem 6. Suppose $|Z_p| \geq p^2$ and $M < G$, $|M| = m$.

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If M is abelian, then $ZM \geq Z_pM$ is abelian of order $\geq p^2m > |G|^{\frac{1}{2}}$, if $n \leq 4$, contradicting $G \notin C$. If $M \in C$, then for some $x \in M - Z(G)$ we have $|C_G(x)| > p^2m^{\frac{1}{2}} \geq |G|^{\frac{1}{2}}$, if $n \leq 4$, again contradicting $G \notin C$.

(b) If $n \leq 4$ then $G \in C$ by Theorem 7, since p is minimal and (by Theorem 5(b)) $p^n > m$. So assume that $5 \leq n \leq 6$. Let $M < G$, $|M| = m$. By Theorem 1 we may assume $|\pi(M)| \geq 2$. Since p is minimal and $p^n > m$, it follows by Theorem 7 applied to M that either M is abelian or $M \in C$. If M is abelian, then $G \in C$ by Theorem 1. Thus suppose $M \in C$. If $|Z(G_p)| = |Z_p| \geq p^3$, then for some $x \in M - Z$ we find $|C_G(x)| > m^{\frac{1}{2}} |Z_p(G)| \geq m^{\frac{1}{2}}p^3 \geq |G|^{\frac{1}{2}}$, and $G \in C$. So assume that $1 < |Z_p| < p^3$.

Case 1. $p = 2$.

Since $m \cdot 2^5 \leq |G| < |O_p|^2$, if $|O_p| \leq 2^5$ then $m < 2^5$. As m is odd and $m < 32$, either every group of order m is nilpotent (and $G \in C$ by Theorem 1) or $m = 3 \cdot 7$. Thus $|G| = 2^5 \cdot 3 \cdot 7$, or $|G| = 2^6 \cdot 3 \cdot 7$. In either case, $|O_2|^2 > |G|$ yields $|O_2| = |G_2|$, and $G \in C$ by Theorem 8. So suppose that $|O_p| = 2^6 = |G_p|$. By Theorem 6, we may assume that $|Z_p| > p$, and since $|Z_p| < p^3$ we have $|Z_p| = p^2$. Also, we may suppose that $|\pi(m)| \geq 2$ and not every group of order m is abelian. As $m < 64$ m odd, we have two cases: $m = 3 \cdot 7$ and $m = 3^2 \cdot 7$. This is because if $r|m$, $r > 7$ a prime, then an element x of order r acts trivially on O_p/Z_p, of order 16, and $x \in Z$ by Theorem 5(e). But now $\pi(m) = 2$, so every group of order m is abelian, a contradiction. By Theorem 5(d), O_p/Z_p is abelian, and since $7 \nmid (16-1)$ an element of order 7 centralizes some element

$\bar{x} \in (O_p/Z_p)^{\#}$. Thus $|C_G(x)| \geq |O_p/Z_p| \cdot 7 = 16 \cdot 7 > |G|^{\frac{1}{2}}$, and $G \in C$.

Case 2. $p > 2$.

If $|Z_p| = p$ then by Theorem 6 we may assume that $n = 5$, and $|G_p/Z_p| = |O_p/Z_p| = p^4$. If $r \in \pi(m)$ then either every element of order r is in $Z(G)$, or (by Theorem 5(e)) r divides

$\prod_{i=1}^{4} (p^i - 1) = (p^2 + 1)(p^2 + p + 1)(p - 1)^4(p + 1)^2$. As $p < r$, r

divides either $p^2 + 1$ or $p^2 + p + 1$. Thus there are at most two primes $r_1 \neq r_2 \in \pi(m)$ for which there exist r_i-elements outside $Z(G)$. Since $p < r_i$, r_i^2 does not divide the above product; but $[G_{r_i} : Z_{r_i}]$ does divide $\prod_{i=1}^{4} (p^i - 1)$, again by Theorem 5(e), so $[G_{r_i} : Z_{r_i}] \leq r_i$ and G_{r_i} is abelian. If only one such r_i exists then there exists an abelian subgroup of G, of order m, and $G \in C$. So suppose that such $r_1 \neq r_2$ exist, $r_1 | p^2 + 1$ and $r_2 | p^2 + p + 1$. Since $r_2 \nmid p^4 - 1$ there exists an $x \in O_p - Z_p$ such that $r_2 \,\big|\, |C_G(x)|$.

Since O_p/Z_p is abelian (Theorem 5), $|C_G(x)| \geq r_2 p^4 > p^5 = |O_p| > |G|^{\frac{1}{2}}$.

Case 2 is complete and the theorem is proved. □

COROLLARY 9.1. *Suppose* $G \in S$ *and* $|G| = p^n q^m r^\ell$ *where* p, q, r *are distinct primes. If* $p^n > |G|^{\frac{1}{2}}$ *and* $n \leq 4$, *then* $G \in C$.

Finally, using many of the previous theorems and corollaries, together with a few more specialized results, the authors have proved the following:

THEOREM 10. *Every non-abelian group of odd order* $< 10^6$ *is a member of* C. *Every non-abelian solvable group of even order* $\leq 10^4$ *is a member of* C.

In the odd order case we rely on the theorem of Feit-Thompson that every group of odd order is solvable.

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