ON THE INCLUSION OF A BOUNDED CONVERGENCE FIELD IN THE SPACE OF ALMOST CONVERGENT SEQUENCES

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Introduction. Let $T = (t_{mn})$ be a regular matrix, and C_T be its bounded convergence field. Necessary and sufficient conditions for C_T to contain the space of almost convergent sequences are well known. (See, e.g., [7, p. 62]). G. M. Petersen has suggested as a problem for research the discovery of necessary and sufficient conditions for the reverse inclusion: When is C_T contained in the space of almost convergent sequences? [7, p. 137, research problem 9]. In this paper we deal with this question in a more general context. First we need some notation.

Letting T be as above, let $C^*(N)$ be the bounded real valued functions on the positive integers. We define m_T as the set of " T-invariant means ", i.e. the set of positive linear functionals ϕ on $C^*(N)$ such that $\phi(e) = 1$ (where e is the unit function), $\phi \circ T = \phi$, and $\phi(f) = 0$ whenever $\lim_{n \to \infty} f(n) = 0$. We define $V_T = \{f \in C^*(N) : \phi_1(f) = \phi_2(f)$ for all $\phi_1, \phi_2 \in m_T\}$, and write $m_T(f)$ for this common value. It follows easily from our assumptions on m_T that $C_T \subset V_T$ *consistently*, in the sense that $m_T(f) = T\text{-}\lim(f)$ for each $f \in C_T$. Notice that V_T is the usual space of almost convergent functions when T is the shift matrix transformation $Tf(n) = f(n+1)$.

Now let $T = (t_{mn})$ and $S = (s_{mn})$ be non-negative regular matrices, and consider the following possible relations.

(I) $V_T \subset C_S$ consistently,
(II) $V_T \subset V_S$ consistently,
(III) $C_T \subset V_S$ consistently.

Since $C_T \subset V_T$ and $C_S \subset V_S$, these three relations are in decreasing order of strength, i.e. (I) implies (II), and (II) implies (III). If S is the shift matrix, then (III) is related to Petersen's problem.

In Section 2, we give necessary and sufficient conditions for (I) and (II). Actually this has already been done for a certain class of permutation matrices by Raimi [8] and Dean and Raimi [4], and the proofs in the general case are largely adaptations of their proofs. An interesting side result is obtained (2.3 below), namely that, for sequences of linear operators induced on $C(\beta N \backslash N)$ by regular matrix operators on $C^*(N)$, convergence in norm is equivalent to convergence in the strong operator topology [3, p. 34].

In Section 3 we give a condition sufficient for (III). In Section 4 we show that, if T is suitably restricted, then this condition is also necessary.

1. Preliminaries. In this section we introduce some notations and lemmas needed in the sequel.

1.1. Notation. If $f \in C^*(N)$, let f' be its extension to βN, and $f^* = f' \mid \beta N \backslash N$. If $A \subset N$, let A' be its closure in βN and $A^* = A' \cap \beta N \backslash N$. Then $N^* = \beta N \backslash N$, and $C(N^*)$ is the space of continuous real functions on N^*.

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If $T = (t_{mn})$ is non-negative regular and C_0 the elements of $C^*(N)$ with limit zero, then $T(C_0) \subset C_0$. Hence T induces a positive linear operator T_1 on $C(N^*)$ defined by $T_1(f^*) = (Tf)^*$. The norm of this operator on $C^*(N)$ is given by $\| T_1 \| = \limsup_{m \to \infty} \sum_k t_{mk} = 1$ (cf. [1] or [2]). The regularity of T means that $T_1 e = e$, where e is the unit in $C(N^*)$.

$T = (t_{mn})$ and $S = (s_{mn})$ are called *equivalent* if $\lim_{m \to \infty} \sum_k | t_{mk} - s_{mk} | = 0$.

If T is regular, then T is equivalent to a *truncated matrix* S, i.e. one such that there exists $m(1) < m(2) < \dots$ and $n(1) < n(2) < \dots$ such that, if $m \in [m(k), m(k+1))$, then $s_{mn} = 0$ for $n \notin [n(k), n(k+2))$ [7, p. 82]. There is no loss in generality in assuming that our matrices have truncated form, for if S and T are equivalent, then for each $f \in C^*(N)$, $\lim_{m \to \infty} (Tf - Sf)(m) = 0$, and hence $T_1 = S_1$ on $C(N^*)$, $C_T = C_S$, $V_T = V_S$, and $m_T = m_S$. Note that the sum and product of truncated matrices are truncated. Note also that $(S \circ T)_1 = S_1 \circ T_1$ for S and T regular.

If $n \in N$, ε_n is the functional on $C^*(N)$ defined by $\varepsilon_n(f) = f(n)$; and if $w \in N^*$, ε_w is defined on $C(N^*)$ by $\varepsilon_w(f) = f(w)$. Then $\varepsilon_n \circ T(f) = Tf(n)$, etc.

If T is a matrix, let $T_n = (1/n) \sum_{k=1}^{n} T^k$, where T^k is the kth iterate of T. If T is regular, so is T_n.

Finally we define

$$\begin{aligned} Z_T &= \{ f \in C^*(N) : m_T(f) = 0 \}, \\ K(T) &= \{ f \in C_T : T\text{-}\lim(f) = 0 \} \\ &= \{ f \in C_T : T_1 f^* = 0 \}. \end{aligned}$$

The following lemmas are mainly slight modifications of those which occur in [4] and [8]. The main difference is that we assume an extra property for our T-invariant means m_T, namely that $m_T(f) = 0$ for all $f \in C_0$, and hence $\phi(f) = \phi(g)$ for all $\phi \in m_T$, whenever $f - g \in C_0$. This leads to our substituting the norm of f^* on N^* for that of f on N, and that of T_1 on $C(N^*)$ for that of T on $C^*(N)$.

1.2. Lemma. *m_T is the weak-* closed convex hull (in the dual $C^*(N)'$ of $C^*(N)$) of the collection of all functionals of the form*

$$\phi = \lim \{ \varepsilon_{p(a)} \circ T_{n(a)} : a \in A \},$$

where A is a directed set, and

$$\lim \{ p(a) : a \in A \} = \lim \{ n(a) : a \in A \} = \infty.$$

Proof. On page 471 of [4] it is noted that $\{ T_n : n \in N \}$ is a " net of averages converging to T-invariance ", where the natural order is taken for N. Hence the result is just a restatement of Theorem 2.1 of [4], except for the assumption that $\lim \{ p(a) : a \in A \} = \infty$. But this is an easy consequence of our assumption that $m_T(f) = 0$ for all $f \in C_0$. (Using this fact it is easy to check that in the first and second lemmas in the proof of Theorem 2.1 of [4], the expression " $\lim_a \sup_{p \in X} (p', S_a f)$ " may be replaced by " $\lim_a \limsup_{p \to \infty} (p', S_a f)$ ".)

1.3. LEMMA. (a) $C_T \subset V_T$,
(b) $V_T = Z_T \oplus \{ke : k \, scalar\}$,
(c) $C_T = K(T) \oplus \{ke : k \, scalar\}$.

Proof. (a) if $\lim_{n\to\infty} (Tf)(n) = k$, then by regularity of T, $\lim_{n\to\infty} (T(f-ke))(n) = 0$, whence for each $\phi \in m_T$, $\phi(f-ke) = \phi \circ T(f-ke)$, or $\phi(f) = \phi(ke) = k$. (b) This follows since $\phi(e) = 1$ for all $\phi \in m_T$. (c) This follows by regularity of T.

1.4. LEMMA (cf. Theorem 3.2 of [4]). *If $f \in C^*(N)$, the following are equivalent*:

(a) $f \in Z_T$,
(b) $\lim_{n\to\infty} (T_n)_1 f^* = 0$, *where $(T_n)_1$ is the operator on $C(N^*)$ induced by T_n*,
(c) *f^* belongs to the norm closed linear hull of $\{T_1 g^* - g^* : g \in C^*(N)\}$.*

Proof. (a) *implies* (b). If (b) fails, then there exists $\varepsilon > 0$ and $n(1) < n(2) < \dots$ such that $(T_{n(k)})_1 f^* > \varepsilon$.

Hence there exist $p(1) < p(2) < \dots$ such that $|\varepsilon_{p(k)} \circ T_{n(k)}(f)| = |T_{n(k)} f(p(k))| > \varepsilon$. Let $\phi \in C^*(N)'$ be a weak-* cluster point of the set $\{\varepsilon_{p(k)} \circ T_{n(k)} : k = 1, 2, \dots\}$. By 1.2, $\phi \in m_T$, and clearly $|\phi(f)| \geqq \varepsilon$; so $f \notin Z_T$.

(b) *implies* (c). Suppose that $\lim_{n\to\infty} (T_n)_1 f^* = 0$. Then $f^* = \lim_{n\to\infty} (f^* - (T_n)_1 f^*)$ uniformly, and it suffices to show that for all n,

$$f^* - (T_n)_1 f^* \in \text{linear hull of } \{T_1 g^* - g^* : g \in C^*(N)\}.$$

But, as in Lemma 3.1 of [4],

$$f^* - (T_n)_1 f^* = f^* - (1/n) \sum_{k=1}^{n} T_1^k f^* = (1/n) \sum_{k=1}^{n} (f^* - T_1^k f^*),$$

and each term may be written

$$\begin{aligned} f^* - T_1^k f^* &= \sum_{j=0}^{k-1} T_1^j f^* - T_1^{j+1} f^* \\ &= \sum_{j=0}^{k-1} (g_j^* - T_1 g_j^*), \end{aligned}$$

where $g_j^* = T_1^j f^*$.

(c) *implies* (a). If $\phi \in m_T$, then $\phi(Tg - g) = 0$ for all $g \in C^*(N)$. If (c) holds, then given $\varepsilon > 0$, there exists $g \in C^*(N)$ with $\| f^* - (T_1 g^* - g^*) \| < \varepsilon$.

Hence there exists n_0 such that $n \geqq n_0$ implies that $|f(n) - (Tg - g)(n)| < \varepsilon e(n)$, where e is the unit function; whence, by positivity of each $\phi \in m_T$, we have

$$\begin{aligned} |\phi(f)| &= |\phi(f) - \phi(Tg - g)| \\ &= |\phi((f) - (Tg - g))| \\ &\leqq \varepsilon \phi(e) = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\phi(f) = 0$ for all $\phi \in m_T$; so $f \in Z_T$.

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2. The conditions (I) and (II).

2.1. THEOREM (cf. Theorems 23 and 24 of [8]). *The following are equivalent*:

(a) $V_T \subset C_S$ *consistently*,
(b) $\| S_1 \circ T_1 - S_1 \| = 0$, *i.e.* $S_1 \circ T_1 = S_1$ *on* $C(N^*)$.

Proof. (b) *implies* (a). By (b) and (c) of Lemma 1.3, it is enough to show that $Z_T \subset K(S)$. If $k = Tf - f$ for some $f \in C^*(N)$, then (b) implies that $Sk \in C_0$; whence $k \in K(S)$. By Lemma 1.4, if $g \in V_T$ and $\varepsilon > 0$, then there exists a function h which is a finite linear combination of functions of the form $Tf - f$, and such that $\| g^* - h^* \| < \varepsilon$. Hence there exists n_0 such that $n \geqq n_0$ implies that $|g(n) - h(n)| < \varepsilon$. Since $h \in K(S)$,

$$\overline{\lim_{n \to \infty}} |Sg(n)| \leqq \overline{\lim_{n \to \infty}} |S(g-h)(n)| + \overline{\lim_{n \to \infty}} |Sh(n)| \leqq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $g \in K(S)$.

(a) *implies* (b). If $V_T \subset C_S$ *consistently*, then $Z_T \subset K(S)$. Hence if $f \in C^*(N)$, then $Tf - f \in Z_T \subset K(S)$; so $(S \circ T - S)(f) = S(Tf - f) \in C_0$. It follows that the matrices $S \circ T$ and S are equivalent; whence $\| S_1 \circ T_1 - S_1 \| = 0$.

2.2. THEOREM (cf. Theorem 3.3 of [4]). *The following are equivalent*:

(a) $V_T \subset V_S$ *consistently*,

(b) $\lim_{n \to \infty} \| (S_n)_1 \circ T_1 - (S_n)_1 \| = 0$.

Proof. (b) *implies* (a). From 1.3(b) it follows that $V_T \subset V_S$ consistently iff $Z_T \subset Z_S$. Now condition (b) implies that $\lim_{n \to \infty} ((S_n)_1 \circ T_1 - (S_n)_1)(f^*) = 0$ for each $f \in C^*(N)$. If $g \in V_T$ and $g = Th - h$ for some h, then we have immediately that $\lim_{n \to \infty} \| (S_n)_1 g^* \| = 0$; so $g \in V_S$, by 1.4. In general, if $f \in V_T$ and $\varepsilon > 0$, then by 1.4 there is a $g \in V_T$ which is a finite linear combination of elements of the form $Th - h$, and such that $\| f^* - g^* \| < \varepsilon$. Clearly, $\lim_{n \to \infty} \| (S_n)_1 g^* \| = 0$, and so there exists n_0 such that $n \geqq n_0$ implies (since $\| (S_n)_1 \| = 1$ for all n) that

$$\| (S_n)_1 f^* \| \leqq \| (S_n)_1 (f^* - g^*) \| + \| (S_n)_1 g^* \| < 2\varepsilon.$$

Hence $\lim_{n \to \infty} \| (S_n)_1 f^* \| = 0$; so $f \in V_S$, by 1.4.

(a) *implies* (b). If $V_T \subset V_S$ *consistently*, then $Z_T \subset Z_S$, or, equivalently, $m_S \subset m_T$. We show that this implies that

$$\lim_{n \to \infty} \| ((S_n)_1 \circ T_1 - (S_n)_1) f^* \| = 0$$

for each $f \in C^*(N)$. The desired conclusion then follows from Lemma 2.3 below.

We suppose, to obtain a contradiction, that there exist $f \in C^*(N)$, $\varepsilon > 0$, and $n(1) < n(2) < \ldots$ such that

$$\| ((S_{n(k)})_1 \circ T_1 - (S_{n(k)})_1) f^* \| > \varepsilon \qquad (k = 1, 2, \ldots).$$

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Then there exist $p(1) < p(2) < \ldots$, with $|S_{n(k)}(Tf-f)(p(k))| > \varepsilon$, $k = 1, 2, \ldots$. By Lemma 1.2, if ϕ is a weak-* cluster point in $C^*(N)'$ of the functionals $\{\varepsilon_{p(k)} \circ S_{n(k)} : k = 1, 2, \ldots\}$, then $\phi \in m_S$, and clearly $|\phi(Tf-f)| \geqq \varepsilon$. This implies that $\phi \notin m_T$, contradicting our assumption that $m_S \subset m_T$.

2.3. LEMMA. *Let $A_1, A_2, \ldots$ be operators on $C(N^*)$ induced by matrix operators. Then $\lim_{n\to\infty} \|A_n\| = 0$ iff $\lim_{n\to\infty} \|A_n f\| = 0$ for each $f \in C(N^*)$.*

Proof. Sufficiency is obvious. For necessity, suppose that $\|A_n\| \nrightarrow 0$ as $n \to \infty$. Taking subsequences if need be, we may assume that, for some $\varepsilon > 0$, $\|A_n\| \geqq \varepsilon$ for all n. Let T^n be a matrix operator inducing A_n, which we assume to be in truncated form, and let T^n_m be the mth row of T^n, considered as a linear functional (with finite support, because of truncation) on $C^*(N)$.

Now for each n there exists $f_n \in C(N^*)$ with $\|f_n\| \leqq 1$ and $\sup\{|A_n f_n(w)| : w \in N^*\} > \varepsilon$. Let g_n be an extension of f_n to all βN such that $\|g_n\| \leqq 1$. Now choose $n(1)$ such that $|T^1_{n(1)} g_1| > \varepsilon$, with, say, support $(T^1_{n(1)}) \subset [0, N(1))$; choose $n(2)$ with $|T^2_{n(2)} g_2| > \varepsilon$, and support $(T^2_{n(2)}) \subset [N(1), N(2))$; choose $n(3)$ with $|T^1_{n(3)} g_1| > \varepsilon$, and support $(T^1_{n(3)}) \subset [N(2), N(3))$; choose $n(4)$ with $|T^2_{n(4)} g_2| > \varepsilon$, and support $(T^2_{n(4)}) \subset [N(3), N(4))$; choose $n(5)$ with $|T^3_{n(5)} g_3| > \varepsilon$, and support $(T^3_{n(5)}) \subset [N(4), N(5))$; and so on in the pattern 1–2, 1–2–3, 1–2–3–4, etc.

Define $g \in C^*(N)$ so that g agrees with g_1 on $[0, N(1))$, with g_2 on $[N(1), N(2))$, with g_1 on $[N(2), N(3))$, with g_2 on $[N(3), N(4))$, with g_3 on $[N(4), N(5))$, and so on. Then clearly $\limsup_{m\to\infty} |T^n_m g| \geqq \varepsilon$ for each n; hence, if f denotes the extension of g to $\beta N \backslash N$, then $\|A_n f\| \geqq \varepsilon$ for each n, so that $\lim_{n\to\infty} \|A_n f\| = 0$ fails. This completes the proof.

3. Sufficiency for (III).

3.1. NOTATION. As a linear function, $\varepsilon_n \circ T$ is the same as the nth row of T. If $T = (t_{mn})$ and $S = (s_{mn})$ are non-negative regular (and assumed to be in truncated form), we write

$$d(\varepsilon_m \circ S, \varepsilon_n \circ T) = \sum_k |s_{mk} - t_{nk}|.$$

We write $S_{n,p} = (1/n) \sum_{k=1}^{n} \varepsilon_p \circ S^k$, where S^k is the kth iterate of S. If $L \subset C^*(N)'$, then L_C is the weak-* closed convex hull of L in $C^*(N)'$.

REMARK. In this section we deal with the inclusion $C_T \subset V_S$ *consistently*. The following lemma gives some convenient restatements of this inclusion.

3.2. LEMMA.

(a) *$C_T \subset V_S$ consistently iff $K(T) \subset Z_S$.*
(b) *$K(T) \subset Z_S$ iff $K(T)^* \subset (Z_S)^*$ (where $K(T)^* = \{f^* : f \in K(T)\}$, etc.),*
(c) *$K(T)^* \subset (Z_S)^*$ iff m_S is contained in the closed linear hull in $C^*(N)'$ of $\{\varepsilon_p \circ T_1 : p \in N^*\}$.*

Proof. (a) Immediate from (b) and (c) of 1.3. (b) Immediate from the definitions.

(c) If $\phi \in m_T$, and ϕ' is the Borel measure on βN representing ϕ, then since $\phi(f) = 0$ for all $f \in c_0$, ϕ' is supported by N^*. Since $K(T)^* = \{f^* : T_1 f^* = 0\}$, where T_1 is the operator on $C(N^*)$ induced by T, we have $f^* \in K(T)^*$ iff $\varepsilon_p \circ T_1(f^*) = 0$ for all $p \in N^*$; and $f^* \in (Z_S)^*$ iff $\phi'(f^*) = 0$ for all $\phi \in m_T$. By [3, p. 20, (8)] the following are equivalent: (i) $\varepsilon_p \circ T_1(f^*) = 0$ for all $p \in N^*$ implies that $\phi'(f^*) = 0$ for all $\phi \in m_T$; (ii) $\{\phi' : \phi \in m_S\}$ is contained in the weak-* closed linear hull of $\{\varepsilon_p \circ T_1 : p \in N^*\}$. This completes the proof of the lemma.

It is part (c) of the lemma that suggests the form of Theorems 3.4 and 4.3 below. Our problem is to translate (c) into a condition involving the rows of the matrices S and T.

3.3. Lemma.

$$\{\varepsilon_p \circ T_1 : p \in N^*\}_c = \bigcap_{n=1}^{\infty} W_n,$$

where $W_n = \{\varepsilon_m \circ T : m = n, n+1, \dots\}_c$.

Proof. According to Theorem 2 in Jerison's paper [6], if K_n is a sequence of compact convex sets in a locally convex space E, with $K_{n+1} \subset K_n$, and A_n is the set of extreme points of K_n, then

$$\bigcap_n K_n = \text{closed convex hull of} \bigcap_n A_n.$$

Let $B_n = \{\varepsilon_m \circ T : m \geqq n\} \cup \{\varepsilon_p \circ T_1 : p \in N^*\}$. It follows easily from a theorem of Milman (Theorem 1 of [6]) that B_n is the set of extreme points of W_n. (Milman's theorem says that, if C is compact and convex and $S \subset C$, then the closure of S contains all the extreme points of C iff $\sup\{f(x) : x \in C\} = \sup\{f(x) : x \in S\}$ for each continuous linear functional f.) Jerison's theorem yields $\bigcap_n W_n = (\bigcap_n B_n)_c$. But it is easy to see that $\bigcap_n B_n = \{\varepsilon_p \circ T_1 : p \in N^*\}$.

Definition. R_m is the norm closed convex hull of $\{\varepsilon_n \circ T : n = m, m+1, \dots\}$ (cf. Remark 4.2).

3.4. Theorem. *If for each* m, $\lim_{n,p\to\infty} d(S_{n,p}, R_m) = 0$, *then* $C_T \subset V_S$ *consistently.*

Proof. Since $R_m \subset W_m$, the condition of the theorem implies that $\lim_{n,p\to\infty} d(S_{n,p}, W_m) = 0$ for each m. We shall show that this implies that $C_T \subset V_S$.

As noted in 3.2, we must show that $m_S \subset \{\varepsilon_p \circ T : pN^*\}_c$ or, by Lemma 3.3, that $m_S \subset \bigcap_m W_m$. Since m_S is the weak-* closed convex hull of functionals of the form

$$\phi = \lim\{\varepsilon_{p(a)} \circ S_{n(a)} : a \in A\} = \lim\{S_{n(a),p(a)} : a \in A\},$$

with $\lim\{p(a) : a \in A\} = \lim\{n(a) : a \in A\} = \infty$, it suffices to show that each such ϕ belongs to $\bigcap_m W_m$. By hypothesis, and since $n(a) \to \infty, p(a) \to \infty$, we have $\lim(d(S_{n(a),p(a)}, W_m) : a \in A) = 0$ for each m. Fix m and choose $\phi_a \in W_m$ such that $\lim\{\| S_{n(a),p(a)} - \phi_a \| : a \in A\} = 0$. Since W_m

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is weak-* compact, there exist $\phi_1 \in W_m$ and a subnet $\{\phi_b : b \in B\}$ such that $\phi_1 = \lim\{\phi_b : b \in B\}$. Now if $f \in C^*(N)$, then

$$\begin{aligned} |\phi(f) - \phi_1(f)| &= \lim\{|S_{n(b),p(b)}f - \phi_b f| : b \in B\} \\ &\leqq \lim\{\|S_{n(b),p(b)} - \phi_b\| : b \in B\}\,\|f\| = 0. \end{aligned}$$

Hence $\phi = \phi_1 \in W_m$ for arbitrary m.

4. Necessity for (III). In this section we prove, under a restriction on T, a converse to 3.4. The author does not know to what degree the restriction can be weakened. First we need a technical lemma.

4.1. LEMMA. *For each* m, $\lim_{n,p\to\infty} d(S_{n,p}, R_m) = 0$ *iff* $\lim_{n,p\to\infty} d(S_{n,p}, W_m) = 0$.

Proof. Sufficiency has already been noted. For necessity, assume that $\lim_{n,p\to\infty} d(S_{n,p}, W_m) = 0$. We show that for fixed m, n and p, if $d(S_{n,p}, W_m) < \alpha < \frac{1}{2}$, then $d(S_{n,p}, R_m) < 4\alpha$. The desired result follows easily from this.

Write $\psi = S_{n,p}$. Since our matrices are assumed truncated,

$$\text{support}\,(\psi) \subset [0, N] \text{ for some } N.$$

Now there is a $\phi \in W_m$ with $d(\phi, \psi) < \alpha$. For some net, $\phi = \text{weak-* } \lim(a \in A)\phi_a$, where $\phi_a = \sum_i t_{a,i} T_{p(a,i)}$, and the combination is convex. Write

$$\phi_a = \Sigma' t_{a,i} T_{p(a,i)} + \Sigma'' t_{a,i} T_{p(a,i)} = \phi_{1,a} + \phi_{2,a}$$

where Σ'' is summation over those terms such that the support of $T_{p(a,i)}$ is disjoint from $[0, N]$, and Σ' is summation over the rest of the terms. Let g be the characteristic function of $[N+1, \infty)$. Then $\psi(g) = 0$, so that

$$|\phi(g)| = |\phi(g) - \psi(g)| \leqq \|\phi - \psi\|\,\|g\| < \alpha.$$

Hence, for some $a_0 \in A$, $a \geqq a_0$ implies (since $T \geqq 0$) that

$$k(a) = \Sigma'' t_{a,i} = |\phi_{2,a}(g)| \leqq |\phi_a(g)| < \alpha.$$

Let $\eta_a = \Sigma' t_{a,i}(1 - k(a))^{-1} T_{p(a,i)}$, a convex combination of rows of T, each of whose supports meet $[0, N]$. Now it follows from the truncated form of T that, since the support of each $T_{p(a,i)}$ involved in the sum meets $[0, N]$, there exists fixed $M \geqq N$ such that

$$\text{support } T_{p(a,i)} \subset [0, M].$$

Hence

$$\text{support}\,(\eta_a) \subset [0, M] \quad \text{for} \quad a \geqq a_0.$$

Let η be a weak-* cluster point of the η_a, and η_b a subnet of η_a which converges to η. Then

$$\text{support}\,(\eta) \subset [0, M].$$

For each b,

$$\begin{aligned}\|\eta_b-\phi_b\| &= \|(1-k(b))^{-1}\phi_{1,b}-\phi_{1,b}-\phi_{2,b}\| \\ &\leqq ((1-k(b))^{-1}-1)\|\phi_{1,b}\|+\|\phi_{2,b}\| \\ &< k(b)(1-k(b))^{-1}+k(b) \\ &< 3k(b) < 3\alpha,\end{aligned}$$

since $\alpha < \frac{1}{2}$. Hence $\|\eta-\phi\| \leqq 3\alpha$, and $\|\eta-\psi\| < 4\alpha$. But

$$\text{support } \eta \subset [0, M] \quad \text{and} \quad \text{support } \eta_b \subset [0, M].$$

This implies that the convergence of η_b to η is essentially finite-dimensional. But in this case weak-* convergence is equivalent to norm convergence [3, p. 39]; so $\eta \in R_m$, and $d(\psi, R_m) \leqq d(\psi, \eta) < 4\alpha$, where $\psi = S_{n,p}$.

4.2. Remark. It should be noted that each element of R_m is a functional on $C(\beta N)$ whose support is contained in N, hence may be represented by a vector in l^1. This is by no means the case for W_m, and is the reason why R_m is preferable to W_m.

4.3. Theorem. *Assume that the induced operator T_1 maps $\{f\in C(N^*): f \geqq 0\}$ onto itself. If $C_T \subset V_S$ consistently, then* $\lim_{n,p\to\infty} d(S_{n,p}, R_m) = 0$ *for all m.*

Proof. First we show that, under our hypotheses on T, if $\phi \in C(N^*)'$ (the dual space of $C(N^*)$), $\phi(e) = 1$, $\phi \geqq 0$ and $\phi \in$weak-* closed linear hull of $\{\varepsilon_w \circ T_1 : w \in N^*\}$, then $\phi \in \{\varepsilon_w \circ T_1 : w \in N^*\}_c$. But our hypotheses imply readily that the range of T_1 is all of $C(N^*)$; hence the range is closed, and it follows from [5, Theorem 2, p. 487] that the range of the adjoint map T_1' consists of those $\psi \in C(N^*)'$ for which $T_1 f = 0$ implies that $\psi(f) = 0$. Since our ϕ satisfies this latter condition, we have $\phi = \psi \circ T_1$ for some $\psi \in C(N^*)'$. Now since $T_1(e) = e$, we have $\psi(e) = \psi(T_1 e) = \phi(e) = 1$. If $f \geqq 0$, then $f = T_1 g$ for some $g \geqq 0$, so that $\psi(f) = \psi(T_1 g) = \phi(g) \geqq 0$. It follows that ψ is the weak-* limit of functionals of the form $\sum t_i \varepsilon_{w(i)}$, where $w(I) \in N^*$ and the combination is convex. Since the adjoint T_1' is weak-* continuous, it follows that ϕ is the weak-* limit of functionals of the form $\sum t_i \varepsilon_{w(i)} \circ T_1$, so that $\phi \in \{\varepsilon_w \circ T_1 : w \in N^*\}_c$.

Suppose now that for some m, $d(S_{n,p}, R_m)$ does not go to 0. Then, by Lemma 4.1, neither does $d(S_{n,p}, W_m)$, so there exist $\varepsilon > 0$, $n(1) < n(2) < \dots$ and $p(1) < p(2) < \dots$ with $d(S_{n(k),p(k)}, W_m) > \varepsilon$ for all k. Let ϕ be a weak-* cluster point of the functionals $\{S_{n(k),p(k)}\}$. Then $\phi \in m_S$ by Lemma 1.2. Clearly, $d(\phi, W_m) \geqq \varepsilon$, and since, by Lemma 3.3, $\{\varepsilon_w \circ T: w \in N^*\}_c \subset W_m$, we have $\phi \notin \{\varepsilon_w \circ T: w \in N^*\}_c$. From the first part of the proof it follows that ϕ does not belong to the weak-* closed *linear* hull of $\{\varepsilon_w \circ T: w \in N^*\}$; so by Lemma 3.2 the inclusion $C_T \subset V_S$ *consistently* fails. This completes the proof.

5. Examples.

5.1. For an example in which inclusion (I) fails but inclusion (II) holds, let S be the shift matrix and let $T = S^2$. Clearly, $V_T \subset V_S$ *consistently*, while the criterion of Theorem 2.1 shows that $V_T \subset C_S$ *consistently* fails.

5.2. For an example in which (II) fails while (III) holds, let S and R be any pair of matrices such that $C_R \subset V_S$ *consistently*. From R we form a new matrix T by letting the odd rows of T be all the rows of R, and choosing the even rows so as to cause the inclusion $V_T \subset V_S$ to fail. Then $C_T \subset C_R \subset V_S$, so that (III) continues to hold.

5.3. The extra hypothesis on T in Theorem 4.3 is actually fulfilled by a reasonably large class of matrices. For instance, let T be a non-negative regular matrix whose rows have disjoint support, i.e. if $m \neq q$, then $t_{mk} \neq 0$ implies that $t_{qk} = 0$. Let the support of the mth row be contained in the interval $[k(m), k(m+1))$, where, for distinct m, the intervals are disjoint. If $\{r_n\}$ is any bounded sequence of non-negative constants, define $f \in C^*(N)$ by the formula $f(k) = r_m$ whenever $k \in [k(m), k(m+1))$, and let f be 0 elsewhere. Then, assuming each row sum of T is 1, we have $Tf(m) = r_m$ for all m. Thus T maps the space of non-negative elements of $C^*(N)$ onto itself, and it is easy to see that T_1 does the same for $C(N^*)$.

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