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HOMOGENEOUS SOLUTIONS OF THE GENERALIZED HEAT EQUATION

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Abstract

We discuss expansions of solutions of the generalized heat equation which have a singularity at zero in terms of two sequences of homogeneous solutions.

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1. Introduction.

A solution of the n-dimensional heat equation

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t) = \sum_{j=0}^{n} \frac{\partial^2}{\partial x_j^2}u(x,t), \tag{1.1}$$
$$x = (x_1, \dots, x_n) \in R^n, \quad t > 0,$$

is called *homogeneous of degree m* if for all $\lambda > 0$, $x \in R^n$ and $t > 0$ we have

$$u(\lambda x, \lambda^2 t) = \lambda^m u(x,t). \tag{1.2}$$

As part of a program outlining analogies between temperature functions and analytic functions, Rosenbloom and Widder [9] introduced two sequences of temperature functions homogeneous of integer degree. Their first sequence, the 'heat polynomials' homogeneous of degree m, are defined by:

$$v_m(x,t) = (-2t)^{m/2} H_m\left(x/\sqrt{-2t}\right), \quad m = 0, 1, \dots \tag{1.3}$$

where $H_m(x)$ is the mth Hermite polynomial orthogonal on R^1 with respect to $e^{-x^2/2}\,dx$. These may be considered as an analogue of $\{z^m\}_{m=0}^{\infty}$ for analytic functions.

The natural region of convergence for an expansion in terms of the $\{v_m\}$ is a time strip $|t| < \sigma$. Furthermore, a temperature function $u(x,t)$ has an expansion in terms of the v_m valid for $|t| < \sigma$ if and only if the *Huygens property* holds there: in other words,

$$(1.4)\qquad u(x,t) = \int_{-\infty}^{\infty} k(x-y,t-t')u(y,t')\,dy, \quad -\sigma < t' < t < \sigma$$

where $k(x,t)$ is the heat kernel:

$$(1.5)\qquad k(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

The 'associated functions' homogeneous of degree $-m-1$, are defined using the *Appell transform*:

$$(1.6)\qquad w_m(x,t) = \mathscr{A} v_m(x,t) = k(x,t)v_m(x/t,-1/t), \quad m = 0, 1, \ldots.$$

Rosenbloom and Widder showed that a temperature function has an expansion in terms of the $\{w_m\}$ valid for $t > \sigma$ if and only if it satisfies the Huygens principle for $t > \sigma$ and satisfies an additional integrability condition.

On the other hand, the Appell transform maps solutions of the heat equation in $|t| < 1/\sigma$ into solutions of the heat equation in $|t| > \sigma$. In addition, a temperature function satisfies the Huygens property in $|t| < 1/\sigma$ if and only if its Appell transform satisfies a Huygens property in $|t| > \sigma$; that is,

$$(1.7)\qquad u(x,t) = \int_R k(x-y,t-t')u(y,t')\,dy$$

whenever $t' < t < -\sigma$, $\sigma < t' < t$ or both $t' > \sigma$ and $t < -\sigma$. Thus expansions in terms of the $\{w_m\}$ are valid in time domains $|t| > \sigma$, and a function $u(x,t)$ can be expanded in terms of the $\{w_m\}$ if and only if $u(x,t)$ satisfies the Huygens property in $|t| > \sigma$: see [6].

In a continuation of Widder's program, two related papers [1, 4] concerned expansions in homogenous solutions of the generalized heat equation

$$(1.8)\qquad \frac{\partial}{\partial t}u(r,t) = \Delta_\mu u(r,t) = \frac{\partial^2 u}{\partial r^2} + \frac{\mu-1}{r}\frac{\partial u}{\partial r}, \quad r > 0,\ \mu > 1.$$

If $\mu = n$ is an integer, then Δ_μ is the Laplacian in radial coordinates in R^n.

Both authors considered expansions in terms of two basic sequences. The radial heat polynomials homogeneous of degree $2m$ were defined by:

$$(1.9)\qquad R_m^\mu(r,t) = m!(4t)^m L_m^{(\mu-2)/2}\left(\frac{-r^2}{4t}\right), \quad m = 0, 1, \ldots$$

where $\{L_m^{(\mu-2)/2}(x)\}$ are the Laguerre polynomials orthogonal with respect to $x^{(\mu-2)/2}e^{-x}\,dx$ on $(0, \infty)$.

The associated functions homogeneous of degree $-2m-1$ are obtained using the Appell transform:

$$\tilde{R}_m^\mu(r,t) = \mathscr{A}_\mu R_m^\mu(r,t) = k_\mu(r,t)R_m^\mu(r/t, -1/t) \tag{1.10}$$

where k_μ is the 'fundamental source solution':

$$k_\mu(r,t) = (4\pi t)^{-\mu/2}e^{-r^2/4t}. \tag{1.11}$$

Expansion theory for $\{R_m^\mu\}$ and $\{\tilde{R}_m^\mu\}$ mirrors expansion theory for $\{v_m\}$ and $\{w_m\}$.

In this case the kernel in the Huygens principle is related to the radialization of translations of the heat kernel in R^n. Let $u(x_1, x_2, \dots, x_n, t)$ be a function defined on $R^n \times (a, b)$ which is radial in $x = (x_1, x_2, \dots, x_n)$, satisfies (1.1) and satisfies the Huygens principle in R^n. In other words,

$$u(x,t) = \int_{R^n} u(y,t')k(x-y, t-t')\,dy, \quad a < t' < t < b, \tag{1.12}$$

where $k(x, t)$ is the heat kernel in R^n

$$k(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/4t}, \quad |x|^2 = x_1^2 + \dots + x_n^2. \tag{1.13}$$

Let $U(r,t) = u(x_1, x_2, \dots, x_n, t)$, $r = |x|$. Then $U(r,t)$ is a solution of (1.8) with $\mu = n$, and switching to polar coordinates with $|y| = \rho$, $y' = y/\rho$ and Σ_{n-1} the unit sphere in R^n we have

(1.14)

$$\begin{aligned} U(r,t) &= \int_0^\infty \left(\int_{\Sigma_{n-1}} U(\rho,t')k(x-\rho y', t-t')\,dy' \right) \rho^{n-1}\,d\rho \\ &= \int_0^\infty U(\rho,t')K_n(r,\rho,t-t')\rho^{n-1}\,d\rho \end{aligned}$$

where

$$\begin{aligned} K_n(r,\rho,t) &= \int_{\Sigma_{n-1}} k(x-\rho y', t)\,dy' \\ &= \frac{(r\rho)^{(2-n)/2}}{2t}e^{-(r^2+\rho^2)/4t}I_{(n-2)/2}\left(\frac{r\rho}{2t}\right) \end{aligned} \tag{1.15}$$

and $I_\nu(z)$ is the Bessel function of complex argument of order ν (see [1]).

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Therefore, for arbitrary $\mu > 1$, the Huygens property in (a, b) is defined by

$$(1.16) \qquad u(r,t) = \int_0^\infty u(\rho, t') K_\mu(r, \rho, t-t') \rho^{\mu-1}\, d\rho, \quad a < t' < t < b$$

where

$$(1.17) \qquad K_\mu(r, \rho, t) = \frac{(r\rho)^{(2-\mu)/2}}{2t} e^{-(r^2+\rho^2)/4t} I_{(\mu-2)/2}\left(\frac{r\rho}{2t}\right).$$

In [1, 4], it was shown that a function $u(r, t)$ had an expansion in terms of $R_m^\mu(r, t)$ converging in a strip $|t| < \sigma$ if and only if $u(r, t)$ satisfied the Huygen's property in that strip. Further, it was shown that a function had an expansion in terms of $\tilde{R}_m^\mu(r, t)$ converging in a domain $t > \sigma$ if and only if it satisfied the Huygen's principle in $t > \sigma$ and satisfied an additional integrability condition.

For $\mu = 1$, $\{R_m^\mu\}$ and $\{\tilde{R}_m^\mu\}$ reduce to the sequences $\{v_{2m}\}_{m=0}^\infty$, $\{w_{2m}\}_{m=0}^\infty$ in [8] because of the connection between the Hermite and Laguerre polynomials. Thus [1,4] included part of the theory of Rosenbloom and Widder.

On the other hand, many elementary solutions of (1.8) cannot be expanded in terms of the radial heat polynomials or their associated functions. All such solutions were entire funcions of r^2. This precludes expansions for solutions which have a singularity at zero, for example, $r^{2-\mu}$.

However, as we will see below, the approach used in [1] to derive the heat polynomials leads to two other linearly independent sequences of homogenous solutions. We define solutions homogeneous of degree $2m + 2 - \mu$ by:

$$(1.18) \qquad V_m^\mu(r,t) = r^{2-\mu}(4t)^m m! L_m^{(2-\mu)/2}\left(\frac{-r^2}{4t}\right), \quad m = 0, 1, \ldots,$$

where the $L_m^{(2-\mu)/2}$ are the Laguerre polynomials of order $(2-\mu)/2$. Note that for $\mu > 4$, these polynomials do not form an orthogonal system in the traditional sense.

The associated functions are obtained using the Appell transform:

$$(1.19) \qquad \tilde{V}_m^\mu(r,t) = \mathscr{A}_\mu V_m^\mu(r,t) = k_\mu(r,t) V_m^\mu\left(\frac{r}{t}, \frac{-1}{t}\right), \quad m = 0, 1, \ldots.$$

These are homogenous of degree $-2m - 2$.

For $\mu = 1$, the sequences $\{V_m^\mu\}$ and $\{\tilde{V}_m^\mu\}$ reduce to the sequences $\{v_{2m+1}\}_{m=0}^\infty$, $\{w_{2m+1}\}_{m=0}^\infty$ of Rosenbloom and Widder ([8]).

We will show that the natural region of convergence for expansions of V_m^μ are time strips $|t| < \sigma$ and that expandibility is equivalent to a certain Huygens principle. We will also prove the corresponding results for the associated functions $\tilde{V}_m^\mu$. Our main difficulty is working with nonintegrable functions.

Another aspect of Widder's program was a characterization of all homogeneous solutions of integer degree. The underlying observation is that analytic functions have two linearly independent solutions of degree n for each n, namely $\mathrm{Re}(z^n)$ and $\mathrm{Im}(z^n)$. This aspect was extended to (1.8) in [5], and this paper overlaps that one to some extent. Specifically, if $\mu = 2, 3, \ldots, 2m + 2 - \mu \geq 0$, then our V_m^μ, $\tilde{V}_m^\mu$ may be expressed as linear combinations of solutions in [5]. However, we characterize expansions in terms of a Huygens property, while the characterization in [5] is of a function theoretic nature.

We organize the paper as follows. In Sections 2 – 4 we develop tools needed later in the paper. In Sections 5 – 7 we derive our homogeneous solutions and discuss their elementary properties including regions of convergence for expansions. Finally, in Sections 8 – 10 we discuss Poisson integrals and characterize expansions in terms of the Huygens property.

2. Hadamard's 'finite part' integral

To handle divergent integrals, we will use Hadamard's 'finite part' integral.

DEFINITION 2.1. ([3]) Let n be a positive integer, $-(n+1) \leq \nu < -n$. Let $f(x)$ be a given function, and suppose there exists a polynomial $P_{n-1}(x)$ of degree at most $n-1$ so that

$$\int_0^\infty |f(x) - P_{n-1}(x)| x^\nu \, dx < \infty. \tag{2.1}$$

Then Hadamard's finite part (f.p.) integral is defined by

$$\text{f.p.} \int_0^\infty f(x) x^\nu \, dx = \int_0^\infty (f(x) - P_{n-1}(x))\, x^\nu \, dx. \tag{2.2}$$

Clearly, if such a polynomial exists, then it is unique. Furthermore, for $-(n+1) < \nu < -n$, if $f(x), f'(x), \ldots, f^{(n)}(x)$ are defined on $[0, a]$ for some $a > 0$, if $f^{(n)}(x) \in L([0,a]; x^{\nu+n}\, dx)$ and if $f(x) \in L((a,\infty); x^\nu \, dx)$ then Hadamard's integral will exist with P_{n-1} the $(n-1)^{\text{st}}$ Taylor polynomial of f centered at 0. Note that in general Hadamard's integral will not exist if $\nu = -(n+1)$ unless $f^{(n)}(0) = 0$.

For future reference, note that if the first integral below exists, then

$$\text{f.p.} \int_0^\infty f(x) x^{(\nu-1)/2} \, dx = 2\, \text{f.p.} \int_0^\infty f(x^2) x^\nu \, dx. \tag{2.3}$$

For $\nu > -1$, we adopt the convention

$$\text{f.p.} \int_0^\infty f(x) x^\nu \, dx = \int_0^\infty f(x) x^\nu \, dx.$$

Frequently one is able to replace Hadamard's integral with an ordinary integral.

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LEMMA 2.2. *Let n be a positive integer, $-(n+1) < \nu < -n$ and suppose $f(x)$ is n-times differentiable on $[0, \infty)$. Then if $f \in L((1,\infty); x^\nu\, dx)$ and $f^{(n)} \in L((0,\infty); x^{n+\nu}\, dx)$ we have*

$$\text{f.p.} \int_0^\infty f(x)\, x^\nu\, dx = \frac{\Gamma(-n-\nu)}{\Gamma(-\nu)} \int_0^\infty f^{(n)}(x)\, x^{\nu+n}\, dx. \tag{2.4}$$

PROOF. Let P_{n-1} denote the $(n-1)$st Taylor polynomial of f centered at 0. We have

$$\begin{aligned}
\text{f.p.} \int_0^\infty f(y)\, y^\nu\, dy &= \int_0^\infty (f(y) - P_{n-1}(y))\, y^\nu\, dy \\
&= \int_0^\infty \left(\int_0^y \frac{f^{(n)}(t)}{(n-1)!} (y-t)^{n-1}\, dt \right) y^\nu\, dy \\
&= \frac{1}{(n-1)!} \int_0^\infty \left(\int_t^\infty (y-t)^{n-1} y^\nu\, dy \right) f^{(n)}(t)\, dt \\
&= \frac{\Gamma(-n-\nu)}{\Gamma(-\nu)} \int_0^\infty f^{(n)}(y)\, y^{\nu+n}\, dy.
\end{aligned}$$

We now consider iterated integrals.

THEOREM 2.3. *Let $-(n+1) < \nu < -n$, $a > 0$. Suppose $f(x, y)$ has continuous mixed partials up to degree n in each variable in $[0, a] \times [0, a]$,*

$$\int_a^\infty \int_a^\infty |f(x,y)| (xy)^\nu\, dx\, dy < \infty, \tag{2.5}$$

$$\int_a^\infty |\partial_y^j f(x,0)| x^\nu\, dx < \infty, \quad j = 0, 1, \dots, n-1, \tag{2.6}$$

and

$$\int_a^\infty |\partial_x^j f(0,y)| y^\nu\, dy < \infty, \quad j = 0, 1, \dots, n-1. \tag{2.7}$$

Suppose also that there exist functions $g(x) \in L((a,\infty); x^\nu\, dx)$ and $h(y) \in L((a,\infty); y^\nu\, dy)$, $j = 0, 1, \dots, n$ such that

$$|\partial_y^n f(x,y)| \le g(x), \quad \textit{for all } y \in [0, a], \tag{2.8}$$

and

$$|\partial_x^n f(x,y)| \le h(y), \quad \textit{for all } x \in [0, a], \tag{2.9}$$

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Then the iterated integrals

$$\text{f.p.} \int_0^\infty \left(\text{f.p.} \int_0^\infty f(x, y)\, x^\nu\, dx \right) y^\nu\, dy \tag{2.10}$$

and

$$\text{f.p.} \int_0^\infty \left(\text{f.p.} \int_0^\infty f(x, y)\, y^\nu\, dy \right) x^\nu\, dx \tag{2.11}$$

both exist and are equal.

PROOF. Let

$$P_{n-1}(x, y) = \sum_{i=0}^{n-1} \frac{\partial_x^i f(0, y)}{i!} x^i, \qquad Q_{n-1}(x, y) = \sum_{i=0}^{n-1} \frac{\partial_y^i f(x, 0)}{i!} y^i,$$

$$R_{n-1}(x, y) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{\partial_y^i \partial_x^j f(0, 0)}{j!\, i!} x^j y^i,$$

and

$$M = \max\left\{ |\partial_y^j \partial_x^i f(x, y)| : 0 \le y \le a,\ 0 \le x \le a,\ 0 \le j \le n,\ 0 \le i \le n \right\}.$$

By symmetry and Fubini's theorem, it suffices to show that

$$\begin{aligned} &\text{f.p.} \int_0^\infty \left(\text{f.p.} \int_0^\infty f(x, y) x^\nu\, dx \right) y^\nu\, dy \\ &\quad = \int_0^\infty \int_0^\infty \big(f(x, y) - P_{n-1}(x, y) - Q_{n-1}(x, y) + R_{n-1}(x, y) \big) (xy)^\nu\, dx\, dy \end{aligned}$$

where the latter integral converges absolutely. To prove the absolute convergence, we consider separately the four regions $(a, \infty) \times (a, \infty)$, $(a, \infty) \times (0, a)$, $(0, a) \times (a, \infty)$, and $(0, a) \times (0, a)$.

Clearly, (2.5), (2.6) and (2.7) imply that

$$\int_a^\infty \int_a^\infty \left| f(x, y) - P_{n-1}(x, y) - Q_{n-1}(x, y) + R_{n-1}(x, y) \right| (xy)^\nu\, dx\, dy < \infty.$$

Secondly, from (2.9) we have

$$\begin{aligned} &\int_a^\infty \int_0^a |f(x, y) - P_{n-1}(x, y)| (xy)^\nu\, dx\, dy \\ &\quad = \int_a^\infty \int_0^a \left| \frac{1}{(n-1)!} \int_0^x \partial_t^n f(t, y)(x - t)^{n-1}\, dt \right| (xy)^\nu\, dx\, dy \\ &\quad \le \frac{1}{n!} \left(\int_a^\infty h(y) y^\nu\, dy \right) \left(\int_0^a x^{n+\nu}\, dx \right) < \infty. \end{aligned}$$

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Also, since

$$R_{n-1}(x, y) = \sum_{i=0}^{n-1} \frac{\partial_x^i Q_{n-1}(0, y)}{i!} x^i$$

we have

$$\begin{aligned}
&\int_a^\infty \int_0^a |Q_{n-1}(x, y) - R_{n-1}(x, y)|(xy)^\nu \, dx\, dy \\
&\quad = \int_a^\infty \int_0^a \left| \frac{1}{(n-1)!} \int_0^x \partial_t^n Q_{n-1}(t, y)(x-t)^{n-1}\, dt \right| (xy)^\nu \, dx\, dy \\
&\quad \le \frac{1}{(n-1)!} \sum_{i=0}^{n-1} \frac{1}{i!} \int_a^\infty \int_0^a \left(\int_0^x |\partial_t^n \partial_y^i f(t, 0)|(x-t)^{n-1}\, dt \right) x^\nu y^{\nu+i}\, dx\, dy \\
&\quad \le \frac{M}{n!} \sum_{i=0}^{n-1} \frac{1}{i!} \left(\int_a^\infty y^{i+\nu}\, dy \right) \left(\int_0^a x^{n+\nu}\, dx \right) < \infty.
\end{aligned}$$

Therefore,

$$\int_a^\infty \int_0^a \left| f(x, y) - P_{n-1}(x, y) - Q_{n-1}(x, y) + R_{n-1}(x, y) \right| (xy)^\nu \, dx\, dy < \infty.$$

The integral over $(0, a) \times (a, \infty)$ is finite by a similar argument. Finally,

$$\begin{aligned}
&\int_0^a \int_0^a \left| f(x, y) - P_{n-1}(x, y) - Q_{n-1}(x, y) + R_{n-1}(x, y) \right| (xy)^\nu \, dx\, dy \\
&\quad = \frac{1}{((n-1)!)^2} \int_0^a \int_0^a \left| \int_0^y \left(\int_0^x \partial_s^n \partial_t^n f(t, s)(x-t)^{n-1}\, dt \right) (y-s)^{n-1}\, ds \right| \\
&\qquad\qquad (xy)^\nu \, dx\, dy \\
&\quad \le \frac{M}{(n!)^2} \int_0^a \int_0^a (xy)^{n+\nu}\, dx\, dy < \infty.
\end{aligned}$$

This proves the absolute convergence of the integral.

Let

$$k(y) = \text{f.p.} \int_0^\infty f(x, y)\, x^\nu\, dx = \int_0^\infty (f(x, y) - P_{n-1}(x, y))\, x^\nu\, dx.$$

By the first hypothesis and Fubini's theorem this integal exists for a.e. y. Let

$$K_{n-1}(y) = \int_0^\infty (Q_{n-1}(x, y) - R_{n-1}(x, y)) x^\nu\, dx.$$

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Then K_{n-1} is a polynomial of degree $n-1$ for which

$$\int_0^\infty |k(y) - K_{n-1}(y)| y^\nu \, dy < \infty.$$

Therefore

$$\text{f.p.} \int_0^\infty \left(\text{f.p.} \int_0^\infty f(x, y) x^\nu \, dx \right) y^\nu \, dy = \int_0^\infty (k(y) - K_{n-1}(y)) y^\nu \, dy$$
$$= \int_0^\infty \int_0^\infty \big(f(x, y) - P_{n-1}(x, y) - Q_{n-1}(x, y) + R_{n-1}(x, y) \big) (xy)^\nu \, dx \, dy.$$

This completes the proof.

3. Laguerre polynomials

For $\nu > -1$, the Laguerre polynomials $\{L_m^\nu(x)\}$ are defined by orthogonality:

$$\int_0^\infty L_k^\nu(x) L_m^\nu(x) e^{-x} x^\nu \, dx = \frac{\Gamma(k+\nu+1)}{k!} \delta_{km}, \quad k, m = 0, 1, 2, \dots \tag{3.1}$$

and the condition that each $L_m^\nu(x)$ is a polynomial of degree m with coefficient of x^m equal to $(-1)^m$. The definition extends to all $\nu \in C$ using the explicit representation:

$$L_m^\nu(x) = \sum_{j=0}^m \frac{\Gamma(m+\nu+1)}{\Gamma(j+\nu+1)} \frac{(-1)^j x^j}{j!(m-j)!}, \quad m = 0, 1, \dots \tag{3.2}$$

or:

$$L_m^\nu(x) = \frac{e^x x^{-\nu}}{m!} \frac{d^m}{dx^m} (e^{-x} x^{\nu+m}). \tag{3.3}$$

Orthogonality extends to Laguerre polynomials of negative order $\nu \neq -1, -2, \dots$ using Hadamard's integral ([6]):

$$\text{f.p.} \int_0^\infty L_k^\nu(x) L_m^\nu(x) e^{-x} x^\nu \, dx = \frac{\Gamma(k+\nu+1)}{k!} \delta_{k,m}, \quad k, m = 0, 1, 2, \dots. \tag{3.4}$$

For $\nu = -1, -2, \dots$ the orthogonality relation holds for restricted indices. Let $\nu = -l$, where l denotes a positive integer. Then for all $k \geq l$ [10]:

$$L_k^{(-l)}(x) = (-x)^l \frac{(k-l)!}{k!} L_{k-l}^{(l)}(x) \tag{3.5}$$

so that we have for $k, m \geq l$:

(3.6)
$$\int_0^\infty L_k^{(-l)}(x) L_m^{(-l)}(x) e^{-x} x^{-l} \, dx = \frac{(k-l)!(m-l)!}{k!m!} \int_0^\infty L_{k-l}^{(l)}(x) L_{m-l}^{(l)}(x) e^{-x} x^l \, dx$$
$$= \frac{(k-l)!}{k!} \delta_{k,m}.$$

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4. Laguerre polynomials: integral formulae, inequalities, and expansions

It is known that for all $x \in C, m+\nu > -1$ ([10]):

$$L_m^\nu(x) = \frac{e^x x^{-\nu/2}}{m!} \int_0^\infty s^{m+\nu/2} J_\nu(2\sqrt{xs}) e^{-s}\, ds \tag{4.1}$$

where $J_\nu(z)$ is the Bessel function of order ν:

$$J_\nu(z) = \sum_{k=0}^\infty \frac{(-1)^k (z/2)^{\nu+2k}}{k!\Gamma(k+\nu+1)}. \tag{4.2}$$

The Bessel function of complex argument is defined by:

$$I_\nu(z) = i^{-\nu} J_\nu(-iz) = \sum_{k=0}^\infty \frac{(z/2)^{\nu+2k}}{k!\Gamma(k+\nu+1)}. \tag{4.3}$$

For $-\pi/2 + \delta \le \arg(z) \le 3\pi/2 - \delta, \delta > 0$, the following assymptotic formula holds [9]:

$$I_\nu(z) = (2\pi z)^{-1/2}(e^z + e^{-z+(\nu+\frac{1}{2})\pi i})\{1 + O(|z|^{-1})\}, \quad z \to \infty. \tag{4.4}$$

Define

$$I_\nu^*(z) = z^{-\nu/2} I_\nu(2\sqrt{z}) = \sum_{k=0}^\infty \frac{z^k}{k!\Gamma(k+\nu+1)}, \quad z \in C. \tag{4.5}$$

Then for $-\pi/2 + \delta \le \arg(z) \le 3\pi/2 - \delta, \delta > 0$:

$$I_\nu^*(z) = \frac{1}{\sqrt{4\pi}} z^{-\nu/2-1/4}(e^{2\sqrt{z}} + e^{-2\sqrt{z}+(\nu+1/2)\pi i})\{1 + O(|z|^{-1/2})\}, \quad z \to \infty. \tag{4.6}$$

THEOREM 4.1. *For $m = 0, 1, \ldots, x \in C, \nu \in R, m+\nu \ne -1, -2, \ldots$, we have*

$$m! L_m^\nu(-x) = \text{f.p.} \int_0^\infty I_\nu^*(xs) e^{-(s+x)} s^{m+\nu}\, ds. \tag{4.7}$$

PROOF. This is equation (4.1) if $m+\nu > -1$. Therefore, since the left hand side above is an analytic function of ν, it suffices to prove that the right hand side is an analytic function of ν, for $m+\nu \ne -1, -2, \ldots$. We will do this by showing that for all $x \in C$, and $m+\nu \ne -1, -2, \ldots$ we have

$$\text{f.p.} \int_0^\infty s^{m+\nu} I_\nu^*(xs) e^{-s}\, ds = \sum_{k=0}^\infty \frac{x^k \Gamma(k+\nu+m+1)}{k!\Gamma(k+\nu+1)} \tag{4.8}$$

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where the last sum converges almost uniformly with repect to ν. Fix n so that $m+n+\nu > -1$. Then by (3.4) with $k = m = 0$

$$\Gamma(\nu+1) = \text{f.p.} \int_0^\infty s^\nu e^{-s}\, ds, \quad \nu \neq -1, -2, \ldots,$$

so that formally

$$\begin{aligned} \text{f.p.} \int_0^\infty s^{m+\nu} I_\nu^*(xs) e^{-s}\, ds &= \sum_{k=0}^{n-1} \frac{x^k}{k!\Gamma(k+\nu+1)} \text{f.p.} \int_0^\infty s^{m+\nu+k} e^{-s}\, ds \\ &\quad + \int_0^\infty s^{m+\nu} \sum_{k=n}^{\infty} \frac{(xs)^k}{k!\Gamma(k+\nu+1)} e^{-s}\, ds \\ &= \sum_{k=0}^{\infty} \frac{x^k \Gamma(k+\nu+m+1)}{k!\Gamma(k+\nu+1)}. \end{aligned}$$

The exchange of integration and summation is justified, since for fixed ν, m,

$$\frac{\Gamma(m+k+\nu+1)}{\Gamma(k+\nu+1)} = O(k^m), \quad k \to \infty.$$

This completes the proof.

THEOREM 4.2. *For all* $x \in C, \nu \in R, a \neq 1, m = 0, 1, \ldots, m+\nu \neq -1, -2, \ldots,$ *we have*

$$L_m^\nu\left(\frac{ax}{1-a}\right) = \frac{1}{(1-a)^m} \text{f.p.} \int_0^\infty I_\nu^*(xy) L_m^\nu(ay) e^{-(x+y)} y^\nu\, dy. \tag{4.9}$$

PROOF. The case $\nu > -1$ is due to Erdelyi ([2]). For $\nu < -1$, $\nu \neq -2, -3, \ldots$ we have, by Theorem 4.1,

$$\begin{aligned} &\text{f.p.} \int_0^\infty I_\nu^*(xy) L_m^\nu(ay) e^{-(x+y)} y^\nu\, dy \\ &\quad = \sum_{j=0}^{m} \frac{\Gamma(m+\nu+1)}{\Gamma(j+\nu+1)} \frac{(-a)^j}{j!(m-j)!} \text{f.p.} \int_0^\infty I_\nu^*(xy) e^{-(x+y)} y^{\nu+j}\, dy \\ &\quad = \sum_{j=0}^{m} \frac{\Gamma(m+\nu+1)}{\Gamma(j+\nu+1)} \frac{(-a)^j}{(m-j)!} L_j^\nu(-x). \end{aligned}$$

The last quantity is an entire function of ν for each a and x and is equal to $(1-a)^m L_m^\nu(ax/(1-a))$ (which is also entire) for $\nu > -1$; therefore equality holds for all ν.

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If $\nu = -k$, $k = 1, 2, \ldots$, then $m + \nu \neq -1, -2, \ldots$ implies $m \geq k$. Since

$$(4.10) \qquad I^*_{-k}(y) = y^k I^*_k(y) \quad \text{and} \quad L^{(-k)}_m(y) = (-y)^k \frac{(m-k)!}{m!} L^{(k)}_{m-k}(y)$$

the theorem in this case follows from the theorem for $\nu = k$.

By multiplying $(1-a)^m$ onto the left hand side of equation (4.9) and taking the limit as $a \to 1$ we obtain

COROLLARY 4.3. *For all $x \in C$, $m = 0, 1, \ldots$, $\nu \in R$, $m + \nu \neq -1, -2, \ldots$, we have*

$$(4.11) \qquad \frac{(-x)^m}{m!} = \text{f.p.} \int_0^\infty I^*_\nu(xy) e^{-(x+y)} L^\nu_m(y) y^\nu \, dy.$$

Note that for $\nu > -1$, $\gamma \in R$, integrals of the form $\int_0^\infty I^*_\gamma(zy) F(y) y^\nu \, dy$ define analytic functions of z. Similarly, for $\nu < -1$ we have

THEOREM 4.4. *Let $0 < a < b$, $\gamma \in R$, $\nu < -1$ and $\nu \neq -2, -3, \ldots$. Suppose*

$$(4.12) \qquad u(z, x) = \text{f.p.} \int_0^\infty I^*_\gamma(zy) e^{-xy} F(y) y^\nu \, dy$$

is well-defined for $x \in (a, b)$ and all $z \in C$. Then $u(z, x)$ is an entire function of z for each $x \in (a, b)$.

PROOF. Fix x. Suppose $-(n+1) < \nu < -n$. We first prove the existence of the integrals

$$\text{f.p.} \int_0^\infty e^{-xy} F(y) y^{\nu+j} \, dy, \quad j = 0, 1, \ldots.$$

The case $j = 0$ follows from the hypothesis with $z = 0$.

Let $P_{n-1}(y) = a_0 + a_1 y + \cdots + a_{n-1} y^{n-1}$ be the polynomial in the definition of Hadamard's integral for the case $j = 0$. For $j = 1, 2, \ldots, n-1$ let $P_{j-1}(y) = a_0 + a_1 y + \cdots + a_{j-1} y^{j-1}$.

If $j \geq n$, then $\nu + j > -1$, so that

$$\int_0^1 |F(y)| e^{-xy} y^{\nu+j} \, dy \leq \int_0^1 |F(y) e^{-xy} - P_{n-1}(y)| y^\nu \, dy + \int_0^1 |P_{n-1}(y)| y^{\nu+j} \, dy < \infty.$$

If $j = 1, 2, \ldots, n-1$ then $-(n-j+1) < \nu + j < -(n-j)$ and

$$\begin{aligned}\int_0^1 &|F(y)e^{-xy} - P_{n-j-1}(y)|y^{\nu+j}\,dy \\ &\le \int_0^1 |F(y)e^{-xy} - P_{n-1}(y)|y^{\nu}\,dy + \int_0^1 |P_{n-1}(y) - P_{n-j-1}(y)|y^{\nu+j}\,dy < \infty.\end{aligned}$$

Since also

$$\int_1^\infty e^{-xy}|F(y)|y^{\nu+j}\,dy < \infty$$

for all $j \in N$ and $x \in (a, b)$, this proves the claim.

Define

$$I^*_{\gamma,m}(z) = \sum_{j=m}^{\infty} \frac{z^j}{j!\Gamma(\gamma + j + 1)} \tag{4.13}$$

where m is chosen sufficiently large so that both $m + \gamma, m + \nu > -1$. Note that

$$\begin{aligned}|I^*_{\gamma,m}(z)| &\le |z|^m \sum_{j=m}^{\infty} \frac{|z|^{j-m}}{j!\Gamma(\gamma + j + 1)} \\ &= |z|^m \sum_{j=0}^{\infty} \frac{|z|^j}{(j+m)!\Gamma(\gamma + m + j + 1)} \\ &\le |z|^m I^*_{\gamma+m}(|z|).\end{aligned} \tag{4.14}$$

Define

$$v(z, x) = \int_0^\infty I^*_{\gamma,m}(zy)e^{-xy}F(y)y^{\nu}\,dy.$$

Fix ϵ, $M > 0$, and suppose $|z| \le M$. Let $x' \in (a, b)$, $x' < x$. There exists a constant C independent of z such that

$$\begin{aligned}\int_0^\infty |I^*_{\gamma,m}(zy)F(y)|e^{-xy}y^{\nu}\,dy &\le C|z|^m \int_0^\infty e^{(2+\epsilon)\sqrt{y|z|}}e^{-xy}|F(y)|y^{\nu+m}\,dy \\ &\le C \int_0^\infty e^{-x'y}|F(y)|y^{\nu+m}\,dy.\end{aligned}$$

Let ρ be any closed curve in the region $|z| \le M$. By Fubini's Theorem we have

$$\int_\rho v(z, x)\,dz = \int_0^\infty \left(\int_\rho I^*_{\gamma,m}(zy)\,dz\right) e^{-xy}F(y)y^{\nu}\,dy = 0.$$

Therefore, by Morera's Theorem, $v(z,x)$ is entire in z. Since

$$u(z,x) = v(z,x) + \sum_{j=0}^{m-1} \frac{z^j}{j!\Gamma(\gamma+j+1)} \text{f.p.} \int_0^\infty e^{-xy} F(y) y^{\nu+j}\, dy$$

this completes the proof.

We will need size estimates on the Laguerre polynomials for both positive and negative values of x. For positive values of x we have:

LEMMA 4.5. (i) ([10], p. 241) *For any* $\nu \in R$ *and* $c > 0$

$$L_m^\nu(x) = O(m^{\nu/2-1/4}) e^{x/2} x^{-\nu/2-1/4}, \quad m \to \infty \tag{4.15}$$

uniformly in (c, ∞).

(ii) ([10], p. 178) *For any* $\nu \in R$ *and* $w > 0$ *we have*

$$L_m^\nu(x) = O(m^\gamma), \quad m \to \infty \tag{4.16}$$

uniformly in $[0, w]$, *where* $\gamma = \max\{\nu, \nu/2 - 1/4\}$.

For negative values of x we have:

LEMMA 4.6. (i) *For any* $\delta > 1$, $\nu \in R$ *and* $c > 0$ *we have*

$$|L_m^\nu(-x)| = O(m^{\nu/2-1/4}\delta^m) e^{x/(\delta-1)} x^{-\nu/2-1/4}, \quad m \to \infty \tag{4.17}$$

uniformly in (c, ∞).

(ii) *For any* $\delta > 1$, $\nu \in R$ *and* $w > 0$ *we have*

$$|L_m^\nu(-x)| = O(m^\gamma \delta^m), \quad m \to \infty \tag{4.18}$$

uniformly for $x \in [0, w]$, *where* $\gamma = \max\{\nu, \nu/2 - 1/4\}$.

PROOF. (i) From (4.6),

$$I_\nu^*(s) = O(1) s^{-\nu/2-1/4} e^{2\sqrt{s}}, \quad s \in R,\ s \to \infty. \tag{4.19}$$

Thus, there exists a constant C independent of $x > 0$ such that

$$|I_\nu^*(xs)| \le C \begin{cases} 1 & \text{if } s < 1/x \\ (xs)^{-\nu/2-1/4} e^{2\sqrt{xs}} & \text{if } s \ge 1/x. \end{cases}$$

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Let $m + \nu > -1$. By Theorem 4.1

(4.20)
$$\begin{aligned}|L_m^\nu(-x)| &\leq \frac{1}{m!}\int_0^\infty |I_\nu^*(xs)|e^{-(s+x)}s^{m+\nu}\,ds\\ &= \frac{1}{m!}\left\{\int_0^{\frac{1}{x}} |I_\nu^*(xs)|e^{-(s+x)}s^{m+\nu}\,ds + \int_{\frac{1}{x}}^\infty |I_\nu^*(xs)|e^{-(s+x)}s^{m+\nu}\,ds\right\}\\ &\leq \frac{C}{m!}\left\{\int_0^{\frac{1}{x}} e^{-(s+x)}s^{m+\nu}\,ds + x^{-\nu/2-1/4}\int_0^\infty e^{-(\sqrt{s}-\sqrt{x})^2}s^{m+\nu/2-1/4}\,ds\right\}.\end{aligned}$$

Therefore, since for any $\delta > 1$ and all $x, s \geq 0$:

$$e^{-(\sqrt{s}-\sqrt{x})^2} \leq e^{x/(\delta-1)}e^{-s/\delta} \tag{4.21}$$

we have uniformly for $x \geq c$, $c > 0$

$$\begin{aligned}|L_m^\nu(-x)| &\leq \frac{O(1)}{m!}x^{-\nu/2-1/4}e^{x/(\delta-1)}\int_0^\infty e^{-s/\delta}s^{m+\nu/2-1/4}\,ds\\ &= \frac{O(1)}{m!}x^{-\nu/2-1/4}e^{x/(\delta-1)}\delta^m\Gamma(m+\nu/2+3/4)\\ &= O(m^{\nu/2-1/4}\delta^m)x^{-\nu/2-1/4}e^{x/(\delta-1)}.\end{aligned}$$

(ii). If $\nu \geq -1/2$, then (4.19) implies that we have $|I_\nu^*(s)| \leq Ce^{2\sqrt{s}}$, $s \geq 0$ so that we have uniformly for $x \in [0, w]$

$$\begin{aligned}|L_m^\nu(-x)| &\leq \frac{C}{m!}\int_0^\infty e^{-(\sqrt{s}-\sqrt{x})^2}s^{m+\nu}\,ds\\ &\leq \frac{C}{m!}\int_0^\infty e^{-s/\delta}s^{m+\nu}\,ds\\ &= O(m^\nu\delta^m).\end{aligned}$$

If $\nu < -1/2$, then $-\nu/2 - 1/4 \geq 0$, so from (4.20) we have uniformly for $x \in [0, w]$

$$\begin{aligned}|L_m^\nu(-x)| &\leq \frac{C}{m!}\left\{\int_0^\infty e^{-(s+x)}s^{m+\nu}\,ds + \int_0^\infty e^{-(\sqrt{s}-\sqrt{x})^2}s^{m+\nu/2-1/4}\,ds\right\}\\ &\leq \frac{C}{m!}\left\{\Gamma(m+\nu+1) + \delta^{m+\nu/2+3/4}\Gamma(m+\nu/2+3/4)\right\}\\ &= O(m^{\nu/2-1/4}\delta^m).\end{aligned}$$

This concludes the proof.

THEOREM 4.7. *For* $n \in N$, $-(n+1) < \nu < -n$ *and* $\gamma_n = \max\{\nu+n, (\nu+n)/2 - 1/4\}$ *suppose*

$$\sum_{m=0}^{\infty} |c_m| m^{\gamma_n} < \infty. \tag{4.22}$$

Let $c > 0$ *and let* f *be a function for which*

$$\int_c^{\infty} |f(x)| e^{-x/2} x^{\nu/2-1/4}\, dx < \infty \tag{4.23}$$

and for which

$$\text{f.p.} \int_0^{\infty} f(x) e^{-x} x^{\nu}\, dx \tag{4.24}$$

exists. Then

(4.25)

$$\text{f.p.} \int_0^{\infty} f(x) \left(\sum_{m=0}^{\infty} c_m L_m^{\nu}(x) \right) e^{-x} x^{\nu}\, dx = \sum_{m=0}^{\infty} c_m \,\text{f.p.} \int_0^{\infty} f(x) L_m^{\nu}(x) e^{-x} x^{\nu}\, dx.$$

PROOF. Since $d(L_m^{\nu}(x))/dx = -L_{m-1}^{\nu+1}(x)$, we have for $j = 0, 1, \ldots, n$:

$$\sum_{m=0}^{\infty} \left| c_m \frac{d^j}{dx^j} L_m^{\nu}(x) \right| = O(1) \begin{cases} 1 & \text{if } x \in [0, c] \\ e^{x/2} x^{-(\nu+j)/2-1/4} & \text{otherwise.} \end{cases}$$

Therefore by Lemma 2.2 the theorem clearly holds if f is sufficiently nice, for example, a polynomial. For more general f, since (4.24) exists, there exists a polynomial

$$Q_{n-1}(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$$

such that

$$\int_0^1 |f(x) e^{-x} - Q_{n-1}(x)| x^{\nu}\, dx < \infty.$$

Let $\mathscr{P}_j$ denote the jth Taylor polynomial of e^x centered at 0, and let

$$P_{n-1}(x) = \sum_{j=0}^{n-1} a_j x^j \mathscr{P}_{n-1-j}(x).$$

Then clearly

$$\int_0^1 |f(x) - P_{n-1}(x)| x^{\nu}\, dx < \infty.$$

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By orthogonality, we have for $m \geq n$,

$$\begin{aligned}
\text{f.p.} \int_0^\infty & f(x) L_m^\nu(x) e^{-x} x^\nu \, dx \\
&= \int_0^\infty (f(x) - P_{n-1}(x)) L_m^\nu(x) e^{-x} x^\nu \, dx + \text{f.p.} \int_0^\infty P_{n-1}(x) L_m^\nu(x) e^{-x} x^\nu \, dx \\
&= \int_0^\infty (f(x) - P_{n-1}(x)) L_m^\nu(x) e^{-x} x^\nu \, dx.
\end{aligned}$$

Furthermore, for $c > 0$, using Lemma 4.5 we have

$$\begin{aligned}
\sum_{m=n}^\infty & |c_m| \int_0^\infty |(f(x) - P_{n-1}(x)) L_m^\nu(x)| e^{-x} x^\nu \, dx \\
&\leq C \sum_{m=n}^\infty |c_m| \left\{ m^{\gamma_n} \int_0^c |f(x) - P_{n-1}(x)| x^\nu \, dx \right. \\
&\qquad \left. + m^{\nu/2-1/4} \int_c^\infty |f(x) - P_{n-1}(x)| e^{-x/2} x^{\nu/2-1/4} \, dx \right\} \\
&< \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{m=n}^\infty c_m \, \text{f.p.} & \int_0^\infty f(x) L_m^\nu(x) e^{-x} x^\nu \, dx \\
&= \sum_{m=n}^\infty c_m \int_0^\infty (f(x) - P_{n-1}(x)) L_m^\nu(x) e^{-x} x^\nu \, dx \\
&= \int_0^\infty (f(x) - P_{n-1}(x)) \left(\sum_{m=n}^\infty c_m L_m^\nu(x) \right) e^{-x} x^\nu \, dx \\
&= \text{f.p.} \int_0^\infty f(x) \left(\sum_{m=n}^\infty c_m L_m^\nu(x) \right) e^{-x} x^\nu \, dx \\
&\quad - \text{f.p.} \int_0^\infty P_{n-1}(x) \left(\sum_{m=n}^\infty c_m L_m^\nu(x) \right) e^{-x} x^\nu \, dx \\
&= \text{f.p.} \int_0^\infty f(x) \left(\sum_{m=n}^\infty c_m L_m^\nu(x) \right) e^{-x} x^\nu \, dx.
\end{aligned}$$

This proves the theorem.

One application of Theorem 4.7 is:

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THEOREM 4.8. *For all $\nu \in R$, $x, s \in C$:*

$$\text{(4.26)} \qquad \text{f.p.} \int_0^\infty I_\nu^*(ys) I_\nu^*(yx) e^{-y} y^\nu \, dy = e^{s+x} I_\nu^*(xs).$$

PROOF. This is well-known if $\nu > -1$. By multiplying power series we obtain:

$$\text{(4.27)} \qquad e^{-s} I_\nu^*(ys) = \sum_{m=0}^\infty \frac{(-s)^m}{\Gamma(m+\nu+1)} L_m^\nu(y).$$

Therefore for $\nu \neq -1, -2, \ldots$, we have, by Theorem 4.7 and Corollary 4.3,

$$\begin{aligned}
&\text{f.p.} \int_0^\infty I_\nu^*(ys) I_\nu^*(yx) e^{-y} y^\nu \, dy \\
&\quad = e^s \, \text{f.p.} \int_0^\infty \left(\sum_{m=0}^\infty \frac{(-s)^m}{\Gamma(m+\nu+1)} L_m^\nu(y) \right) I_\nu^*(yx) e^{-y} y^\nu \, dy \\
&\quad = e^s \sum_{m=0}^\infty \frac{(-s)^m}{\Gamma(m+\nu+1)} \text{f.p.} \int_0^\infty L_m^\nu(y) I_\nu^*(yx) e^{-y} y^\nu \, dy \\
&\quad = e^{s+x} I_\nu^*(xs).
\end{aligned}$$

For $\nu = -k$, $k = 1, 2, \ldots$, we have

$$\begin{aligned}
\int_0^\infty I_{-k}^*(ys) I_{-k}^*(yx) e^{-y} y^{-k} \, dy &= (sx)^k \int_0^\infty I_k^*(ys) I_k^*(yx) e^{-y} y^k \, dy \\
&= (sx)^k e^{s+x} I_k^*(sx) \\
&= e^{s+x} I_{-k}^*(sx).
\end{aligned}$$

COROLLARY 4.9. *For all $a > 0$, $x, s, \nu \in R$:*

$$\text{(4.28)} \qquad \text{f.p.} \int_0^\infty I_\nu^*(ys) I_\nu^*(yx) e^{-ay} y^\nu \, dy = a^{-\nu-1} e^{(x+s)/a} I_\nu^*(xs/a^2).$$

5. Homogeneous solutions

We consider solutions of equation (1.8) whose boundary values are powers of r. Following [1], we formally calculate such solutions:

$$\text{(5.1)} \quad e^{t\Delta_\mu} r^{2\gamma} = \sum_{k=0}^\infty \frac{t^k}{k!} \Delta_\mu^k r^{2\gamma} = \sum_{k=0}^\infty \frac{(4t)^k}{k!} \frac{\Gamma(\gamma+1)\Gamma(\gamma+\frac{\mu}{2})}{\Gamma(\gamma+1-k)\Gamma(\gamma+\frac{\mu}{2}-k)} r^{2\gamma-2k}.$$

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The series above terminates if $\gamma = m$ or if $\gamma = m + 1 - \mu/2$, $m = 0, 1, \ldots$, and diverges otherwise. The radial heat polynomials $R_m^\mu(r, t)$ defined in equation (1.9) are obtained by letting $\gamma = m$. Letting $\gamma = m + 1 - \mu/2$, $m = 0, 1, \ldots$, we define for $r \neq 0$:

$$
\begin{aligned}
(5.2)\qquad V_m^\mu(r,t) &= e^{t\Delta_\mu} r^{2\gamma} \\
&= \sum_{k=0}^{m} \frac{(4t)^k}{k!} \frac{\Gamma(m+2-\frac{\mu}{2})m!}{\Gamma(m+2-\frac{\mu}{2}-k)\Gamma(m-k+1)} r^{2m+2-\mu-2k} \\
&= r^{2-\mu} \sum_{k=0}^{m} \frac{(4t)^{m-k}}{(m-k)!} \frac{\Gamma(m+2-\frac{\mu}{2})m!}{\Gamma(k+2-\frac{\mu}{2})k!} r^{2k} \\
&= r^{2-\mu}(4t)^m \sum_{k=0}^{m} \binom{m}{k} \left(\frac{r^2}{4t}\right)^k \frac{\Gamma(m+2-\frac{\mu}{2})}{\Gamma(k+2-\frac{\mu}{2})} \\
&= r^{2-\mu}(4t)^m m! L_m^{(2-\mu)/2}(-r^2/4t).
\end{aligned}
$$

It is easy to verify that $V_m^\mu(r,t)$ are solutions of (1.8) which are homogenous of degree $2m + 2 - \mu$. Note $V_0^\mu(r,t) = r^{2-\mu}$, $V_m^\mu(r,t) = O(r^{2-\mu})$, $r \to 0$, and $V_m^\mu(r, 0) = r^{2-\mu+2m}$, $m = 0, 1, \ldots$, except in the case $\mu = 2k + 2$, $m \geq k$.

In the latter case, V_m^μ are identical to R_{m-k}^μ:

$$
\begin{aligned}
(5.3)\qquad V_m^\mu(r,t) &= r^{-2k}(4t)^m m! L_m^{(-k)}(-r^2/4t) \\
&= (4t)^{m-k}(m-k)! L_{m-k}^{(k)}(-r^2/4t) \\
&= R_{m-k}^\mu(r,t).
\end{aligned}
$$

We define the associated functions using the Appell transform:

$$
\begin{aligned}
(5.4)\qquad \tilde{V}_m^\mu(r,t) &= k_\mu(r,t) V_m^\mu(r/t, -1/t) \\
&= k_\mu(r,t) t^{\mu-2-2m} V_m^\mu(r, -t) \\
&= r^{2-\mu} k_\mu(r,t) t^{\mu-2-2m} (-4t)^m m! L_m^{(2-\mu)/2}(r^2/4t).
\end{aligned}
$$

The $\tilde{V}_m^\mu(r,t)$ are homogeneous of degree $-2m-2$ and $\tilde{V}_m^\mu(r,t) = O(r^{2-\mu})$, $r \to 0$, except for $\mu = 2k + 2$, $k = 0, 1, \ldots, m \geq k$, in which case

$$(5.5)\qquad \tilde{V}_m^\mu(r,t) = \tilde{R}_{m-k}^\mu(r,t).$$

For $\mu = 1$, $\{V_m^\mu\}$ and $\{\tilde{V}_m^\mu\}$ coincide with $\{v_{2m+1}\}$ and $\{w_{2m+1}\}$ of Rosenbloom-Widder. As in the case $\mu = 1$, the solutions for $\mu > 1$ are bi-orthogonal:

THEOREM 5.1. *Define $W_\mu(r) = 2\pi^{\mu/2} r^{\mu-1}$. For all $t > 0$, $\mu \neq 2, 4, \ldots$, we have*

$$(5.6)\qquad \text{f.p.} \int_0^\infty V_n^\mu(r, -t) \tilde{V}_m^\mu(r,t) W_\mu(r)\, dr = \lambda_m^\mu \delta_{m,n}, \quad m, n = 0, 1, \ldots$$

where

$$\lambda_m^\mu = 4^{2m+2-\mu} m!\Gamma(m+2-\mu/2).$$

For $\mu = 2k+2$, $k = 0, 1, \ldots$, the same result holds provided $m, n \geq k$.

PROOF. From (2.3), (3.1), (3.4) and (3.6) we have

$$\begin{aligned}
&\text{f.p.} \int_0^\infty V_n^\mu(r,-t)\tilde{V}_m^\mu(r,t)W_\mu(r)\,dr \\
&\quad = 4^{n+m+2-\mu}(-1)^{n+m} n!m!t^{n-m} \text{f.p.} \int_0^\infty L_n^{(2-\mu)/2}(u)L_m^{(2-\mu)/2}(u)e^{-u}u^{(2-\mu)/2}\,du \\
&\quad = 4^{2m+2-\mu} m!\Gamma(m+2-\mu/2)\delta_{m,n}.
\end{aligned}$$

The result for $\mu = 2k+2$, $k = 0, 1, \ldots$, also follows from (5.3) and (5.5), see [1, 4].

We obtain a generating function for $V_m^\mu(r,t)$ from their connection to the Laguerre polynomials. Since for all $\nu \in C$ ([10]),

$$(5.7) \qquad (1-t)^{-\nu-1}e^{-xt/(1-t)} = \sum_{k=0}^\infty L_k^\nu(x)t^k, \quad |t| < 1,$$

we have

$$(5.8) \qquad \begin{aligned}
\sum_{k=0}^\infty \frac{a^k}{k!} V_k^\mu(r,t) &= r^{2-\mu} \sum_{k=0}^\infty L_k^{(2-\mu)/2}\left(-\frac{r^2}{4t}\right)(4at)^k \\
&= r^{2-\mu}(1-4at)^{(\mu-4)/2} e^{ar^2/(1-4at)}, \quad |a| < 1/4t.
\end{aligned}$$

By taking the Appell transform of both sides above we have

$$(5.9) \qquad (4\pi r)^{2-\mu} k_{4-\mu}(r, t+4a) = \sum_{k=0}^\infty \frac{a^k}{k!}\tilde{V}_k^\mu(r,t), \quad |a| < t/4.$$

It is easily verified that

$$(5.10) \qquad \frac{\partial}{\partial t} V_m^\mu(r,t) = 2m(2m+2-\mu)V_{m-1}^\mu(r,t),$$

$$(5.11) \qquad \frac{\partial}{\partial t}\tilde{V}_k^\mu(r,t) = \frac{1}{4}\tilde{V}_{k+1}^\mu(r,t).$$

6. Regions of convergence for expansions

A useful tool for estimating the size of coefficients in convergent expansions is:

FEJER'S FORMULA. ([10]) *For all $\nu \in R$ and $x > 0$*

(6.1)
$$L_m^\nu(x) = \pi^{-1/2} m^{\nu/2-1/4} e^{x/2} x^{-\nu/2-1/4} \left(\cos\{2(mx)^{1/2} - (2\nu+1)\pi/4\} + \theta_{m,\nu}(x)/m^{1/2}\right)$$

where $\theta_{m,\nu}(x)$ is uniformly bounded for $x \in [a,b]$, $0 < a < b < \infty$, as $m \to \infty$.

From Fejer's formula, it follows easily as in [9, Lemma 5.2] that if $\sum_{m=0}^{\infty} a_m L_m^\nu(x)$ converges for all x in some interval $[a,b]$, $0 < a < b < \infty$, then

$$a_m = o(m^{1/4-\nu/2}), \quad m \to \infty. \tag{6.2}$$

Therefore,

THEOREM 6.1. *For $t = t_0 < 0$, $r^2 \in [a,b]$, $0 < a < b < \infty$, if*

$$\sum_{m=0}^{\infty} a_m V_m^\mu(r,t) \tag{6.3}$$

converges, then

$$a_m = o\left(\frac{m^{(\mu-1)/4}}{(4t_0)^m m!}\right), \quad m \to \infty. \tag{6.4}$$

LEMMA 6.2. *For any $\delta > 1$ and r^2, η^2 in a fixed interval $[a,b]$, $0 < a < b < \infty$, we have as $m \to \infty$:*

$$V_m^\mu(r,t) = O((4t\delta)^m m!), \qquad t > 0 \tag{6.5}$$
$$V_m^\mu(r,t) = O((4|t|)^m m! m^{(1-\mu)/4}), \quad t < 0 \tag{6.6}$$
$$\tilde{V}_m^\mu(\eta,s) = O((4/s)^m m! m^{(1-\mu)/4}), \quad s > 0 \tag{6.7}$$
$$\tilde{V}_m^\mu(\eta,s) = O((4\delta/|s|)^m m!), \qquad s < 0. \tag{6.8}$$

PROOF. This follows from Lemmas 4.5 and 4.6.

COROLLARY 6.3. *If* (6.3) *converges for any $t_0 < 0$, $r^2 \in [a,b]$, $0 < a < b < \infty$, then it also converges for $|t| < |t_0|$, $r \neq 0$. Furthermore, $u(r,t) = \sum_{m=0}^{\infty} a_m V_m^\mu(r,t)$ defines a solution of* (1.8) *in $|t| < |t_0|$ such that $r^{\mu-2}u(r,t)$ is an entire function of r^2.*

PROOF. This follows from (6.4), (6.5), (6.6) and (5.10).

Note that if (6.3) converges for all $|t| < \sigma$ for some $\sigma > 0$, then for any $t_0 \in (0, \sigma)$ we have the simpler estimate

$$a_m = o\left(\frac{1}{(4t_0)^m m!}\right), \quad m \to \infty. \tag{6.9}$$

THEOREM 6.4. *If the series*

$$\sum_{m=0}^{\infty} a_m \tilde{V}_m^{\mu}(r, t) \tag{6.10}$$

converges for some $t = t_0 > 0$ *and* $r^2 \in [a, b]$, $0 < a < b < \infty$, *then*

$$a_m = o\left(\frac{m^{(\mu-1)/4} t_0^m}{4^m m!}\right), \quad m \to \infty. \tag{6.11}$$

PROOF. The series (6.10) converges if and only if $\sum_{m=0}^{\infty} a_m t_0^{-2m} V_m^{\mu}(r, -t_0)$ converges. Therefore (6.11) follows from (6.4).

COROLLARY 6.5. *If the series* (6.10) *converges for some* $t_0 > 0$, $r^2 \in [a, b]$, *then it also converges for* $|t| > |t_0|$ *and defines a solution* $u(r, t)$ *of (1.8) there such that* $r^{\mu-2} u(r, t)$ *is entire in* r^2.

7. The double generating function

Define the double generating function:

$$S_{\mu}(r, \eta; t, s) = \sum_{m=0}^{\infty} \frac{V_m^{\mu}(r, t) \tilde{V}_m^{\mu}(\eta, s)}{4^{2m+2-\mu} m! \Gamma(m + 2 - \frac{\mu}{2})}, \quad |t| < |s|. \tag{7.1}$$

The convergence of (7.1) follows from Lemma 6.2.

THEOREM 7.1. *For all* $r, \eta > 0$ *and* $|t| < |s|$ *we have*

$$S_{\mu}(r, \eta; t, s) = \frac{(\eta r)^{2-\mu}}{\pi^{\mu/2} (4(s + t))^{2-\mu/2}} e^{-(r^2+\eta^2)/4(s+t)} I^{*}_{(2-\mu)/2}\left(\frac{r^2 \eta^2}{16(s + t)^2}\right). \tag{7.2}$$

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PROOF. This follows from making the substitutions $\nu = (2-\mu)/2$, $u = -t/s$, $x = -r^2/4t$, and $y = \eta^2/4s$ into ([8]):

$$(7.3)\qquad (1-u)^{-\nu-1}e^{-(x+y)u/(1-u)}I_\nu^*\left(\frac{xyu}{(1-u)^2}\right) = \sum_{m=0}^{\infty}\frac{m!L_m^\nu(x)L_m^\nu(y)}{\Gamma(m+\nu+1)}u^m, \quad |u|<1.$$

Define for $s, r, \eta \neq 0$:

$$(7.4)\qquad \begin{aligned} K_\mu(r,\eta,s) &= \lim_{t\to 0+} S_\mu(r,\eta;t,s) \\ &= \sum_{m=0}^{\infty}\frac{r^{2m+2-\mu}\tilde V_m^\mu(\eta,s)}{4^{2m+2-\mu}m!\Gamma(m+2-\frac{\mu}{2})} \\ &= \frac{(r\eta)^{2-\mu}}{\pi^{\mu/2}(4s)^{2-\frac{\mu}{2}}}e^{-(r^2+\eta^2)/4s}I^*_{\frac{2-\mu}{2}}\left(\frac{r^2\eta^2}{16s^2}\right). \end{aligned}$$

Note that for $\mu \neq 2k+2$, $k = 0, 1, \ldots$, we have $K_\mu(r,\eta,s) = O((r\eta)^{2-\mu})$, $r\eta \to 0$. For $\mu = 2k+2$, $k = 0, 1, \ldots$,

$$\begin{aligned} K_\mu(r,\eta,s) &= \sum_{m=0}^{\infty}\frac{r^{2m-2k}\tilde V_m^\mu(\eta,s)}{4^{2m-2k}m!\Gamma(m+1-k)} \\ &= \sum_{m=k}^{\infty}\frac{r^{2m-2k}\tilde V_m^\mu(\eta,s)}{4^{2m-2k}m!\Gamma(m+1-k)} \\ &= \sum_{m=0}^{\infty}\frac{r^{2m}\tilde R_m^\mu(\eta,s)}{4^{2m}m!\Gamma(m+1+k)}. \end{aligned}$$

Since this coincides with the kernel given in [1, 4], and because of (5.3) and (5.5), from now on we consider only the case $\mu \in \Omega$, where we define

$$(7.5)\qquad \Omega = \{\mu : \mu > 1,\ \mu \neq 2k+2,\ k = 0, 1, \ldots\}.$$

Note that

$$(7.6)\qquad \lim_{r\to 0+} r^{\mu-2}K_\mu(r,\eta,s) = \frac{(\pi\eta)^{2-\mu}}{\Gamma((4-\mu)/2)}k_{4-\mu}(\eta,s).$$

From the asymptotic formula (4.6) we have as $r\eta/s \to \infty$

(7.7)
$$K_\mu(r,\eta,s) = \frac{1}{\sqrt{4\pi}}\frac{(\eta r)^{(1-\mu)/2}}{\pi^{\mu/2}\sqrt{4s}}\left(e^{-(r-\eta)^2/4s} + e^{(3-\mu)\pi i/2}e^{-(r+\eta)^2/4s}\right)\left\{1 + O\left(\frac{s}{r\eta}\right)\right\}.$$

Thus for fixed s,

$$(7.8)\qquad K_\mu(r,\eta,s) = O(1)(\eta r)^{(1-\mu)/2}e^{-(r-\eta)^2/4s}, \quad r\eta \to \infty.$$

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We will also need estimates on the size of the derivatives of K_μ. Define

$$h_\mu(\xi, \eta) = e^{-(\xi+\eta)} I^*_{(2-\mu)/2}(\xi\eta). \tag{7.9}$$

Note that

$$K_\mu(\xi, \eta, t) = \frac{(\xi\eta)^{2-\mu}}{\pi^{\mu/2}(4t)^{2-\mu/2}} h_\mu\left(\frac{\xi^2}{4t}, \frac{\eta^2}{4t}\right). \tag{7.10}$$

We have for $j = 0, 1, \dots$

$$\partial_\xi^j h_\mu(\xi, \eta) = (-1)^j e^{-(\xi+\eta)} \sum_{k=0}^{j} \binom{j}{k} (-\eta)^k I^*_{(2-\mu)/2+k}(\xi\eta) \tag{7.11}$$

so that by equation (4.19) we have

$$\partial_\xi^j h_\mu(\xi, \eta) = O(1)\eta^j (\xi\eta)^{(\mu-3)/4} e^{-(\sqrt{\xi}-\sqrt{\eta})^2}, \quad \xi\eta \to \infty. \tag{7.12}$$

8. The Huygens property.

DEFINITION 8.1 (The Huygens property). Let $\mu > 1$. Let $u(r, t)$ be a solution of the generalized heat equation (1.8) in a strip $a < t < b$, $r \neq 0$. We say that $u \in H^*(a, b)$ if and only if for $a < t' < t < b$ we have

$$r^{\mu-2} u(r, t) = \text{f.p.} \int_0^\infty r^{\mu-2} K_\mu(r, \xi, t - t') u(\xi, t') W_\mu(\xi) d\xi, \quad r \in R. \tag{8.1}$$

Furthermore, if $u(r, t)$ is a solution of the generalized heat equation in the complement of such a strip, then we say $u(r, t) \in H^*(b, a)$ if and only if (8.1) holds whenever $b < t' < t$, $t' < t < a$ or both $t' > b$ and $t < a$.

For $r = 0$, this is understood to mean that

(8.2)

$$\lim_{r \to 0+} r^{\mu-2} u(r, t) = \frac{\pi^{2-\mu}}{\Gamma((4-\mu)/2)} \text{f.p.} \int_0^\infty k_{4-\mu}(\xi, t - t') u(\xi, t') \xi^{2-\mu} W_\mu(\xi)\, d\xi;$$

see equation (7.6).

Note that the existence of the integral in (8.2) implies that, for any $c > 0$,

$$\int_c^\infty e^{-\xi^2/4(t-t')} |u(\xi, t')|\, \xi\, d\xi < \infty, \quad a < t' < t < b. \tag{8.3}$$

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LEMMA 8.2. *If $u(r,t) \in H^*(-\sigma,\sigma)$ then $r^{\mu-2}u(r,t)$ is an entire function of r^2 for each $t \in (a,b)$.*

PROOF. This follows easily from Theorem 4.4.

THEOREM 8.3. *For all $\mu \in \Omega$, $m = 0, 1, \dots,$ we have*

$$V_m^\mu(r,t) \in H^*(-\infty,\infty). \tag{8.4}$$

PROOF. The identity

$$r^{\mu-2}V_m^\mu(r,t) = \text{f.p.}\int_0^\infty r^{\mu-2}K_\mu(r,\xi,t-t')V_m^\mu(\xi,t')W_\mu(\xi)\,d\xi, \quad t' < t,\ r \in R$$

for $t \neq 0$ follows from letting

$$\nu = \frac{2-\mu}{2}, \qquad a = \frac{t'-t}{t'}, \qquad x = \frac{r^2}{4(t-t')}, \quad \text{and} \quad y = \frac{\xi^2}{4(t-t')}$$

in Theorem 4.2. The case $t = 0$ follows from making the same substitutions into the equality of Corollary 4.3.

COROLLARY 8.4. *For $m = 0, 1, \dots, \mu \in \Omega$,*

$$r^{\mu-2}V_m^\mu(r,t) = \text{f.p.}\int_0^\infty r^{\mu-2}K_\mu(r,\xi,t)\xi^{2m+2-\mu}W_\mu(\xi)\,d\xi, \quad t > 0,\ r \in R. \tag{8.5}$$

THEOREM 8.5. *For every generalized temperature function $u(r,t)$ and every $\sigma > 0$, we have $u(r,t) \in H^*(-\sigma,\sigma)$ if and only if $\mathscr{A}_\mu u(r,t) \in H^*(1/\sigma, -1/\sigma)$.*

PROOF. Assume $r \neq 0$. If $u \in H^*(-\sigma,\sigma)$ then for $-\sigma < -1/t' < t < \sigma$ we have

$$u(r,t) = \text{f.p.}\int_0^\infty K_\mu(r,\xi,t+\frac{1}{t'})u(\xi,-\frac{1}{t'})W_\mu(\xi)\,d\xi.$$

Therefore, if $-\sigma < -1/t' < -1/t < \sigma$, or equivalently if $1/\sigma < t' < t$, $t' < t < -1/\sigma$ or both $t' > 1/\sigma$ and $t < -1/\sigma$, we have

$$\begin{aligned}
\mathscr{A}_\mu u(r,t) &= k_\mu(r,t)\,\text{f.p.}\int_0^\infty K_\mu\left(\frac{r}{t},\xi,\frac{1}{t'}-\frac{1}{t}\right)u\left(\xi,-\frac{1}{t'}\right)W_\mu(\xi)\,d\xi \\
&= k_\mu(r,t)\,\text{f.p.}\int_0^\infty K_\mu\left(\frac{r}{t},\frac{\xi}{t'},\frac{1}{t'}-\frac{1}{t}\right)u\left(\frac{\xi}{t'},-\frac{1}{t'}\right)W_\mu\left(\frac{\xi}{t'}\right)d\left(\frac{\xi}{t'}\right) \\
&= \frac{k_\mu(r,t)}{(t')^\mu}\,\text{f.p.}\int_0^\infty K_\mu\left(\frac{r}{t},\frac{\xi}{t'},\frac{1}{t'}-\frac{1}{t}\right)\frac{1}{k_\mu(\xi,t')}\mathscr{A}_\mu u(\xi,t')W_\mu(\xi)\,d\xi \\
&= \text{f.p.}\int_0^\infty K_\mu(r,\xi,t-t')\mathscr{A}_\mu u(\xi,t')W_\mu(\xi)\,d\xi.
\end{aligned}$$

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For $r = 0$, note that

$$\lim_{r\to 0} r^{\mu-2}\mathscr{A}_\mu u(r,t) = \lim_{r\to 0} r^{\mu-2} u\left(\frac{r}{t}, \frac{-1}{t}\right) k_\mu(r,t)$$
$$= t^{\mu-2} k_\mu(0,t) \lim_{r\to 0} r^{\mu-2} u(r, -1/t)$$

and the rest of the proof follows as before.

The proof in the other direction is similar.

THEOREM 8.6. *For $\mu \in \Omega$, $m = 0, 1, \dots$, we have*

(8.6) $$\tilde{V}_m^\mu(r,t) \in H^*(0+, 0-).$$

PROOF. This follows from Theorems 8.3 and 8.5.

THEOREM 8.7. *For $0 < t' < t$, $t' < t < 0$ or both $t' > 0$ and $t < 0$ we have*

(8.7)
$$r^{\mu-2} K_\mu(r,\eta,t) = \text{f.p.} \int_0^\infty r^{\mu-2} K_\mu(r,\xi,t-t') K_\mu(\xi,\eta,t') W_\mu(\xi)\, d\xi,$$
$$r \in R,\ \ \eta \neq 0.$$

PROOF. Let

$$C(r,\eta,t,t') = \frac{\eta^{2-\mu} e^{-r^2/4(t-t')} e^{-\eta^2/4t'}}{(16t'(t-t'))^{2-\mu/2}}.$$

Since

$$\text{f.p.} \int_0^\infty r^{\mu-2} K_\mu(r,\xi,t-t') K_\mu(\xi,\eta,t') W_\mu(\xi)\, d\xi$$
$$= 2C(r,\eta,t,t')\text{f.p.} \int_0^\infty e^{-\xi^2(1/4(t-t')+1/4t')} I^*_{(2-\mu)/2}\left(\frac{r^2\xi^2}{16(t-t')^2}\right) I^*_{(2-\mu)/2}\left(\frac{\eta^2\xi^2}{16(t')^2}\right) \xi^{3-\mu}\, d\xi$$
$$= C(r,\eta,t,t')\text{f.p.} \int_0^\infty e^{-\xi(1/4(t-t')+1/4t')} I^*_{(2-\mu)/2}\left(\frac{r^2\xi}{16(t-t')^2}\right) I^*_{(2-\mu)/2}\left(\frac{\eta^2\xi}{16(t')^2}\right) \xi^{(2-\mu)/2}\, d\xi$$

we obtain the result from Corollary 4.9.

9. The Huygens Property of Poisson integrals

THEOREM 9.1. *Let* $1 < \mu < 4$. *Define* $u(r,t)$ *by*

(9.1)
$$r^{\mu-2}u(r,t) = \int_0^\infty r^{\mu-2}K_\mu(r,\xi,t-a)G(\xi)W_\mu(\xi)\,d\xi, \quad a < t < b,\ r \in R,$$

where we assume that the integral converges absolutely for $r \in R$. *Then*

(9.2)
$$u(r,t) \in H^*(a,b).$$

PROOF. For $1 < \mu < 4$ we note that $K_\mu(r,\xi,t)$ is non-negative for non-negative values of the arguments. Thus if $G \geq 0$, then for all $r \in R$, $a < t' < t < b$, we have

(9.3)
$$\begin{aligned}
&\int_0^\infty r^{\mu-2}K_\mu(r,\xi,t-t')u(\xi,t')W_\mu(\xi)\,d\xi \\
&\quad = \int_0^\infty r^{\mu-2}K_\mu(r,\xi,t-t')\left(\int_0^\infty K_\mu(\xi,\eta,t'-a)\right)G(\eta)W_\mu(\eta)\,d\eta)W_\mu(\xi)\,d\xi \\
&\quad = \int_0^\infty \left(\int_0^\infty r^{\mu-2}K_\mu(r,\xi,t-t')K_\mu(\xi,\eta,t'-a)W_\mu(\xi)\,d\xi\right)G(\eta)W_\mu(\eta)\,d\eta \\
&\quad = \int_0^\infty r^{\mu-2}K_\mu(r,\eta,t-a)G(\eta)W_\mu(\eta)\,d\eta \\
&\quad = r^{\mu-2}u(r,t),
\end{aligned}$$

where the exchange of integration is justified by Tonelli's theorem. For arbitrary G, write G as the difference of its positive and negative parts, and the theorem follows from the case $G \geq 0$.

THEOREM 9.2. *Suppose* $-(n+1) < (2-\mu)/2 < -n$, $n \in N$. *Define* $u(r,t)$ *by*

(9.4)
$$r^{\mu-2}u(r,t) = \text{f.p.}\int_0^\infty r^{\mu-2}K_\mu(r,\xi,t-a)G(\xi)W_\mu(\xi)\,d\xi, \quad a < t < b,\ r \in R$$

where $G(\xi) = \xi^{2-\mu}F(\xi^2)$, $F(\xi)$ *has n continuous derivatives in some interval* $[0,c]$, $c > 0$, *and*

$$\int_c^\infty |K_\mu(r,\xi,t-a)G(\xi)|\,W_\mu(\xi)\,d\xi < \infty.$$

Then $u(r,t) \in H^*(a,b)$.

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PROOF. Formally, as in Theorem 9.1, the result follows from the interchange of integration. The justification will follow from Theorem 2.3. We may assume $a = 0$, so that $0 < t' < t < b$. Using the change of variables $\eta^2 \to \eta$, $\xi^2 \to \xi$ we may rewrite (9.3) as:

$$\frac{1}{(16t'(t-t'))^{2-\mu/2}}\text{f.p.}\int_0^\infty \left(\text{f.p.}\int_0^\infty f(\xi,\eta)(\xi\eta)^{(2-\mu)/2}\,d\xi\right)d\eta$$

where

$$f(\xi,\eta) = h_\mu\left(\frac{\xi}{4(t-t')},\frac{r^2}{4(t-t')}\right)h_\mu\left(\frac{\xi}{4t'},\frac{\eta}{4t'}\right)F(\eta);$$

see equation (7.10). Firstly, note that $f(\xi,\eta)$ has continuous mixed partials up to degree n in each variable in $[0,c]\times[0,c]$.

Secondly, since by equations (4.21), (7.9), and (7.12) with $j = 0$,

$$h_\mu\left(\frac{\xi}{4t},\frac{\eta}{4t}\right) = O(1)(\xi\eta)^{(\mu-3)/4}e^{-\eta/4\delta t}e^{\xi/4(\delta-1)t}, \quad \xi\eta\to\infty,$$

we have

$$\begin{aligned}\int_c^\infty\int_c^\infty &|f(\xi,\eta)|(\xi\eta)^{(2-\mu)/2}\,d\xi\,d\eta\\ &= O(1)\int_c^\infty e^{-\eta/4\delta t'}|F(\eta)|\eta^{(1-\mu)/4}\,d\eta\\ &\quad\cdot\int_c^\infty e^{\xi/4(\delta-1)t'}\left|h_\mu\left(\frac{\xi}{4(t-t')},\frac{r^2}{4(t-t')}\right)\right|\xi^{(1-\mu)/4}\,d\xi\end{aligned}$$

which is finite provided $\delta\in(t/t', b/t')$.

Next, note that

$$\partial_\xi^j f(\xi,\eta) = F(\eta)\sum_{k=0}^{j}\binom{j}{k}\left(\partial_\xi^k h_\mu\left(\frac{\xi}{4(t-t')},\frac{r^2}{4(t-t')}\right)\right)\left(\partial_\xi^{j-k}h_\mu\left(\frac{\xi}{4t'},\frac{\eta}{4t'}\right)\right)$$

so that since I_ν^* is of exponential growth $1/2$, given $\delta > 1$, we have uniformly for $\xi\in[0,c]$:

$$|\partial_\xi^j f(\xi,\eta)| = O(1)|F(\eta)|e^{-\eta/4\delta t'}, \quad \eta\to\infty.$$

This implies conditions (2.7) and (2.9) of Theorem 2.3.

A similar estimate can be found for $|\partial_\eta^j f(\xi,\eta)|$ for $\eta\in[0,c]$, $\xi\to\infty$. This completes the proof.

10. Expansions and the Huygens property

THEOREM 10.1. *If*

$$u(r,t)=\sum_{m=0}^{\infty}a_m V_m^{\mu}(r,t) \tag{10.1}$$

converges for $|t|<\sigma$, $r\neq 0$, *then*

$$u(r,t)\in H^*(-\sigma,\sigma). \tag{10.2}$$

PROOF. Formally, we have using Theorem 8.3

$$\begin{aligned}
&\text{f.p.}\int_0^{\infty} u(\xi,t')r^{\mu-2}K_\mu(r,\xi,t-t')W_\mu(\xi)\,d\xi\\
&\quad=\text{f.p.}\int_0^{\infty}\left(\sum_{m=0}^{\infty}a_m V_m^{\mu}(\xi,t')\right)r^{\mu-2}K_\mu(r,\xi,t-t')W_\mu(\xi)\,d\xi\\
&\quad=\sum_{m=0}^{\infty}a_m\,\text{f.p.}\int_0^{\infty}V_m^{\mu}(\xi,t')r^{\mu-2}K_\mu(r,\xi,t-t')W_\mu(\xi)\,d\xi\\
&\quad=\sum_{m=0}^{\infty}a_m r^{\mu-2}V_m^{\mu}(r,t)=r^{\mu-2}u(r,t).
\end{aligned} \tag{10.3}$$

Let $c_m=a_m(4t')^m m!$. By equation (6.9), $c_m=O((t'/t_0)^m)$ for any $t_0\in(0,\sigma)$. Using the change of variables $\xi^2\to -4t'\xi$ the sum in (10.3) may be rewritten as

$$\left(\frac{-t'}{t-t'}\right)^{2-\mu/2}\sum_{m=0}^{\infty}c_m\text{f.p.}\int_0^{\infty}L_m^{(2-\mu)/2}(\xi)h_\mu\left(\frac{-t'\xi}{t-t'},\frac{r^2}{4(t-t')}\right)\xi^{(2-\mu)/2}\,d\xi.$$

It is easy to verify that for any $c>0$ and $t'<t<-t'$, $t'\in(-\sigma,0)$ we have:

$$\int_c^{\infty}\left|h_\mu\left(\frac{-t'\xi}{t-t'},\frac{r^2}{4(t-t')}\right)\right|e^{\xi/2}\xi^{(1-\mu)/4}\,d\xi<\infty.$$

Thus the interchange of summation and integration is justified for $t'<t<-t'$ by Theorem 4.7.

Therefore, Theorems 9.1 and 9.2 imply that $u(r,t)\in H^*(t',-t')$, and since t' can be chosen arbitrarily close to $-\sigma$ this proves the theorem.

LEMMA 10.2. *If for* $\sigma>0$, $u\in H^*(-\sigma,\sigma)$ *then*

$$\text{f.p.}\int_0^{\infty}\tilde V_m^{\mu}(\xi,-t)u(\xi,t)W_\mu(\xi)\,d\xi$$

exists and is independent of $t\in(-\sigma,0)$.

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PROOF. Let $-\sigma < t' < t < 0$. Formally, we have

(10.4)

$$
\begin{aligned}
&\text{f.p.} \int_0^\infty \tilde{V}_m^\mu(\xi, -t) u(\xi, t) W_\mu(\xi)\, d\xi \\
&\quad = \text{f.p.} \int_0^\infty \tilde{V}_m^\mu(\xi, -t) \left(\text{f.p.} \int_0^\infty u(\eta, t') K_\mu(\xi, \eta, t - t') W_\mu(\eta)\, d\eta \right) W_\mu(\xi)\, d\xi \\
&\quad = \text{f.p.} \int_0^\infty u(\eta, t') \left(\text{f.p.} \int_0^\infty K_\mu(\xi, \eta, t - t') \tilde{V}_m^\mu(\xi, -t) W_\mu(\xi)\, d\xi \right) W_\mu(\eta)\, d\eta \\
&\quad = \text{f.p.} \int_0^\infty \tilde{V}_m^\mu(\eta, -t') u(\eta, t') W_\mu(\eta)\, d\eta.
\end{aligned}
$$

We now justify the exchange of integration.

By Lemma 8.2, we can write $u(\xi, t) = \xi^{2-\mu} v(\xi^2, t)$ where $v(\xi, t)$ is entire in the first variable. By the change of variables $\eta^2 \to \eta$ and $\xi^2 \to \xi$, the integral in (10.4) can be written

$$
\frac{m! 4^{m-2}}{(-t)^{m+2-\mu/2} (t - t')^{2-\mu/2}} \text{f.p.} \int_0^\infty \left(\text{f.p.} \int_0^\infty f(\xi, \eta) (\xi\eta)^{(2-\mu)/2}\, d\xi \right) d\eta
$$

where

$$
f(\xi, \eta) = v(\eta, t') L_m^{(2-\mu)/2} \left(-\frac{\xi}{4t} \right) e^{\xi/4t} h_\mu \left(\frac{\xi}{4(t - t')}, \frac{\eta}{4(t - t')} \right).
$$

We will justify the interchange of integration using Theorem 2.3. Fix $c > 0$. Firstly, note that $f(\xi, \eta)$ is entire in ξ and η.

Secondly, using equations (4.21) and (7.12) with $j = 0$ we have

$$
\begin{aligned}
&\int_c^\infty \int_c^\infty |f(\xi, \eta)| (\xi\eta)^{(2-\mu)/2}\, d\xi\, d\eta \\
&\quad = O(1) \int_c^\infty |v(\eta, t')| e^{-\eta/4\delta(t - t')} \eta^{(1-\mu)/4}\, d\eta \\
&\qquad \cdot \int_c^\infty \left| L_m^{(2-\mu)/2}(-\xi/4t) \right| e^{\xi/4t} e^{\xi/4(\delta-1)(t-t')} \xi^{(1-\mu)/4}\, d\xi
\end{aligned}
$$

which is finite provided $\delta - 1 \in (-t/(t - t'), (\sigma - t)/(t - t'))$; see equation (8.3).

Finally, since

$$
\partial_\eta^j f(\xi, \eta) = L_m^{(2-\mu)/2} \left(\frac{-\xi}{4t} \right) e^{\xi/4t} \sum_{k=0}^j \binom{j}{k} (\partial_\eta^k v(\eta, t)) \left(\partial_\eta^{j-k} h_\mu \left(\frac{\xi}{4(t - t')}, \frac{\eta}{4(t - t')} \right) \right)
$$

given $\delta > 1$ we have uniformly for $\eta \in [0, c]$,

$$
|\partial_\eta^j f(\xi, \eta)| = O(1) |L_m^{(2-\mu)/2}(-\xi/4t)| e^{\xi/4t} e^{-\xi/4\delta(t-t')}, \quad \xi \to \infty
$$

so that conditions (2.6) and (2.8) of Theorem 2.3 are satisfied. A similar estimate holds for $\xi \in [0, c]$ and $\eta \to \infty$.

This completes the proof.

THEOREM 10.3. *If for* $\sigma > 0$, $u \in H^*(-\sigma, \sigma)$, *then* $u(r, t)$ *has an expansion* $u(r, t) = \sum_{m=0}^{\infty} a_m V_m^\mu(r, t)$ *converging pointwise for* $-\sigma < t < \sigma$, $r \neq 0$, *with*

$$a_m = \frac{1}{4^{2m+2-\mu} m! \Gamma(m + 2 - \mu/2)} f.p. \int_0^\infty \tilde{V}_m^\mu(\xi, -t) u(\xi, t) W_\mu(\xi)\, d\xi, \quad -\sigma < t < 0.$$

PROOF. From Lemma 10.2 the representation of a_m is independent of t. Since for $|t| < |t'|$:

$$K_\mu(r, \xi, t - t') = \sum_{m=0}^{\infty} \frac{V_m^\mu(r, t) \tilde{V}_m^\mu(\xi, -t')}{4^{2m+2-\mu} m! \Gamma(m + 2 - \frac{\mu}{2})}$$

we have formally for $-\sigma < t' < 0$, $t' < t < -t'$, $r \neq 0$

(10.5)

$$\begin{aligned} u(r, t) &= \sum_{m=0}^{\infty} \frac{V_m^\mu(r, t)}{4^{2m+2-\mu} m! \Gamma(m + 2 - \frac{\mu}{2})} \text{f.p.} \int_0^\infty \tilde{V}_m^\mu(\xi, -t') u(\xi, t') W_\mu(\xi)\, d\xi \\ &= \sum_{m=0}^{\infty} a_m V_m^\mu(r, t). \end{aligned}$$

Writing $u(\xi, t) = \xi^{2-\mu} v(\xi^2, t)$, $v(\xi, t)$ entire in ξ, we may rewrite (10.5) as

$$\begin{aligned} &\frac{1}{4^{2-\mu}(-4t')^{2-\mu/2}} \sum_{m=0}^{\infty} \frac{V_m^\mu(r, t)}{(4t')^m \Gamma(m + 2 - \frac{\mu}{2})} \\ &\qquad \text{f.p.} \int_0^\infty e^{\xi/4t'} L_m^{(2-\mu)/2}\left(-\frac{\xi}{4t'}\right) v(\xi, t') \xi^{(2-\mu)/2}\, d\xi. \end{aligned}$$

By (8.3) with $t = -t'$ we have

$$\int_c^\infty e^{\xi/8t'} |v(\xi, t')| \xi^{(1-\mu)/4}\, d\xi < \infty.$$

Thus by Theorem 4.7 and Lemma 6.2 we obtain the result.

THEOREM 10.4. *If for* $\sigma > 0$, $u(r, t) = \sum_{m=0}^{\infty} a_m \tilde{V}_m^\mu(r, t)$ *converges pointwise for* $|t| > \sigma$ *and* $r \neq 0$ *then* $u(r, t) \in H^*(\sigma, -\sigma)$.

PROOF. Since

$$\mathscr{A}_\mu u(r,t) = \frac{e^{-3\pi\mu i/2}}{(4\pi)^\mu}\sum_{m=0}^{\infty} a_m V_m^\mu(r,t)$$

converges in $|t| < 1/\sigma$, Theorem 10.1 implies that $\mathscr{A}_\mu u \in H^*(-1/\sigma, 1/\sigma)$. Thus by Theorem 8.5 we have $u \in H^*(\sigma, -\sigma)$.

THEOREM 10.5. *If for* $\sigma > 0$, $u(r,t) \in H^*(\sigma, -\sigma)$ *then* $u(r,t) = \sum_{m=0}^{\infty} a_m \tilde{V}_m^\mu(r,t)$ *where*

$$a_m = \frac{1}{4^{2m+2-\mu} m!\Gamma(m+2-\frac{\mu}{2})}\text{f.p.}\int_0^\infty V_m^\mu(\xi,t)u(\xi,-t)W_\mu(\xi)\,d\xi, \quad t < -\sigma.$$

PROOF. If $u \in H^*(\sigma, -\sigma)$, then $\mathscr{A}_\mu u(r,t) \in H^*(-1/\sigma, 1/\sigma)$. Thus by Theorem 10.3, $\mathscr{A}_\mu u(r,t) = \sum_{m=0}^{\infty} b_m V_m^\mu(r,t)$, $|t| < 1/\sigma$, where for $-1/\sigma < t < 0$

$$b_m = \frac{1}{4^{2m+2-\mu} m!\Gamma(m+2-\frac{\mu}{2})}\text{f.p.}\int_0^\infty \tilde{V}_m^\mu(\xi,-t)\mathscr{A}_\mu u(\xi,t)W_\mu(\xi)\,d\xi.$$

Since $\tilde{V}_m^\mu(\xi,-t) = k_\mu(\xi,-t)V_m^\mu(-\xi/t, 1/t)$ and $\mathscr{A}_\mu u(\xi,t) = k_\mu(\xi,t)u(\xi/t, -1/t)$ we have

$$\begin{aligned}\text{f.p.}&\int_0^\infty \tilde{V}_m^\mu(\xi,-t)\mathscr{A}_\mu u(\xi,t)W_\mu(\xi)\,d\xi \\ &= \frac{e^{-3\pi i\mu/2}}{(4\pi)^\mu}\text{f.p.}\int_0^\infty V_m^\mu\left(\xi, \frac{1}{t}\right)u\left(\xi, -\frac{1}{t}\right)W_\mu(\xi)\,d\xi\end{aligned}$$

so that

$$b_m = \frac{e^{-3\pi i\mu/2}}{(4\pi)^\mu}\frac{1}{4^{2m+2-\mu} m!\Gamma(m+2-\frac{\mu}{2})}\text{f.p.}\int_0^\infty V_m^\mu(\xi,t)u(\xi,-t)W_\mu(\xi)\,d\xi,$$
$$t < -\sigma.$$

Therefore, since $u(r,t) = (4\pi)^\mu \mathscr{A}_\mu^2 u(r,t)/e^{-3\pi i\mu/2}$, we obtain the result.

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