ON COUNTABLY PARACOMPACT NORMAL SPACES

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1. Introduction. A. H. Stone **(9)**, E. Michael **(3, 4)**, J. L. Kelley and J. S. Griffen **(2)** have established many necessary and sufficient conditions that a regular Hausdorff space be paracompact. It is the purpose of this note to show that if the word "countable" is inserted in the appropriate places in the above-mentioned conditions they become, in general, necessary and sufficient conditions that a normal space be countably paracompact. Making use of one of these conditions, countably paracompact normal spaces are identified as a natural generalization of fully normal spaces.

The following terminology will be used throughout this note. Let X be a topological space. A collection $\mathfrak{R}$ of subsets of X is said to be a *covering* of X if $\bigcup\{R : R \in \mathfrak{R}\} = X$. If $\mathfrak{R}$ and $\mathfrak{S}$ are coverings of X, then $\mathfrak{S}$ is a *refinement* of $\mathfrak{R}$ if each member of $\mathfrak{S}$ is a subset of some member of $\mathfrak{R}$. A collection $\mathfrak{R}$ of subsets of X is *closure-preserving* if for each $\mathfrak{S} \subset \mathfrak{R}$ we have

$$\bigcup\{S^- : S \in \mathfrak{S}\} = (\bigcup\{S : S \in \mathfrak{S}\})^-.$$

The collection $\mathfrak{R}$ is *locally finite* if each $x \in X$ has a neighborhood meeting only a finite number of members of $\mathfrak{R}$; *star-finite* if each member of $\mathfrak{R}$ meets only a finite number of members of $\mathfrak{R}$; *discrete* if each $x \in X$ has a neighborhood meeting at most one member of $\mathfrak{R}$. $\mathfrak{R}$ is *σ-closure-preserving* (resp. *σ-locally finite*, resp. *σ-discrete*) if $\mathfrak{R}$ is the union of countably many closure-preserving (resp. locally finite, resp. discrete) subcollections.

An open covering $\mathfrak{R}$ of a topological space X is said to be *even* if there is a neighborhood U of the diagonal[1] in the cartesian product space $X \times X$ such that for each $x \in X$ the set

$$U[x] = \{y \in X : (x, y) \in U\}$$

is a subset of some member of $\mathfrak{R}$. If $\mathfrak{R}$ and $\mathfrak{S}$ are open coverings of X, then $\mathfrak{S}$ is said to be a *star refinement* of $\mathfrak{R}$ if for each $x \in X$ the set

$$\mathrm{St}(x, \mathfrak{S}) = \bigcup\{S \in \mathfrak{S} : x \in S\}$$

is a subset of some member of $\mathfrak{R}$.[2]

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[1]The *diagonal* in $X \times X$ is the set $\{(x, x) : x \in X\}$.

[2]Our usage of "star refinement," which follows **(2**, Exercise 5.U**)**, differs from that of Tukey **(10)**. Tukey refers to our "star refinement" as a "Δ-refinement," reserving the words "star refinement" to describe a slightly different concept.

Finally, a topological space X is said to be *countably paracompact* if each countable open covering of X admits a (countable) locally finite open refinement. A topological space X is *fully normal* if each open covering of X admits an open star refinement.

Throughout the sequel the letter i, j, or k will always stand for a variable which runs over all the natural numbers.

2. The Theorem. We shall frequently need the following two well-known theorems. Theorem 1 is due to Dowker (**1**, Theorem 2); Theorem 2 is due to Morita (**6**, Theorem 3). The reader is referred to the original papers for the proofs.

THEOREM 1 (Dowker). *The following properties of a normal*[3] *space X are equivalent*:

(a) *The space X is countably paracompact.*

(b) *Every countable open covering $\{U_i\}$ of X admits an open refinement $\{V_i\}$ with $\overline{V_i} \subset U_i$.*

(c) *Given a decreasing sequence $\{F_i\}$ of closed sets with a vacuous intersection, there is a sequence $\{G_i\}$ of open sets with vacuous intersection such that $F_i \subset G_i$.*

THEOREM 2 (Morita). *If $\mathfrak{G} = \{G_i\}$ is a countable open covering of a normal space X, and if there exists a closed covering $\mathfrak{F} = \{F_i\}$ of X such that $F_i \subset G_i$, then $\mathfrak{G}$ admits a countable, star-finite, open, star-refinement.*

In his statement of Theorem 2, Morita has the additional hypothesis that X be a T_1-space; this hypothesis, however, is not used in the proof.

Remark. It follows at once from Theorem 2 and condition (b) of Theorem 1 that a normal space X is countably paracompact if and only if each countable open covering of X admits a (countable) star-finite open refinement.

We are now ready to state and prove the main theorem of this note. It should be remarked that some of the implications involved in the theorem are well known.

THEOREM 3. *If X is a normal space, then the following statements are equivalent.*

(a) *The space X is countably paracompact.*
(b) *Each countable open covering of X admits a countable, locally finite, closed refinement.*
(c) *Each countable open covering of X admits a countable, closure-preserving, closed refinement.*
(d) *Each countable open covering of X admits a σ-discrete closed refinement.*
(e) *Each countable open covering of X admits a σ-locally finite closed refinement.*

[3] In our terminology a normal space need not be a Hausdorff space.

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(f) *Each countable open covering of X admits a σ-closure-preserving closed refinement.*
(g) *Each countable open covering of X admits a countable, open, star refinement.*
(h) *Each countable open covering of X is even.*
(i) *For each countable locally finite collection* $\{A_i\}$ *of subsets of X there is a countable, locally finite collection* $\{G_i\}$ *of open subsets of X such that* $A_i \subset G_i$.

Proof. The pattern of proof is (a) → (b) → (c) → (d) → (e) → (f) → (g) → (h) → (a) → (i) → (a).

(a) *implies* (b): Let $\mathfrak{R} = \{R_i\}$ be a countable open covering of X. We shall show that $\mathfrak{R}$ admits a countable, locally finite, closed refinement. Since X is countably paracompact, there is a locally finite open refinement $\mathfrak{S} = \{S_i\}$ of $\mathfrak{R}$ such that $S_i \subset R_i$ (**1**, proof of Theorem 1). By Theorem 1 (b) there is an open covering $\mathfrak{T} = \{T_i\}$ of X such that $\bar{T_i} \subset S_i$. Evidently $\mathfrak{F} = \{\bar{T_i}\}$ is a countable, locally finite, closed refinement of $\mathfrak{R}$.

(b) *implies* (c) *and* (e) *implies* (f): A locally finite collection of subsets is closure-preserving.

(c) *implies* (d): Any countable collection of subsets is σ-discrete.

(d) *implies* (e): A discrete collection of subsets is locally finite.

(f) *implies* (g): Let $\mathfrak{R} = \{R_i\}$ be a countable open covering of X. We shall show that $\mathfrak{R}$ admits a countable, open, star refinement. By (f), $\mathfrak{R}$ admits a σ-closure-preserving closed refinement $\mathfrak{S} = \bigcup \mathfrak{S}_i$. For each pair of positive integers i, j let

$$T_{i,j} = \bigcup\{S \in \mathfrak{S}_j : S \subset R_i\}.$$

Since each $S \in \mathfrak{S}$ is closed and $\mathfrak{S}_j$ is closure-preserving for each j, we conclude that $T_{i,j}$ is closed.

For each i and any j, $T_{i,j} \subset R_i$. Moreover if $x \in X$, then, since $\mathfrak{S}$ is a covering, there is an integer j and a set $S \in \mathfrak{S}_j$ such that $x \in S$. Since $\mathfrak{S}$ is a refinement of $\mathfrak{R}$, there is an integer i such that $S \subset R_i$. Therefore, $x \in T_{i,j}$. Consequently $\mathfrak{T} = \{T_{i,j}\}$ is a closed refinement of $\mathfrak{R}$.

The family $\mathfrak{T}$ is countable and therefore can be indexed by the natural numbers, say $\mathfrak{T} = \{T_k\}$. Since $\mathfrak{T}$ is a refinement of $\mathfrak{R}$, for each positive integer k there is an integer i_k such that

$$T_k \subset R_{i_k}.$$

Let

$$U_k = R_{i_k}$$

and $\mathfrak{U} = \{U_k\}$. Then $\mathfrak{U}$ is a countable open covering of the normal space X, and $\mathfrak{T}$ is a countable closed covering of X such that $T_k \subset U_k$. Therefore, by Theorem 2, $\mathfrak{U}$ admits a countable, open, star refinement $\mathfrak{B}$. Obviously $\mathfrak{B}$ is also a star refinement of $\mathfrak{R}$.

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(g) *implies* (h): We shall show that if an open covering $\mathfrak{R}$ of a topological space X admits an open star refinement, then $\mathfrak{R}$ is an even covering (**2**, Exercise 5.U). Let $\mathfrak{S}$ be an open star refinement of $\mathfrak{R}$, and put $V = \bigcup\{S \times S : S \in \mathfrak{S}\}$. Clearly V is a neighborhood of the diagonal in $X \times X$.

If $x \in X$, choose $R_x \in \mathfrak{R}$ such that $St(x, \mathfrak{S}) \subset R_x$. If $y \in V[x]$ then, by definition of V, there is an $S \in \mathfrak{S}$ such that $x \in S$ and $y \in S$. Therefore $y \in St(x, \mathfrak{S}) \subset R_x$. Hence $V[x] \subset R_x$. Consequently $\mathfrak{R}$ is an even covering of X.

(h) *implies* (a): Let $\mathfrak{R} = \{R_i\}$ be a countable open covering of X. We shall show that $\mathfrak{R}$ admits a star-finite open refinement. It will then follow from the *Remark* above that X is countably paracompact.

By hypothesis, there is a neighborhood U of the diagonal in $X \times X$ such that $\{U[x] : x \in X\}$ is a refinement of $\mathfrak{R}$. Let

$$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\},$$

and let $V = U \cap U^{-1}$. Evidently V is a *symmetric* neighborhood of the diagonal in $X \times X$; i.e., V is a neighborhood of the diagonal with the property that $(x, y) \in V$ if and only if $(y, x) \in V$.

For each positive integer i let $S_i = \{x \in X : V[x] \subset R_i\}$. Evidently $\{S_i\}$ is a refinement of $\mathfrak{R}$ such that $S_i \subset R_i$. If $x \in S_i^-$, then, since $V[x]$ is a neighborhood of x, $V[x] \cap S_i \neq \phi$, say $y \in V[x] \cap S_i$. Since $y \in S_i$, $V[y] \subset R_i$; but since $y \in V[x]$ and V is symmetric, we conclude that $x \in V[y] \subset R_i$. Hence $S_i^- \subset R_i$. By Theorem 2, $\mathfrak{R}$ admits a star-finite open refinement. Hence X is countably paracompact.

(a) *implies* (i): Let $\{A_i\}$ be a locally finite collection of subsets of X. We must find a locally finite collection $\{G_i\}$ of open sets such that $A_i \subset G_i$.

Let $F_k = \bigcup\{A_p^- : p \geqslant k\}$. Since $\{A_i\}$ is locally finite, F_k is closed for each k. Moreover $F_k \supset F_{k+1}$ and $A_k^- \subset F_k$.

Now if $x \in X$, then x belongs to only a finite number of the A_i^-, say

$$A_{i_1}^-, \ldots, A_{i_{n_x}}^-.$$

Set

$$m_x = 1 + \max\{i_1, \ldots, i_{n_x}\}.$$

Then for $k \geqslant m_x$, $x \notin A_k^-$. Hence

$$x \notin F_{m_x}.$$

Therefore $\bigcap F_k = \phi$.

Since X is countably paracompact and normal, we may apply Theorem 1 (c): there is a sequence $\{H_k\}$ of open sets such that $F_k \subset H_k$ for each k and $\bigcap H_k = \phi$.

Now $F_1 \subset H_1$. Since X is normal, there is an open set G_1 such that $F_1 \subset G_1 \subset G_1^- \subset H_1$. Proceeding inductively, suppose that for each positive integer $p \leqslant n$ open sets G_p have been defined such that $G_1 \supset G_2 \supset \ldots \supset G_n$, and such that

$$F_p \subset G_p \subset G_p^- \subset H_p \qquad (p = 1, 2, \ldots, n).$$

Now $F_{n+1} \subset F_n \subset G_n$ and $F_{n+1} \subset H_{n+1}$. Therefore $F_{n+1} \subset G_n \cap H_{n+1}$. By the normality of X, there is an open set G_{n+1} such that

$$F_{n+1} \subset G_{n+1} \subset G_{n+1}^{-} \subset G_n \cap H_{n+1}.$$

Consequently $F_{n+1} \subset G_{n+1} \subset G_{n+1}^{-} \subset H_{n+1}$ and $G_{n+1} \subset G_n$.

We shall now show that $\{G_k\}$ is locally finite. If $x \in X$, then, since $\bigcap H_k = \phi$, there is an n_x such that

$$x \notin H_{n_x}.$$

Hence, since

$$H_{n_x} \supset G_{n_x}^{-}, \quad x \notin G_{n_x}^{-}.$$

Therefore there is a neighborhood V_x of x such that

$$V_x \cap G_{n_x} = \phi.$$

Consequently, if $k \geqslant n_x$, then $V_x \cap G_k = \phi$, because

$$G_k \subset G_{n_x}.$$

Therefore V_x meets at most

$$G_1, \ldots, G_{n_x - 1}.$$

Hence $\{G_k\}$ is locally finite.

Finally, for each k, $A_k \subset A_k^{-} \subset F_k \subset G_k$.

(i) *implies* (a): Let $\mathfrak{R} = \{R_i\}$ be a countable open covering of an arbitrary space X. Put $S_i = \bigcup\{R_p : p = 1, \ldots, i\}$; let $A_1 = S_1$ and $A_i = S_i - S_{i-1}$ $(i = 2, 3, \ldots)$. Evidently $A_i \subset R_i$ for each i. If $x \in X$, then $x \in R_i$ for some i. If i_x is the smallest such i, then

$$x \in A_{i_x}.$$

Hence $\mathfrak{A} = \{A_i\}$ is a refinement of $\mathfrak{R}$. Finally, observe that $R_i \cap A_j = \phi$ if $j > i$. Hence $\mathfrak{A}$ is locally finite.[4]

Now if X satisfies (i), then there is a locally finite collection $\{G_i\}$ of open sets such that $A_i \subset G_i$ for each i. Let $U_i = R_i \cap G_i$. If $x \in X$, then, for some i, $x \in A_i$. But $A_i \subset R_i$ and $A_i \subset G_i$. Hence $x \in R_i \cap G_i = U_i$. Therefore $\mathfrak{U} = \{U_i\}$ is an open refinement of $\mathfrak{R}$. Moreover, $\mathfrak{U}$ is locally finite: $N \cap U_i \neq \phi$ implies $N \cap G_i \neq \phi$. Hence $\mathfrak{U}$ is a locally finite open refinement of $\mathfrak{R}$. Therefore X is countably paracompact. This completes the proof of Theorem 3.

It is well known that if a topological space X satisfies condition (g) of Theorem 3, then X is normal (**10**, Chapter V, §§ 3.2, 4.1–4.5). Hence countably paracompact normal spaces are obtained as a generalization of fully normal spaces in the following natural way.

COROLLARY. *A topological space X is countably paracompact and normal if and only if each countable open covering of X admits a countable, open, star refinement:*

[4]This construction of $\mathfrak{A}$ is due to Michael (**3**, Lemma **2**).

3. Remarks. The word "closed" in conditions (d), (e), and (f) of Theorem 3 cannot be replaced by "open": a countable open covering of an *arbitrary* space is a σ-discrete (and hence σ-locally finite and σ-closure-preserving) refinement of itself. Thus the validity of Theorem 3 with "closed" replaced by "open" in conditions (d), (e), and (f) would imply that every normal space is countably paracompact. Dowker **(1)** has given an example of a normal (albeit *not* Hausdorff) space which is not countably paracompact.

The word "closed" in condition (c) of Theorem 3 cannot be replaced by "open": the example due to Dowker referred to above has the property that *every* collection of open subsets is closure-preserving. It may well be, however, that a normal Hausdorff space is countably paracompact if and only if each countable open covering admits a (countable) closure-preserving open refinement.

The proof that (i) implies (a) in Theorem 3 shows that a countable open covering of an arbitrary space admits a locally finite refinement. Hence the following assertion is *false*: A normal space X is countably paracompact if and only if each countable open covering of X admits a locally finite refinement.

4. Some applications. Using conditions (b) and (d) of Theorem 3, it can easily be shown that every F_σ-subset of a countably paracompact normal space is countably paracompact (and, of course, normal **(8)**). The proof is analogous to that of **(3**, Proposition 3**)**. Similarly, conditions (b) and (d) can be employed to show that if a topological space X is the union of a locally finite collection of closed, normal, countably paracompact subspaces, then X is countably paracompact and normal. (This result is not new. See **(**5, Theorem 8.2 ff.**)**; observe that a locally finite closed covering of a space X dominates X.)

It follows from Theorem 3 (g) and a result due to Shirota **(**7, p. 23**)** that the family $\mathscr{C}$ of all countable open coverings of a countably paracompact normal space X is a base of a uniformity for X in the sense of Tukey **(10)**. In the Weil-Bourbaki version of uniformity **(2**, Chapter 6**)**, this means that the family $\{V(\mathfrak{R}) : \mathfrak{R} \in \mathscr{C}\}$, where $V(\mathfrak{R}) = \bigcup\{R \times R : R \in \mathfrak{R}\}$, is a base of a uniformity for X.

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