PART V

STATISTICAL MECHANICS

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

Combining Statistical-Thermodynamics and Relativity Theory: Methodological and Foundations Problems

John Earman[1]

University of Minnesota

0. Introduction.

Classical statistical mechanics has commanded a modest but steady amount of attention from philosophers of science. By contrast, there has been an almost total neglect of relativistic statistical mechanics, or more precisely, a neglect of the prospects and problems of producing a relativistic version of classical statistical mechanics. The neglect is undeserved, for this area offers a fascinating array of case studies for those concerned with the history and sociology of science, with the structure and dynamics of scientific theories, or with foundations problems in physics. This paper is dedicated to the goal of ending the neglect. Towards this end, I will survey some of the issues which arise in attempting to marry statistical-thermodynamics with relativity theory. The choice of the issues to be discussed and their treatment naturally reflect my own preferences and prejudices, and I cannot hope for the reader's agreement on all points. But I do hope to convey some sense of how rich a mine this area is for philosophy of science.

In the present section I will briefly outline some of the stages of development of this subject, giving enough references so that the reader interested in the history can find his way into the literature. Section 1 discusses the role of the Hamiltonian formalism in classical and relativistic mechanics. Sections 2 and 3 describe how the concepts of heat and work fare when carried from the classical to the relativistic setting. Sections 4 and 5 deal with the relativistic version of the classical First and Second Laws of Thermodynamics. Section 6 contrasts the nature of equilibrium states in the classical and relativistic cases. Section 7 describes some of the problems in obtaining a relativistic version of the Fourier heat flow law. Section 8 details some of the implications of relativistic statistical mechanics for the Planck-Ott debate

PSA 1978, Volume 2, pp. 157-185

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

about the transformation properties of thermodynamic quantities. And finally, Section 9 provides some concluding remarks. Various mathematical details are to be found in the Appendix.

Once the special theory of relativity was accepted, the drive began to make every branch of physics conform to the requirements of relativity. Some branches proved to be recalcitrant; gravitation was the most notable example, and it took Einstein many years of struggle to create a satisfactory theory, a theory which burst the bounds of special relativity. But thermodynamics seemed to present no great challenge. Einstein [14] and Planck [27] set for themselves the twin tasks of finding how thermodynamic quantities behave under Lorentz transformations and demonstrating that the laws of thermodynamics are Lorentz covariant. Their solution to the first task is contained in the formulae

$$(0.1) \qquad T' = T\sqrt{1 - v^2/c^2}, \qquad \Delta Q' = \Delta Q\sqrt{1 - v^2/c^2}, \qquad \Delta S' = \Delta S.$$

Where T, Q, and S are respectively the temperature, heat, and entropy, and the primes indicate a moving inertial observer whose velocity relative to the unprimed inertial observer is $\underline{v}$. (0.1) was repeated in most of the standard texts on relativity theory (see, for example, [24] and [38]). In 1963 Ott [26] questioned the validity of (0.1), and proposed instead

$$(0.2) \qquad T' = T/\sqrt{1 - v^2/c^2}, \qquad \Delta Q' = \Delta Q/\sqrt{1 - v^2/c^2}, \qquad \Delta S' = \Delta S,$$

so that a moving body should look hotter rather than cooler, and the controversy has been raging ever since.

Subsequent developments came fitfully. In the teens, Jüttner [18] and Tolman [35] applied Maxwell-Boltzmann statistics to a relativistic gas. Then little occurred until the late '20's when Tolman ([36] and [37]) attempted to put thermodynamics in a general relativistic setting. This was followed by a flurry of activity in the '30's and '40's, with pioneering developments by Synge [32], Tolman [38], van Danzig ([5] - [7]), and Eckart [9]. Much of the literature up to this point contains an admixture of phenomenological thermodynamics, statistical mechanics, and relativity theory, and relativistic statistical mechanics was not developed from first principles without taking for granted the so-called laws of thermodynamics. This procedure is understandable when it is remembered that through the early part of this century, thermodynamics was seen as a subject whose existence and validity is independent of any statistical interpretation. This attitude is in sharp contrast to the one taken here; namely, all macro-thermodynamical quantities must be given a statistical interpretation, and the interpretation must explain to what extent and under what

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

conditions the 'laws' of thermodynamics are valid. Thus, I am betting that the lawlike behavior of thermodynamic processes does not turn on any 'emergent' features of macro-systems; and in so doing, I am putting myself at risk since even this mild form of the unity of science could turn out to be false. My motivation here stems from more than a love of danger; for, as I will try to indicate in Section 8, there is a sense in which thermodynamics is conceptually dependent upon statistical mechanics.

A notable exception to the trend of the '30's and '40's was Synge, who introduced the key tool needed for a thorough going relativistic statistical mechanics---the statistical definition of the stress-energy tensor (see [32] and the discussion in Section 2 below). Unfortunately, Synge chose to work with the dynamical definition of macroscopic velocity, a choice that does not easily lend itself to a treatment of non-adiabatic processes (see Section 2 below). In any case, the relativistic version of the Boltzmann equation and relativistic transport theory did not develop until the '50 and '60's (see [10] - [13] and the references given there), and the growth of these branches is far from complete today. The unfinished nature of this subject commends it all the more to those philosophers of science who want to witness first-hand science in the making.

1. Hamiltonian Dynamics and Liouville's Theorem.[2]

The main approach used in classical statistical mechanics is based on ensemble theory, which in turn relies heavily on the Hamiltonian formulation of Newtonian mechanics. The mathematical skeleton of Hamiltonian dynamics can be quickly summarized. For a system with configuration space M (dim (M) = n), the phase space is taken to be the cotangent bundle T*(M). T*(M) comes equipped with a natural symplectic structure, a closed 2-form Ω of maximal rank. This defines a volume element $\tilde{\Omega} \equiv \Omega \wedge \Omega \wedge \cdots \wedge \Omega$ (n times) on T*(M). The allowable histories of a system with Hamiltonian H: T*(m) $\to \mathbb{R}$ is represented by a flow on T*(M) whose tangent vector field is $L \equiv (\partial H/\partial p_i)\partial/\partial x^i - (\partial H/\partial x^i)\partial/\partial p_i$ where x^i, i = 1, 2, ..., n, are local coordinates for M and the (x^i, p_i) are local coordinates for T*(M). Hamiltonian flows conserve volume:

$$(1.1)\qquad \mathcal{L}_L\Omega = d(L\cdot\Omega) + L\cdot d\Omega$$
$$= d(L\cdot\Omega) \quad (\text{since } d\Omega = 0)$$
$$= d(dH) \quad (\text{definition of L and } \Omega)$$
$$= 0 .$$

This is the first and most basic form of Liouville's Theorem.

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

In order to make things as parallel as possible with the relativistic case, let us take M to be Newtonian space-time. An appropriate measure for a hypersurface Σ of $T^*(M)$ which projects down onto a plane of simultaneity of M is $\omega \equiv L\cdot\tilde{\Omega}$ since ω assigns a non-zero measure to any such Σ and since we again have $\mathcal{L}_L\omega = 0$. Assume now that there is a probability density function f such that $\int_\Sigma f\omega$ is the probability, at the 'time' corresponding to Σ, of finding the system in a state lying in Σ. Choosing a cylindrical region D of $T^*(M)$ with walls Λ tangent to L and with bottom and top respectively the hypersurfaces Σ_1 and Σ_2 (see Fig. 1), we have

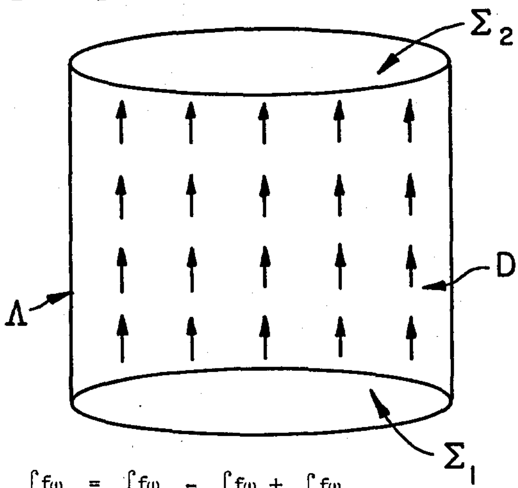

Fig. 1

(1.2) $$\int_{\partial D} f\omega = \int_{\Sigma_2} f\omega - \int_{\Sigma_1} f\omega + \int_{\Lambda} f\omega .$$

The last term on the rhs vanishes since $L\cdot\omega = 0$, and if probability is conserved, the first and second terms cancel out. Hence, we have

(1.3) $$0 = \int_{\partial D} f\omega = \int_D d(f\omega) \quad \text{(Stokes' Theorem)}$$
$$= \int_D df\wedge\omega = \int_D df\wedge L\cdot\tilde{\Omega} \quad \text{(using } d\omega = 0 \text{ and the definition of } \omega)$$
$$= \int_D L(f)\tilde{\Omega} .$$

From this we can conclude that

(1.4) $$L(f) = 0,$$

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

which is another form of Liouville's Theorem. This form is less fundamental than the first, since when probability is not conserved we will want the first but not the second form.

Actually things are a bit more complicated than indicated above since the appropriate one-particle phase space is not the eight dimensional T*(M) but the seven dimensional subspace obtained by imposing the condition that when the time coordinate is normalized so as to coincide with the absolute time t, the fourth component of momentum $p^4 = m =$ constant (m = mass of the particle). But it is easy to define the approprate $\tilde{\Omega}_m$ and ω_m for this seven dimensional subspace, which corresponds to the more familiar augmented ($\underline{x}$, $\underline{p}$, t) phase space. In the relativistic case we have a similar constraint since $g_{ij}p^i p^j = -m^2$ (see Appendix 2).

This emphasis on Hamiltonian dynamics should not be taken to mean that it is the Hamiltonian formalism itself which is essential to classical statistical mechanics. But it is hard to see how statistical mechanics, at least in its ensemble form, can be done without the help of certain features of the Hamiltonian dynamics; namely, a natural phase space, a natural volume element for the phase space, and a description of the dynamics that generates a measure preserving flow. If a Hamiltonian formulation of the underlying particle mechanics should prove not to be feasible, we could seek to preserve the ensemble approach by preserving these features. But this preservation task may not be an easy one to perform; for even if we could prove that there is an appropriate volume measure that is preserved under the non-Hamiltonian dynamics, we cannot automatically proceed with the statistics if the measure is not unique or if it cannot be constructed without first having to solve the dynamics.

These worries are not merely academic, for it is not at all clear how to construct a suitable Hamiltonian dynamics for relativistic particle mechanics; indeed, the No-Interaction Theorems of Currie, Jordan, Sudarshan [4] and Leutwyler [22], proved in the 1960's, were thought to demonstrate that non-trivial particle interactions cannot be given a Hamiltonian description in Minkowski space-time. However, the folk version of these theorems, which emphasized the requirement of canonical representations of the Lorentz group in deducing the negative results, has turned out to be misleading. Also essential to the no-interaction results is the requirement that the particle coordinates are canonical or, equivalently, that their Poisson bracket vanishes (Commutation Condition). And there is evidence that this condition is the real culprit. Droz-Vincent [8] and Künzle [20] have shown that in certain situations, the Commutation Condition alone, without any assumptions on the group structure of the isometries of the underlying configuration space, can lead to a kind of no-interaction in which the particle trajectories are independent (i.e., the N-particle solution is obtain by taking a product of the 1-particle solutions). This form of no-interaction is weaker than that of Currie _et al_., where the world lines of the particles are geodesics of Minkowski space-time, but it is still disturbing.

On the other hand, if the Commutation Condition is dropped, there is an _embarrass de riches_ since many different symplectic structures then become available. Bel [1] proposes to get a unique symplectic structure by imposing asymptotic conditions. Künzle [20] has recommended maintaining the Commutation Condition on a null surface. The details of these and other proposals are too technical to review here.

To some extent, this problem can be side-stepped in general relativity. The gravitational interactions among the particles can be viewed in terms of a background field while the short range interactions can be treated in terms of instantaneous collisions. Thus, the world lines of the particles will be broken geodesic segments in the background metric g. (Allowance can also be made for electromagnetic interactions, but this complication will not be considered here; see [10]). The phase space, which can be taken to be either the tangent or cotangent bundle of the space-time, can be equipped with a natural volume measure which is invariant under the flow (see Appendix 2). Though the idealizations involved in this side-stepping maneuver are crude, quite good numbers can emerge from the theory. But from the point of view of foundations, the idealizations are severe since, in effect, we are restricted to a 1-particle probability density function. Some of the implications of this restriction will be discussed below in Section 5.

2. Heat.

For sake of simplicity, the discussion is restricted to systems consisting of a single species of the particle with mass $m > 0$.

The fundamental tool of relativistic fluid mechanics is Synge's statistical definition of the stress-energy tensor

$$T^{ij} \equiv \int p^i p^j f \pi_m . \tag{2.1}$$

In order to decompose T^{ij} into factors associated with pressure, shear, etc., and to make contact with macroscopic observations, we also need to define the notion of the average or macroscopic velocity of the fluid. This definition and (2.1) justify the term 'fluid' since together they affect a transition from the discrete-particle description to a continuum description. This form of 'reduction' deserves more attention than it has received thus far in the philosophical literature.

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

In the classical setting, the mean velocity can be defined as the velocity of the center of mass of an element; but this concept does not have an invariant relativistic counterpart. Alternatively, the mean velocity can be defined by the condition that an observer who is at rest with respect to the mean flow should see no net average particle flux and no net average momentum flux. Each part of this definition does have a suitable relativistic counterpart, but the counterparts need not coincide. This leads to a distinction between what Synge [34] has called the 'dynamic' and the 'kinematic' concepts of macroscopic velocity.

With plausible regularity conditions on f, it can be shown that

$$T^{ij}U_iU_j > 0 \quad (2.2)$$

for any timelike vector U^i. It then follows from a theorem of Synge's that T^{ij} can be decomposed as

$$T^{ij} = \mu_D V_D^i V_D^j + A^{ij} \quad (2.3)$$

where

$$V_D^i V_{Dj} = -1, \quad A^{ij}V_{Dj} = 0 \quad (2.4)$$

and μ_D is the energy density as measured by an observer with velocity V_D^i. If V_D^i is chosen to be future oriented, then the decomposition in (2.3) is unique and, thus, defines the dynamic velocity. If the momentum flux as measured by an observer whose velocity coincides with V_D^i is taken to be $P_{V_D}^i \equiv -T^{ij}V_{Dj} = \mu_D V_D^i$ then it is obvious that the observer sees no net momentum flux in his rest frame. Synge [34], Landau and Lifshitz [21], and others recommend V_D^i as the explication of the mean macroscopic velocity.

By contrast, Eckart [9] and his followers have proposed that the mean velocity is best explicated by the concept of the kinematic velocity. The particle current density is defined by

$$N^i \equiv \int p^i f \pi_m . \quad (2.5)$$

Using the first form of Liouville's Theorem, it can be shown that

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

(2.6) $$N^i{}_{;i} = \int p^i L(f)\pi_m$$

so that the number of particles is conserved if and only if the second form of Liouville's Theorem holds. Further, from the fact that N^i is timelike, we can set

(2.7) $$N^i = n_K V^i_K , \quad V^i_K V_{Ki} = -1, \quad V^i_K \text{ future oriented}$$

where n_K is the particle number density as measured by an observer with the kinematic velocity V^i_K. So an observer moving with velocity V^i_K will see no net average particle flux in his rest frame.

The condition for V^i_D and V^i_K to coincide is that V^i_K be an eigenvector of T^{ij}, i.e.,

(2.8) $$T^{ij}V_{Kj} = \lambda V^i_K$$

or equivalently, that

(2.9) $$V_K^{[k}T^{i]}{}_j V^j_K = 0 .$$

One would like to know the necessary and sufficient conditions on the *probability function f for (2.8)-(2.9) to hold , but this is* apparently an open problem. When f characterizes equilibrium, then $V^i_D = V^i_K$ (see Sec. 6 below), but outside of equilibrium one cannot expect the equality to hold in general. Thus, transport theory, which is concerned with non-equilibrium processes, must face up to a choice between these two possible definitions.

The point I want to emphasize here is that this choice has profound consequences for the description of heat. Relative to any normalized timelike vector, T^{ij} can be split up as

(2.10) $$T^{ij} = \mu V^i V^j + ph^{ij} + S^{ij} + 2q^{(i}V^{j)}$$

where

(2.11) $$q^j V_j = S^{ij}V_j = S^i{}_i = 0$$

and μ and p are interpreted respectively as the energy density and pressure as measured by an observer with velocity V^i, q^i is the heat flow vector, S^{ij} is due to shear stresses (see below), and

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

$h^{ij} \equiv g^{ij} + V^iV^j$ is the metric of the instantaneous three space orthogonal to V^i. The interpretation of q^i as heat flow is justified by the fact that the total momentum flow $-T^{ij}V_j$ as measured relative to V^i is $\mu V^i + q^i$ so that q^i is the difference between the total flow and the 'organized' flow μV^i and, thus, is equal to the 'unorganized' flow. It is an immediate consequence of the definitions that $q^i_K = 0$ if and only if $V^i_K = V^i_D$. But by the same token, it follows that $q^i_D = 0$ always. This appears to have the counter-intuitive consequence that on the V^i_D description, there is never any heat flow. Landau and Lifshitz tried to avoid this result by describing heat in terms of a particle flux ν^i in the rest frame of V^i_D, where

(2.12) $$N^i = n_D V^i + \nu^i, \quad V^i_D \nu_i = 0.$$

However, they go on to say that pure thermal conduction corresponds to the case of an energy flux with no particle flux, and they take the latter to mean that

(2.13) $$N^\alpha = n_D V^\alpha_D + \nu^\alpha = 0, \quad \alpha = 1,2,3.$$

At best, this procedure can give only a coordinate effect since one can always choose a coordinate system so that locally $V^\alpha_D = 0$, which by (2.13) means that $\nu^\alpha = 0$ and by (2.12) that $\nu^i = 0$; consequently, $V^i_D = V^i_K$, i.e., there is no heat flow.

The proponent of the dynamic velocity description can, of course, take the tough line and maintain that heat is forever banished. Alternatively, if he is unwilling to assume such a radical stance, he can follow Landau and Lifshitz part way by assimilating thermal conduction to particle diffusion but can part company with them in refusing to draw a distinction between pure and impure thermal conduction. While the latter option sounds less radical than the former, it still involves a significant break with standard classical theory where the essential role heat plays is in the law of conservation of energy. And the fact that these options are available at all is an interesting consequence of the application of relativistic consideration.

But are these options really live ones? How, for example, can the V^i_D description be applied to systems which are closed to particle transfer but not to heat transfer? In the present setting, this is not really a fair question since the definition (2.1) of the stress energy tensor is not designed to accommodate such cases. However, the changes needed for the accommodation force a modification in the V_D^i description. The first and most radical option must now be read as saying that there is never any heat flow unless there is, say, radiative transfer, while the second option must now admit both particle transfer and radiative transfer as sources of heat conduction; and both of these modified options are awkward in their non-uniform treatment of heat. But the real test for the V_D^i description concerns how well it hooks up with the relativistic kinetic theory of non-equilibrium processes. I will not put V_D^i to this test here, and in what follows I will work with the V_K^i description; for I am less interested in the question of how far the concepts of relativistic statistical mechanics can be made to depart from their classical counterparts than in the opposite question of how closely they can be made to parallel one another.

3. Work.

The relativistic conservation law

$$(3.1) \qquad T^{ij}{}_{;j} = 0$$

and the symmetry of T^{ij} imply that

$$(3.2) \qquad (T^{ij}V_j)_{;i} = T^{ij}V_{(i;j)}.$$

Using (2.10) and (2.11), this in turn implies that

$$(3.3) \qquad (\mu V^i)_{;i} = -q^i{}_{;i} - T^{ij}V_{(i;j)}.$$

If the second term on the right hand side (hereafter abbreviated rhs) of (3.3) is interpreted as the density of the rate of working, then (3.3) looks like something which we may be able to integrate to give a relativistic analogue of the classical First Law

$$(3.4) \qquad \Delta E = \Delta Q + \Delta W$$

where ΔE is the change in the internal energy, ΔQ is the heat absorbed, and ΔW is the work done by the system. Of course, (3.3) could be used to define the work done, but that would give the relativistic First Law a definitional status so it is worth investigating whether an independent justification can be given for interpreting $-T^{ij}V_{(i;j)}$ as the rate of working.

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

To simplify the investigation, let us assume in analogy with the classical case that the stress term in (2.10) is given by

$$S^{ij} = -2\eta\sigma^{ij} - \zeta\theta h^{ij} \tag{3.5}$$

where η and ζ are respectively the coefficients of shear and bulk viscosity, and the expansion θ and shear σ^{ij} are defined by

$$\theta \equiv V^{i}{}_{;i} \tag{3.6}$$

$$\sigma^{ij} \equiv h^{im}h^{jn}V_{(m;n)} - (1/3)h^{ij}\theta \,.$$

Ultimately, the form of S^{ij} must be derived from (2.1) and non-equilibrium relativistic kinetic theory; while not an exact expression, (3.5) can be justified as a first approximation (see [30]). A calculation from (2.10) and (3.5) gives

$$-T^{ij}V_{(i;j)} = -p\theta + 2\eta\sigma^{ij}\sigma_{ij} + \zeta\theta^{2} - q^{i}\dot{V}_{i} \tag{3.7}$$

$$\dot{V}^{i} \equiv V^{i}{}_{;j}V^{j} \,.$$

Each of the terms on the rhs of (3.7), except the last, has a classical analogue. The final term, $-q^{i}\dot{V}_{i}$, can be motivated by the remark that, according to relativity theory, all forms of energy have an inertia, so that work has to be done in accelerating the heat energy. Such arguments, however, have to be handled with care. By way of illustration, Tolman [38] and Møller [24] once maintained that, on the same grounds, work has to be done to keep a system moving at a constant velocity when heat is added; but no such effect is contained in (3.7).

Unfortunately, such intuitive motivations are the best we can hope to do since no satisfactory independent definition of work is available in this context. For example, we might split off the mechanical part of the stress-energy tensor

$$M^{ij} \equiv ph^{ij} - 2\eta\sigma^{ij} - \zeta\theta^{ij} \tag{3.8}$$

and define the rate of working by the mechanical forces $M^{ij}{}_{;j}$ as $M^{ij}{}_{;j}V_{i}$.

We find that

$$M^{ij}{}_{;j}V_{i} = (M^{ij}V_{i})_{;j} - M^{ij}V_{(i;j)} \tag{3.9}$$

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

$$= -M^{ij}V_{(i;j)}$$

$$= -p\theta + \zeta\theta^2 + 2\eta\sigma^{ij}\sigma_{ij}.$$

This is nice in that it gives back the first three terms on the rhs of (3.7). To get the last term, we might try to add by analogy the work generated by the heat tensor $H^{ij} \equiv 2q^{(i}V^{j)}$. We find

(3.10) $$H^{ij}{}_{;j}V_i = \dot{q}^iV_i - q^i{}_{;i} + V_iV^i{}_{;j}q^j.$$

Using the identities

(3.11) $$V^i{}_{;k}V_i = 0, \quad q^i{}_{;k}V_i + q^iV_{i;k} = 0$$

reduces (3.10) to

(3.12) $$H^{ij}{}_{;j}V_i = -q^i\dot{V}_i - q^i{}_{;i}.$$

But (3.12) contains not only the term we want but also the additional divergence term which corresponds to the Tolman-Møller conception. Alernatively, we could try to define the rate of working through the momentum flow $P^i \equiv -T^{ij}V_j$ as

(3.13) $$\dot{P}^iV_i = P^i{}_{;k}V^kV_i$$

$$= (\mu V^i)_{;k}V^kV_i + q^i{}_{;k}V^kV_i.$$

Again using (3.11), (3.13) reduces to

(3.14) $$\dot{P}^iV_i = \dot{\mu} - q^i\dot{V}_i$$

which is even further from what we want, as can be seen from the equation

(3.15) $$\dot{\mu} + \mu\theta + p\theta - 2\eta\sigma^{ij}\sigma_{ij} - \zeta\theta^2 + q^i{}_{;i} + q^i\dot{V}_i = 0$$

which follows from (3.3) and (3.7). And even if one of these calculations should produce a result which is in agreement with the expression on the rhs of (3.7), it would provide only a numerological justification for interpreting this expression as rate of working, for the 'forces' involved are not true forces (see the discussion in [25]). To obtain true forces, which are proportional to accelerations, it is necessary to project onto the hyperplane orthogonal to the velocity.

The result, which is obtained by writing our $h^k_{\ i}T^{ij}_{\ \ ;j} = 0$ is

$$\mu\dot{V}^k = F^k_M + F^k_H \tag{3.16}$$

where the mechanical force F^k_M and the heat force F^k_H are defined by

$$F^k_M \equiv h^k_{\ i}M^{ij}_{\ \ ;j}, \quad F^k_H \equiv h^k_{\ i}H^{ij}_{\ \ ;j}. \tag{3.17}$$

It appears then that we must simply accept (3.7) as defining the rate of working. The upshot is a non-classical effect; because of the term $-q^i\dot{V}_i$, heat and work are inextricably bound up in a manner not found in the classical case.

4. The Relativistic First Law.

In some of the early treatments, it was simply assumed that a relativistic First Law could be written in the form (3.4) (see, for example, Tolman [38]). According to the adequacy conditions announced in Section 0, this is not an admissable assumption. We have already derived the relativistic energy balance relation (3.15), and this could be taken as the expression of the relativistic First Law. But it is natural to go on to ask whether this relation can be integrated to give something more closely resembling the classical First Law.

Very special relativistic space-times admit a covariantly constant unit timelike vector field U^i (see Appendix 3)

$$U^i_{\ ;j} = 0 \iff [U_{(i;j)} = 0 \text{ (Killing condition)} \text{ and } \omega_{ij} = 0 \text{ (non-rotating)}] \tag{4.1}$$

where the rotation tensor is defined by

$$\omega^{ij} \equiv h^{im}h^{in}U_{[m,n]}. \tag{4.2}$$

Such a field permits the integration of the divergence law (3.1) into the more familiar conservation law in the form of the constancy of the energy. With U^i in place of V^i in (3.2), the Killing condition implies that $P^i_{U;i} = 0$, where $P^i_U \equiv -T^{ij}U_j$ is the momentum flow as measured relative to U^i. The non-rotating condition means that U^i is hypersurface orthogonal, so that if we choose a domain D of space-time such that the bottom H_1 and top H_2 consist of spacelike

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

hypersurfaces orthogonal to U^i and the walls Λ are formed by a timelike surface on which

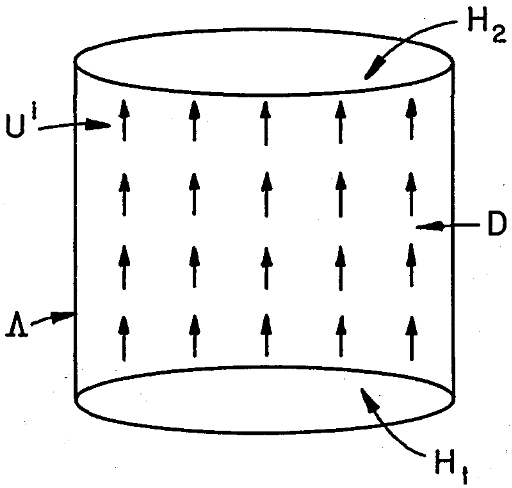

Figure 2

P^i_U vanishes or to which P^i_U is tangent (see Fig. 2), then Stokes' Theorem gives

$$(4.3)\qquad \int_D P^i_{U;i}\,d\mathcal{V} = \int_{\partial D} P^i_U d\sigma_i = \int_{H_2} P^i_U d\sigma_i - \int_{H_1} P^i_U d\sigma_i \equiv E(H_2) - E(H_1) \equiv \Delta E = 0\,.$$

Because of the orthogonality of U^i to H_1 and H_2, we are justified in taking the defined ΔE as the change in the internal energy from the 'time' H_1 to the 'time' H_2 since the surface integrals do indeed give the spatial integration of the energy density $T^{ij}U_iU_j$ as measured relative to U^i.

In the case we are interested in, the relevant velocity field V^i_K cannot be expected to satisfy the Killing condition; but from the point of view of statistical thermodynamics, this does not cause any

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

problems since the failure of the Killing condition is to be interpreted in terms of work. Nor is the failure of the energy-momentum current to vanish on the spatial boundary disconcerting if this failure can be interpreted as heat flow across the boundary. What does cause problems is the failure of the non-rotating condition; and a relativistic fluid can rotate even in equilibrium (see, however, Section 6 below). The point is this: (3.3) can be integrated to give an equation of the form (3.4), if the quantities in (3.4) are interpreted as

$$(4.4) \qquad \Delta E \equiv \int_{H_2} \mu V^i_K d\sigma_i - \int_{H_1} \mu V^i_K d\sigma_i$$

$$\Delta Q \equiv -\int_{\partial D} q^i_K d\sigma_i , \qquad \Delta W \equiv -\int_D T^{ij} V_{K(i;j)} d\mathcal{V} .$$

But the resemblance to the classical First Law may be only a formal one when V^i_K is rotating. For then ΔE as defined in (4.4) may not be equal to the change in the internal energy density μ, and ΔQ may not be equal to the heat transferred across the spatial boundary Λ.

5. The relativistic Second Law.

One of the earliest investigations of the implications of general relativity for statistical thermodynamics was undertaken by Tolman ([36] and [37]). He wrote the relativistic Second Law in the form

$$(5.1) \qquad S^i_{\ ;i} \Delta\mathcal{V} \geq \Delta Q/T$$

where ΔQ is the heat absorbed by the volume element $\Delta\mathcal{V}$ at the absolute temperature T, and the entropy current vector is defined by

$$(5.2) \qquad S^i \equiv sV^i$$

where V^i is the macroscopic velocity and s is the proper entropy density. Tolman's presentation was incomplete in that it did not supply independent statistical interpretations for ΔQ and V^i, and it simply assumed the validity of the ordinary laws of thermodynamics. Furthermore, if V^i is identified with V^i_K, then the entropy current will not satisfy (5.2) except in very special cases, e.g., equilibrium where we should have $\Delta Q = 0$.

The modern approach begins with the definition of the entropy current as

$$(5.3) \qquad S^i \equiv -\int p^i f \log f \pi_m .$$

Here we face exactly the same paradox which arises in classical

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

statistical mechanics. Taking the covariant derivative of (5.3) we find that

(5.4) $$s^i{}_{;i} = -\int(\log f + 1)L(f)\pi_m .$$

And upon application of the Liouville Theorem (1.4), we have

(5.5) $$s^i{}_{;i} = 0 ,$$

i.e., no entropy production.

One way to resolve the paradox in the classical case is to point out that while $L(f) = 0$ for the N-particle probability density, it is generally non-zero for various reduced densities. Indeed, the most promising tack (and, one is tempted to say, the only legitimate tack within the ensemble theory of isolated systems) for justifying the Boltzmann equation is to show that $L(f) = 0$ for the N-particle density, plus certain 'plausible' assumptions, entail the Boltzmann equation for the 1-particle density.[3] This tack is not available in the present relativistic formalism since we are constrained from the beginning to work with the 1-particle density.

Thus, it will have to be taken for granted here that the 1-particle density f does not obey (1.4) but rather a relativistic Boltzmann equation of the form

(5.6) $$L(f) = C(f)$$

where $C(f)$ depends upon the nature of the collisions among the particles. The problem for the relativist is to find an appropriate relativistic expression for the collision term $C(f)$ and to show that the resultant equation leads to the 'relativistic H-Theorem'

(5.7) $$s^i{}_{;i} \geq 0 .$$

The problem has been solved (see [10] - [13] and [30]). (5.7) then provides one expression of the relativistic Second Law.

The entropy current can be split up into components parallel and orthogonal to the chosen mean velocity V^i

(5.8) $$s^i = sV^i + s^i, \quad s^i V_i = 0$$

where s is the proper entropy density and s^i is the entropy diffusion vector. If the diffusion is zero on the spatial boundaries of the system, as will be the case for an isolated system, then (5.7) can be integrated to give the classical form of the Second Law.

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

$$\int_D s^i{}_{;i}\,dV = \int_{\partial D} s^i d\sigma_i \tag{5.9}$$

$$= \int_{H_2} s^i d\sigma_i - \int_{H_1} s^i d\sigma_i$$

$$\equiv S(H_2) - S(H_1) \equiv \Delta S \geq 0.$$

In the general case where s^i does not vanish on the walls Λ, we have

$$\Delta S \geq -\int_\Lambda s^i d\sigma_i . \tag{5.10}$$

And a crude first order calculation (see Appendix 4) based on an f which is close to equilibrium shows that

$$s^i \approx q^i/T \tag{5.11}$$

so that for constant T we recover the usual expression

$$\Delta S \geq \Delta Q/T \tag{5.12}$$

for the Second Law. But as we will see below, causes of entropy production other than the absorption of heat must be countenanced. Moreover, the T that appears in (5.12) is not the temperature of the fluid in the non-equilibrium state f but rather the temperature in the fictitious equilibrium state f_o which f approximates; and in fact, no meaning has been assigned so far to temperature in non-equilibrium states. One way to remedy this situation is to show that for an equilibrium state is an 'equation of state' $\mu = \mu(s, n)$ which gives the energy density μ as a function of the entropy density s and the particle density n and which obeys the Gibbs relation

$$d\mu = Tds + \text{constant}\times Tdn. \tag{5.13}$$

Next, one argues that the equation of state remains valid to first order in near equilibrium situations and that (5.13) can be used to define the temperature for these cases. This remedy is somewhat contrary to the spirit of a thoroughgoing statistical foundation, and there is obviously a need for more investigation of how the temperature concept fares in non-equilibrium states.

6. Equilibrium.

Classically, equilibrium is defined by the absence of entropy production. Relativistically, this condition is expressed by

$$s^i{}_{;i} = 0 . \tag{6.1}$$

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

Using the relativistic Boltzmann equation (5.6), it follows that for binary collisions

$$(6.2) \qquad S^{i}{}_{;i} = -\int(\log f)C(f)\pi_m = 0 ,$$

i.e., logf is a collision invariant. Thus, in the case where only binary collisions are taken into account, this leads to the relativistic Maxwell-Boltzmann distribution

$$(6.3) \qquad f_o = C\exp(\alpha(x) + \beta_i(x)p^i).$$

In order that f_o have finite moments, β^i must be a future directed timelike vector, so that we can write $\beta_i = \beta\overset{\circ}{U}_i$ where $\overset{\circ}{U}{}^i\overset{\circ}{U}_i = -1$. Calculating the moments of f_o gives

$$(6.4) \qquad \overset{\circ}{T}{}^{ij} = \mu\overset{\circ}{U}{}^i\overset{\circ}{U}{}^j + ph^{ij}, \quad \overset{\circ}{N}{}^i = n\overset{\circ}{U}{}^i, \quad \overset{\circ}{U}{}^i = V^i_K = V^i_D$$

so that the behavior is that of a perfect fluid. From the analogy with ordinary thermodynamics, we identify β with 1/KT, T being the absolute temperature.

In classical setting, one often hears the assertion that temperature is mean kinetic energy. This cannot be true if the 'is' here is the is of identity. For by symmetry of identity, mean kinetic energy would be identical with temperature, but this is clearly not the case since in states far from equilibrium the concept of temperature may not be well-defined while that of mean kinetic energy is. Moreover, in the relativistic case temperature (or some multiple of it) is not even coextensive with mean kinetic energy in equilibrium states, though an approximate coincidence may hold for extreme temperatures (see [33]). It seems to me that the best way to accommodate these facts is to follow the lead of those mind-body identity theorists who want to give a 'functional' characterization of mental states; thus, temperature would be seen as 'the quantity which plays such-and-such a role in the equilibrium distribution'. One advantage of such a treatment is that it explains how we can make cross-theoretical identifications of the quantity of temperature. I hope to work out this suggestion in detail in another place.

Returning to (6.3) and applying the condition $L(f_o) = 0$ leads to

$$(6.5) \qquad \beta_{(i;j)} = 0, \text{ i.e., } V^i/T \text{ is a Killing field.}$$

Ehlers [11] has shown that in some circumstances, the move from (6.1) to (6.5) can be reversed. Assume that the space-time admits a group $\mathcal{G}$ of fixed-point free isometries, with timelike orbits, which leave f invariant. If X is the generating field, the conditions are that $\mathcal{L}_X g = \mathcal{L}_X f = 0$. This implies that $\mathcal{L}_X S^i = 0$. Assume further that for some $G \in \mathcal{G}$ the walls Λ of the world tube in Fig. 2 are invariant under G, i.e., $G(\Lambda) = \Lambda$, and that $G(H_1) = H_2$. Then if all the flow lines of S^i are intersected by H_1 and H_2, we can conclude that

(6.6) $$\int_{H_2} S^i d\sigma_i = \int_{H_1} S^i d\sigma_i .$$

Then if the system is isolated so that S^i vanishes on the walls, Stokes' Theorem together with the relativistic Second Law (5.7) imply (6.1).

To see some of the consequences of the relativistic condition for equilibrium, note that (see Appendix 3)

(6.7) $$V^i/T \text{ is a Killing field} \Longleftrightarrow [\theta = 0 \text{ and } \sigma_{ij} = 0 \text{ and } \dot{V}_i = -(\log T)_{,i} \text{ and } \dot{T} \equiv T_{,i}V^i = 0].$$

Thus, an expanding or contracting relativistic fluid cannot be in equilibrium. This does not necessarily mean that equilibrium is not possible in an expanding or contracting universe, though the implication is usually safe since only exceptional expanding or contracting universes admit timelike Killing fields. Lichnerowicz [23] and Carter [2] have further shown that in an asymptotically flat and source free universe satisfying Einstein's field equations, the rotation of the equilibrium fluid must vanish.

In a stationary coordinate system adapted to the Killing field V^i/T we get the Tolman temperature law

(6.8) $$T\sqrt{-g_{44}} = \text{constant} .$$

Here the g_{44} component of the metric is independent of time but may depend upon the spatial coordinates; thus, in contrast to the classical case, the temperature of a relativistic equilibrium fluid need not be a constant.

It is sometimes said that since $\theta \neq 0$ implies that $S^i{}_{;i} \neq 0$, adiabatic expansion of a relativistic fluid is not possible (see [29]).

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

This is not correct since, for example, in a closed expanding universe the entropy production cannot be due to heat flow into the system from an external source. But such an example raises the new problem of accounting for the entropy production in terms of non-thermal causes. This is a problem which is rarely, if ever, considered in the classical setting. Its solution depends upon the techniques of non-equilibrium relativistic kinetic theory mentioned in the next section.

7. The Relativistic Fourier Law.

As we saw in Sec. 6, relativistic equilibrium does not require that T = constant; in fact, (6.7) says that T is constant if and only if the fluid is in geodesic motion. This situation seems to lead to a paradox since temperature gradients should set up a heat flow, implying that the fluid is not in equilibrium after all. What this paradox shows is that the correct generalization of the Fourier heat flow law

$$\underline{q} = -\varkappa \underline{\mathrm{grad}}T \tag{7.1}$$

is not the most obvious relativistic analogue

$$q^i = -\varkappa h^{ij} T_{,j}. \tag{7.2}$$

Eckart [9] proposed instead

$$q^i = -\varkappa h^{ij}(T_{,j} + T\dot{V}_j). \tag{7.3}$$

This resolves the paradox since (7.3) implies that

$$q^i = 0 \Longleftrightarrow \dot{V}_i = -(\log T)_{,i} - (\dot{T}/T)V_i . \tag{7.4}$$

And the application of (6.7) then guarantees that $q^i = 0$ in equilibrium. In fact, the combination of (6.7) and (7.4) show that the condition that V^i/T is a Killing field is equivalent to the more intuitive definition of equilibrium

(7.5) No heat flow ($q^i = 0$), no work ($\theta = 0, \sigma_{ij} = 0$), and T constant along the flow lines.

Alas, the Eckart Law cannot be quite correct, for it is a parabolic equation and, thus, allows arbitrarily great velocities of heat wave propagation. This consequence led some physicists to simply neglect heat conduction (see [39]) while others treated heat flow by means of (7.3) despite the apparent inconsistency with relativity theory (see [16]). Still others proposed to modify (7.3) by adding a time derivative term that converts the equation into a hyperbolic one (see [3] and [19]). Kranyš proposed

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

(7.6) $$\hat{q}^i = -\lambda \dot{\hat{q}}^i - \kappa h^{ij}(T_{,j} + T\dot{V}_j).$$

This modification is ad hoc, and further, unless it is assumed that $\hat{q}^i V_i = 0$, $\hat{q}^i$ cannot be identified with the heat flow vector q^i defined in (2.10)-(2.11). Kranyš did not impose this assumption and, thus, he was led to identify $\hat{q}^i$ as the heat flow and q^i as the component of $\hat{q}^i$ orthogonal to V^i. This treatment is somewhat awkward since only q^i enters into conservation laws.

What is needed then is a derivation of the relativistic heat flow law from relativistic kinetic theory. A good deal of progress has been made on this and related problems in the past few years, most notably by Israel [17] and Stewart [31]. They obtain transport equations for a dilute relativistic gas and show that these equations form a hyperbolic system in which the propagation velocities do not exceed $\sqrt{3/5}$ the velocity of light. The details are too technical to report here.

8. The Planck-Ott Debate.

The relativistic statistical mechanics sketched above does not justify either side in the Planck-Ott debate; in fact, it tends to show that both sides are wrong. In relativistic kinetic theory, the temperature T does not transform either by the Planck law (0.1) or by the Ott law (0.2); rather, it enters into the equilibrium distribution (6.3) as a scalar parameter.

Once the alleged transformation law for T or ΔQ was decided upon, both sides in the debate used the same method to fix the transformation law for the other quantity. The idea was to make use of situations where $\Delta S = \Delta Q/T$ and to appeal to the fact that entropy, because of its statistical interpretation, is an 'invariant quantity'. Then, the argument went, ΔQ and T must transform in the same manner in order that the Second Law be relativistically invariant. Now in exactly what sense is entropy an invariant? On the kinetic theory approach, the entropy current S^i as defined in (5.3) in an invariant in that it is a 4-vector, and further, every observer associates the same current vector with the system. On the other hand, the entropy change ΔS computed by integrating S^i will be the same for two observers only if they both use the same domain of integration. But assuming a fixed domain, not only is ΔS an 'invariant' but so is ΔQ as computed from integration of the 4-vector heat flow q^i. Thus, it would seem that all the thermodynamic quantities are 'invariants'.

It might be objected that there is an important difference between ΔS and ΔQ; for the decomposition (2.10) of the stress-energy tensor and, hence, the value of q^i depends on the choice of the velocity field V^i. So it might seem that the resultant ΔQ could transform in a manner prescribed by Planck or Ott, or perhaps in some even more complicated fashion. While this objection ultimately fails, it does serve to raise some valuable points. T^{ij} can be mathematically decomposed relative to any unit timelike vector field V^i, but for purposes at hand, only those decompositions in which V^i can be interpreted as the mean velocity of the fluid will allow contact to be made with thermodynamics. To emphasize this point, let us assume that we are in the context of special relativity where a timelike Killing V^i is always available. Then the rate of working $-T^{ij}V_{(i;j)}$ as computed relative to this field is identically zero. This is a wholly misleading result since from the macro-thermodynamical point of view, the fluid is performing work if it is expanding or shearing. Of course, we have seen that there are different ways of defining the mean velocity of the fluid and that these different definitions lead to different values for heat and work. But once the definition is chosen, there is a preferred way to evaluate ΔW and ΔQ, and the evaluation does not conform to law of Planck, or Ott, or any similar law. It is also instructive to compare the implications of relativistic kinetic theory for q^i with those for pressure p, which is often held to be an 'invariant' par excellance; but I will leave this exercise to the reader.

The pioneers of 'relativistic thermodynamics' were led astray by two attitudes. First, they acted as if thermodynamics were a self-contained subject, existing independently of any statistical mechanical interpretation. Within this setting, many different 'transformation laws' for the thermodynamical quantities are possible. But the attitude was never carried through consistently since, for example, appeal was made to the probabilistic interpretation of entropy in order to determine its transformation law. Second, in attempting to demonstrate that thermodynamics can be made to conform to the requirements of relativity, they adopted a passive view of relativity principles according to which relativity transformations connect different descriptions of the same physical system. Thus, their program consisted in finding how thermodynamic quantities transform under Lorentz coordinate transformations and showing that the laws of thermodynamics are covariant under these transformation rules. Finding the transformation properties of thermodynamic quantities is an admirable task if it is taken to mean finding what geometrical objects underlie thermodynamics. But this is a task which has an unambiguous solution only when the statistical mechanical interpretation is employed. Furthermore, the passive view of symmetry tends

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

to obscure the fact that the rest frame of the fluid provides a preferred way of defining thermodynamic quantities like heat and work and a preferred domain of integration for evaluating total changes in these quantities.

In closing this section, I want to make explicit a thesis which is lurking in the above remarks; namely, the thesis that meaningful physical quantities correspond to geometric object fields in space-time.[4] If this thesis is correct then, as I believe this paper illustrates, almost all of the standard presentations of classical thermodynamics are non-perspicuous because they fail to make clear what geometric objects underlie the thermodynamic quantities. In a way, one can view the work on 'relativistic thermodynamics' by Einstein, Planck, Tolman, Møller, Ott, etc., as attempts to fill this gap. But, as I have argued, their attempts were failures; and it seems that success can be achieved only when we know the statistical mechanical basis for thermodynamics. If this is so, we have the somewhat paradoxical situation that thermodynamics cannot even be properly stated without being micro-reduced.

9. Concluding Remarks.

Some of the foundations problems reviewed here, like that of finding a satisfactory relativistic Hamiltonian formalism, belong to the upper reaches of mathematical physics. But many of the interpretations problems are ones to which philosophers of science can contribute. If this paper helps to stimulate such contributions, I will count it as a success.

While it would be premature to draw any positive morals for the philosophy of scientific methodology, I feel safe in reaching a negative conclusion; namely, none of the standard views about the meaning of theoretical terms, the reduction of scientific theories, the nature of scientific revolutions, etc., do justice to the cases treated here. This may have something to do with the special nature of relativistic statistical thermodynamics, or it may be a more general reflection on the current state of philosophy of science. Only time and a lot more study will tell.

Appendices

1. Notation and Conventions.

The signature of the metric tensor g is chosen to be (+ + + −). Latin indices run from 1 to 4 while Greek indices run from 1 to 3. The Einstein summation convention on repeated indices is used throughout. The comma denotes ordinary derivatives and the semi-colon denotes covariant derivatives. Units are chosen so that $c \equiv 1$.

2. The Relativistic Phase Space.

Let (M, g) be a relativistic space-time. Because the metric g gives a correspondence between tangent and cotangent vectors, there is a canonical isomorphism between the tangent bundle T(M) and the cotangent bundle T*(M). And for a particle free falling in the gravitational field described by g, we can pass freely between the Lagrangian description with Lagrangian $\mathcal{L} = (1/2)g_{ij}\dot{x}^i\dot{x}^j$ and the Hamiltonian description with Hamiltonian $H = (1/2)g^{ij}p_ip_j$. The Liouville operator is

(A2.1) $$L = (\partial H/\partial p_i)\partial/\partial x^i - (\partial H/\partial x^i)\partial/\partial p_i$$

[1] $$= p^i\,\partial/\partial x^i - \Gamma^i_{jk}p^jp^k\,\partial/\partial p^i$$

where Γ^i_{jk} are the components of the unique affine connection compatible with g. A volume element can be obtained either by taking the canonical 8-form on T*(M) or, equivalently, by taking the natural measure on T(M) consisting of the exterior product $\eta \wedge \pi$ of the space-time volume measure $\eta \equiv \sqrt{-g}\,dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$, $g \equiv \det(g)$, and the volume measure $\pi \equiv \sqrt{-g}\,dp^1 \wedge dp^2 \wedge dp^3 \wedge dp^4$ on a tangent space. The physical phase P_m, however, is not T*(M) but the 7-dimensional subspace consisting of future directed timelike momentum vectors with constant mass $m = \sqrt{-g_{ij}p^ip^j}$. The volume element for P_m is $\eta \wedge \pi_m$ with $\pi_m \equiv 2H(p^i)\delta(p^2 + m^2)\sqrt{-g}\,dp^1 \wedge dp^2 \wedge dp^3 \wedge dp^4$ where δ is the Dirac function and $H(p^i)$ is 1 if p^i is future directed and 0 otherwise. Since $L(m) = 0$, L is tangent to P_m, and the restriction L_m to p_m is the physical Liouville operator. The unique (up to a numerical factor) volume element which gives a non-zero measure to any hypersurface of P_m not tangent to L_m is $\omega_m \equiv L_m \cdot (\eta \wedge \pi_m)$. For a hypersurface Σ which projects down onto a spacelike hypersurface of (M, g), this reduces to the form

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

(A2.2) $$\omega_m = p^i d\sigma_i \wedge \pi_m$$

where $d\sigma_i$ is the usual surface element.

3. Reference Frames.

A <u>reference frame</u> is a unit future directed timelike vector field. The frame V^i is said to be <u>stationary</u> iff there is a positive function f and a Killing field X^i such that $V^i = fX^i$. To derive some consequences of stationarity, we write out the stationarity condition

(A3.1) $$V_{(i;j)} = f_{,(i}X_{j)} + fX_{(i;j)}$$ [2]

$$= f_{,(i}X_{j)} = (\log f)_{,(i}V_{j)}.$$

Contracting with V^j and using the identities

(A3.2) $$V^iV_i = -1, \qquad V_{i;k}V^i = 0$$

we get

(A3.3) $$A_i \equiv V_{i;j}V^j = -(\log f)_{,i} + (\dot{f}/f)V_i.$$ [3]

Contracting once again and using $A_iV^i = 0$ gives [4]

(A3.4) $$0 = -2\dot{f}/f$$

with the result that

(A3.5) $$A_i = -(\log f)_{,i}.$$

This shows that the curl of the acceleration vanishes. Moreover, if (A3.5) is substituted back into (A3.1) and comparison is made with the identity

(A3.6) $$V_{(i;j)} = \sigma_{ij} + (1/3)\theta h_{ij} - A_{(i}V_{j)}$$

we see that stationarity requires rigidity, i.e., $\theta = \sigma_{ij} = 0$. The argument can also be reversed: $A_{[i,j]} = \theta = \sigma_{ij} = 0$ implies stationarity. The proof is left as an exercise.

A frame V^i is <u>covariantly constant</u> iff $V^i{}_{;j} = 0$. This obviously entails that $V_{(i;j)} = 0$ and $V_{[i,j]} = 0$. Further, it follows that $A_i = V_{i;j}V^j = V_{(i;j)}V^j = 0$. Then from the decomposition of the

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

rotation tensor as

(A3.7) $$\omega_{ij} = V_{[i,j]} + V_{[i}A_{j]}$$

it is seen that covariant constancy is equivalent to $V_{(i;j)} = \omega_{ij} = 0$.

4. Entropy and Heat Flow.

Consider a fluid not far from equilibrium; in particular, suppose that the probability density is $f = f_o(1 + g)$ where f_o is the equilibrium density and g is 'small'. We impose the matching conditions

(A4.1) $$\tilde{V}^i_K = \mathring{V}^i_K, \quad \tilde{N}^i_K = \mathring{N}^i_K$$

and for ease of calculation, we assume that $V_{[i,j]} = 0$; this means that we can choose a coordinate system such that $V^i_K = (0, 0, 0, 1)$, $g_{\alpha 4} = g^{\alpha 4} = 0$, and $g_{44} = -1$. Then using the equations (5.3), (5.8), and (6.3) and neglecting terms multiplied by log(1 + g), we have

(A4.2) $$\tilde{s}^4_K = 0, \quad \tilde{s}^\alpha_K = (\tilde{T}^{\alpha 4} - \mathring{T}^{\alpha 4})/T.$$

Then substituting from (2.10) and (6.4), we have

(A4.3) $$\tilde{s}^\alpha_K = q^\alpha_K/T.$$

This calculation is very crude, and a more detailed treatment shows that entropy production can result from factors neglected here.

Notes

[1]There are many people whom I would like to thank for help and encouragement on this project; but so far, they have not given any.

[2]Comprehensive reviews of the relevant mathematics and physics are to be found in Ehlers ([10],[11],[12],[13]), Havas [15], and Stewart [30]. In this section and the Appendices I present only the minimal amount of technical apparatus needed for an understanding of the foundations problems.

[3]The reason for the scare quotes is that the assumptions turn out to involve conditions very similar to Boltzmann's molecular chaos hypothesis.

[4]For a characterization of geometric objects, see Schouten [28].

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

References

[1] Bel, L. "Formes hamiltoniennes et systems conservatifs." Annales de l'Institut Henri Poincaré 22(1975): 173-199.

[2] Carter, B. "Black Hole Equilibrium States." In Black Holes. Edited by C. DeWitt and B.S. DeWitt. New York: Gordon and Breach, 1973. Pages 57-214.

[3] Cattaneo, C. "Sur une forme de l'equation de la chaleur eliminante la paradoxe d'une propagation instantanee." Comptes Rendus 247(1958): 431-433.

[4] Currie, D.G., Jordan, T.F., and Sudarshan, E.C.G. "Relativistic Invariance and Hamiltonian Theories of Interacting Particles." Reviews of Modern Physics 35(1963): 350-375.

[5] Danzig, D. van. "On the Phenomenological Thermodynamics of Moving Water." Physica 6(1939): 673-704.

[6] --------------. "On Relativistic Thermodynamics." Koninklijke Nederlansche Akademie van Wetenschappen 42(1939): 601-607.

[7] ---------------. "On Relativistic Gas Theory." Koninklijke Nederlansche Akademie van Wetenschappen 42(1939): 608-625.

[8] Droz-Vincent, P. "Hamiltonian Systems in Relativistic Dynamics." Nuovo Cimento B 12(1972): 1-9.

[9] Eckart, C. "Thermodynamics of Irreversible Processes III." Physical Review 58(1940): 919-924.

[10] Ehlers, J. "General Relativity and Kinetic Theory." In General Relativity and Cosmology. Edited by R.K. Sachs. New York: Academic Press, 1971. Pages 1-70.

[11] ---------. "General-Relativistic Kinetic Theory of Gases." In Relativistic Fluid Dynamics. Edited by C. Cattaneo. Rome: Edizioni Cremonese, 1971. Pages 301-388.

[12] ---------. "Survey of General Relativity Theory." In Relativity, Astrophysics, and Cosmology. Edited by W. Israel. Dordrecht-Holland: D. Reidel, 1973. Pages 1-125.

[13] ---------. "Progress in Relativistic Statistical Mechanics, Thermodynamics and Continuum Mechanics." In General Relativity and Gravitation. Edited by G. Shaviv and J. Rosen. Pages 213-232.

[14] Einstein, A. "Über das Relätivitatsprinzip und die aus demselben gezogenen Folgerungen." Jahrbuch der Radioaktivitat und Electronik 4(1907): 411-462.

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press

[15] Havas, P. "Some Basic Problems in the Formulation of Relativistic Statistical Mechanics of Interacting Particles." In Statistical Mechanics of Equilibrium and Non-Equilibrium. Edited by J. Meixner. Amsterdam: North-Holland, 1965. Pages 1-19.

[16] Hughes, W.F. "Relativistic Magnetohydrodynamics and Irreversible Thermodynamics." Proceedings of the Cambridge Philosophical Society 57(1961): 878-889.

[17] Israel, W. "Nonstationary Irreversible Thermodynamics: A Causal Relativistic Theory." Annals of Physics 100(1976): 310-331.

[18] Jüttner, F. "Das Maxwellsche Gesetz der Geschwindigkeitverteilung in der Relativtheorie." Annalen der Physik 34(1911): 856-882.

[19] Kranyš, M. "Relativistic Hydrodynamics with Irreversible Thermodynamics without the Paradox of Infinite Velocity of Heat Conduction." Nuovo Cimento B 41(1966): 51-70.

[20] Künzle, H.P. "Galilei and Lorentz Invariance of Classical Particle Interactions." Symposia Mathematica 14(1974): 53-84.

[21] Landau, L.D. and Lifshitz, E.M. Fluid Mechanics. New York: Pergamon Press, 1975.

[22] Leutwyler, H. "A No-Interaction Theorem in Classical Relativistic Hamiltonian Particle Mechanics." Nuovo Cimento 37 (1965): 556-567.

[23] Lichnerowicz, A. Theories Relativistes de la Gravitation et de L'Électromagnétisme. Paris: Masson et Cie, 1955.

[24] Møller, C. The Theory of Relativity. Oxford: Clarendon Press, 1962.

[25] ---------. The Theory of Relativity. 2nd ed. Oxford: Clarendon Press, 1972.

[26] Ott, H. "Lorentz Transformation der Wärme und der Temperature." Zeitschrift fur Physik 175(1963): 70-104.

[27] Planck, M. "Zur Dynamik Bewegter System." Annalen der Physik 26 (1908): 1-34.

[28] Schouten, J.A. Tensor Analysis for Physicists. Oxford: Oxford University Press, 1959.

[29] Schucking, E.L. and Spiegel, E.A. "Thermodynamics and Cosmology." Comments on Astrophysics and Space Science. 2(1970): 121-125.

[30] Stewart, J.M. Non-Equilibrium Relativistic Kinetic Theory. New York: Springer-Verlag, 1971.

[31] ------------. "On Transient Relativistic Thermodynamics and Kinetic Theory." Proceedings of the Royal Society of London A 357(1977): 59-75.

[32] Synge, J.L. "The Energy Tensor of a Continuous Medium." Proceedings of the Royal Society of Canada 28(1934): 127-171.

[33] ----------. The Relativistic Gas. Amsterdam: North Holland, 1957.

[34] ----------. Relativity: The General Theory. Amsterdam: North Holland, 1964.

[35] Tolman, R.C. "Relativity Theory: The Equipartition Law in a System of Particles." Philosophical Magazine 28(1914): 583-600.

[36] -----------. "On the Extension of Thermodynamics to General Relativity." Proceedings of the National Academy of Sciences 14 (1928): 268-272.

[37] -----------. "Further Remarks on the Second Law of Thermodynamics in General Relativity." Proceedings of the National Academy of Sciences 14(1928): 701-706.

[38] -----------. Relativity, Thermodynamics, and Cosmology. Oxford: Clarendon Press, 1934.

[39] Zumino, B. "Some Questions in Relativistic Hydrodynamics." Physical Review 108(1957): 1116-1121.

https://doi.org/10.1086/psaprocbienmeetp.1978.2.192467 Published online by Cambridge University Press