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$\mathfrak{m}$-STONE LATTICES

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A *Stone lattice* is a distributive, pseudo-complemented lattice L such that $a^* \vee a^{**} = 1$, for all a in L; or equivalently, a bounded distributive lattice L in which, for all a in L, the annihilator $a^\perp = \{b \in L | a \wedge b = 0\}$ is a principal ideal generated by an element of the centre of L, namely a^*.

Thus it is natural to define an $\mathfrak{m}$-*Stone lattice* to be a bounded distributive lattice L in which, for each subset A of cardinality less than or equal to $\mathfrak{m}$, the annihilator $A^\perp = \{b \in L | a \wedge b = 0$, for all $a \in A\}$ is a principal ideal generated by an element of the centre of L.

In this paper we characterize $\mathfrak{m}$-Stone lattices, and show, by considering the lattice of global sections of an appropriate sheaf, that any bounded distributive lattice can be embedded in an $\mathfrak{m}$-Stone lattice, the embedding being a left adjoint to the forgetful functor.

1. The definition. Let L be a bounded distributive lattice. Then the lattice of ideals of L is denoted by $I(L)$, and the centre of L, the set of complemented elements, is denoted by $\mathscr{C}(L)$. The principal ideal generated by x is denoted by $(x]$.

For basic facts concerning Stone lattices we refer to [**1**].

THEOREM 1. *Let L be a bounded distributive lattice. Then the following are equivalent:*

(i) *for each subset A of L with $|A| \leqq \mathfrak{m}$, there is an element $a_1 \in \mathscr{C}(L)$ such that $A^\perp = (a_1]$;*

(ii) *L is a Stone lattice and $\mathscr{C}(L)$ is an $\mathfrak{m}$-complete Boolean lattice;*

(iii) *$L^{\perp\perp} = \{(a^\perp)^\perp | a \in L\}$ is an $\mathfrak{m}$-complete Boolean sublattice of $I(L)$;*

(iv) *for all a, b in L, $(a \wedge b)^\perp = a^\perp \vee b^\perp$; and L satisfies the condition*

$\mathfrak{m}(*)$: *For each subset A of L with $|A| \leqq \mathfrak{m}$ there is an element a_1 of L with $A^{\perp\perp} = (a_1)^\perp$;*

(v) *For each subset A of L with $|A| \leqq \mathfrak{m}$, $A^\perp \vee A^{\perp\perp} = L$.*

Proof. (i) $\Rightarrow$ (ii). For all a in L, $a^\perp = (a_1]$ for some a_1 in $\mathscr{C}(L)$ and hence L is a Stone lattice. Further, by Speed [**6**, p. 734, Theorem 2], $L^{\perp\perp}$ is an $\mathfrak{m}$-complete Boolean lattice. The result follows if we observe that, for any Stone lattice, $\mathscr{C}(L) \cong L^{\perp\perp}$.

(ii) $\Rightarrow$ (iii). Since $\mathscr{C}(L) \cong L^{\perp\perp}$, it only remains to show that $L^{\perp\perp}$ is a sublattice of $I(L)$, but this is true for any Stone lattice by Frink [**2**, p. 512, Theorem 3].

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(iii) $\Rightarrow$ (iv). This follows by the results of Speed and Frink quoted above.

(iv) $\Rightarrow$ (v). If A is a subset of L with $|A| \leqq \mathfrak{m}$, then $A^{\perp\perp} = (a_1)^{\perp}$ and $A^{\perp} = A^{\perp\perp\perp} = (a_1)^{\perp\perp} = (a_2)^{\perp}$, for some a_1, a_2 in L, by (iv). Hence

$$A^{\perp} \vee A^{\perp\perp} = (a_2)^{\perp} \vee (a_1)^{\perp} = (a_2 \wedge a_1)^{\perp} = ((a_2)^{\perp\perp} \cap (a_1)^{\perp\perp})^{\perp} = ((a_1)^{\perp} \cap (a_1)^{\perp\perp})^{\perp} = \{0\}^{\perp} = L,$$

as required.

(v) $\Rightarrow$ (i). Since $A^{\perp} \vee A^{\perp\perp} = L$, we have $a_1 \vee a_2 = 1$, with $a_1 \in A^{\perp}$ and $a_2 \in A^{\perp\perp}$; further $a_1 \wedge a_2 = 0$ since $a_1 \wedge a_2 \in A^{\perp} \cap A^{\perp\perp} = \{0\}$. If $x \in A^{\perp}$, then $x \wedge a_2 = 0$, and thus

$$x \vee a_1 = (x \vee a_1) \wedge (a_2 \vee a_1) = (x \wedge a_2) \vee a_1 = 0 \vee a_1 = a_1.$$

Hence $A^{\perp} = (a_1]$, with a_1 in $\mathscr{C}(L)$.

A bounded distributive lattice will be called an $\mathfrak{m}$-*Stone lattice* if the conditions of Theorem 1 hold.

Apart from viewing $\mathfrak{m}$-Stone lattices as the objects of a full subcategory of Stone lattices, we may also consider a category $\mathbf{S}(\mathfrak{m})$ whose objects are $\mathfrak{m}$-Stone lattices and whose morphisms, which we call $\mathfrak{m}$-*Stone homomorphisms*, are Stone homomorphisms which are Boolean $\mathfrak{m}$-homomorphisms when restricted to central elements. Although the definitions and results of this paper are stated for the category $\mathbf{S}(\mathfrak{m})$, very little alteration is needed to make them valid for the former category.

The category of bounded distributive lattices with zero and unit preserving lattice homomorphisms (called simply homomorphisms henceforth) is denoted by $\mathbf{D}$.

2. Sheaves of distributive lattices. Definitions in this section follow those in [**9**] (see also [**4**]).

Definition. A *sheaf of sets* is a triple (S, π, X) where

(i) S and X are topological spaces; and

(ii) $\pi: S \rightarrow X$ is a local homeomorphism of S onto X.

A sheaf of sets (S, π, X) is a *sheaf of bounded distributive lattices* if it also satisfies

(iii) for each x in X, $S_x = x\pi^{-1}$ is a bounded distributive lattice;

(iv) the functions $(a, b) \mapsto a \vee b$ and $(a, b) \mapsto a \wedge b$ from the set $\{(a, b) \in S \times S | a\pi = b\pi\}$ into S are continuous; and

(v) the functions $[0]$ and $[1]$, which assign to each x in X the zero (0_x) and unit (1_x) of S_x respectively, are continuous.

X is known as the *base space*, and the S_x are known as the *stalks*. If Y is a subset of X, then $\Gamma(Y, S)$ denotes the set of continuous functions $\sigma: Y \rightarrow S$ satisfying $y\sigma\pi = y$ for all y in Y. Elements of $\Gamma(Y, S)$ are called *sections over* Y

and elements of $\Gamma(X, S)$ are called *global sections*. It is immediate from the definition that $\Gamma(Y, S)$ is a distributive lattice for each subset Y of X.

The following lemma is well known (see [**9**, p. 25, Lemma 1]).

LEMMA 1. *Let (S, π, X) be a sheaf of sets, and let Y be a subset of X. Then for each pair σ, τ in $\Gamma(Y, S)$, the set $\{y \in Y | y\sigma = y\tau\}$ is open in Y.*

Let X be a topological space and let $\mathscr{T}$ be the category of open sets of X with inclusion maps. Then a contravariant functor $\mathscr{F}$ from $\mathscr{T}$ to the category $\mathbf{D}$ of bounded distributive lattices is called a *pre-sheaf of bounded distributive lattices*. Given a pre-sheaf $\mathscr{F}$ of bounded distributive lattices a sheaf of bounded distributive lattices is constructed as follows:

For each x in X, let

$$S_x = \lim_{\rightarrow} \mathscr{F}(U),$$

where the limit is taken over the set of open sets U which contain x. Now let S be the disjoint union of the S_x and define $\pi: S \rightarrow X$ in the obvious way.

For x in U, let $\phi_x{}^U : \mathscr{F}(U) \rightarrow S_x$ be the natural homomorphism, and for each $a \in \mathscr{F}(U)$ define a function $[a]: U \rightarrow S$ by $x[a] = a\phi_x{}^U$.

It is not difficult to see that (S, π, X) becomes a sheaf of bounded distributive lattices when S is endowed with the topology having $\{U[a] | U$ is open in X, $a \in \mathscr{F}(U)\}$ as a basis.

Now let (S, π, X) be a sheaf of bounded distributive lattices, and let Y be a subset of X. The characteristic function of Y, denoted by χ_Y, is given by

$$x\chi_Y = \begin{cases} 1_x & (x \in Y). \\ 0_x & (x \notin Y). \end{cases}$$

Clearly χ_Y is in (the centre of) $\Gamma(X, S)$ if and only if Y is clopen in X. Further if each stalk has a trivial centre then each central global section σ is equal to the characteristic function of its *support*, $supp(\sigma) = \{x \in X | x\sigma \neq 0_x\}$.

A bounded distributive lattice L is called *dense* if $a \wedge b = 0$ implies $a = 0$ or $b = 0$. Clearly if L is dense it has a trivial centre.

THEOREM 2. *Let (S, π, X) be a sheaf of bounded distributive lattices for which each stalk S_x is dense, and assume that X has a basis of clopen subsets. Then the following are equivalent*:

(i) $\Gamma = \Gamma(X, S)$ *is an $\mathfrak{m}$-Stone lattice*;

(ii) Γ *satisfies the condition* $\mathfrak{m}(*)$;

(iii) *The clopen subsets of X form an $\mathfrak{m}$-complete Boolean lattice and for each σ in Γ, $supp(\sigma)$ is clopen in X.*

Proof. (i) $\Rightarrow$ (ii). This follows by Theorem 1.

(ii) $\Rightarrow$ (iii). Let $\{U_i | i \in I\}$ be a family of clopen subsets of X with $|I| \leqq \mathfrak{m}$. Let $\sigma_i = \chi_{U_i}$, and let $A = \{\sigma_i | i \in I\}$. By (ii) there is a σ_1 in Γ with $A^{\perp\perp} = (\sigma_1)^{\perp}$. It is easily seen that $U = X \backslash supp(\sigma_1)$ is the required upper bound. Finally, $supp(\sigma)$ is always closed by Lemma 1, and it follows from (ii), with $A = \{\sigma\}$, that $supp(\sigma) = X \backslash supp(\sigma_1)$. Thus $supp(\sigma)$ is clopen.

(iii) $\Rightarrow$ (i). Let $\sigma \in \Gamma$, and let $Y = X \backslash supp(\sigma)$. It is clear that χ_Y is the central generator of $\sigma^\perp$. The centre of Γ is $\mathfrak{m}$-complete since it is isomorphic to the Boolean lattice of clopen subsets of X. The result follows.

COROLLARY. *Under the assumptions of the Theorem, if S is a Hausdorff space and the clopen subsets of X form an $\mathfrak{m}$-complete Boolean lattice, then Γ is an $\mathfrak{m}$-Stone lattice.*

Proof. $Supp(\sigma)$ is always open and since S is a Hausdorff space it is also closed.

3. The $\mathfrak{m}$-Stone extension. For any bounded distributive lattice L, $L^{\perp\perp} = \{(a^\perp)^\perp | a \in L\}$ is a disjunctive distributive lattice; further $\mathscr{A}(L) = \{(A^\perp)^\perp | A \subseteq L\}$ is the complete Boolean lattice generated by $L^{\perp\perp}$, i.e. the Dedekind-MacNeille completion of the Boolean extension of $L^{\perp\perp}$. Thus we may define $\mathscr{A}_\mathfrak{m} = \mathscr{A}_\mathfrak{m}(L)$ to be the $\mathfrak{m}$-complete Boolean sublattice of $\mathscr{A}(L)$ generated by $L^{\perp\perp}$.

A homomorphism $\beta: L \rightarrow L'$ is called an *R-homomorphism* if $a^\perp = b^\perp$ implies $(a\beta)^\perp = (b\beta)^\perp$. Thus β is an R-homomorphism if and only if the induced homomorphism $\bar{\beta}: L^{\perp\perp} \rightarrow (L')^{\perp\perp}$ given by $(a^{\perp\perp})\bar{\beta} = (a\beta)^{\perp\perp}$, is well defined. An R-homomorphism β is called an *$R_\mathfrak{m}$-homomorphism* if $\bar{\beta}$ preserves all existing $\mathfrak{m}$-ary joins and meets in $L^{\perp\perp}$, or equivalently if $\bar{\beta}$ extends to a Boolean $\mathfrak{m}$-homomorphism $\bar{\beta}: \mathscr{A}_\mathfrak{m}(L) \rightarrow \mathscr{A}_\mathfrak{m}(L')$.

Clearly a Stone homomorphism between two $\mathfrak{m}$-Stone lattices is an $\mathfrak{m}$-Stone homomorphism if and only if it is an $R_\mathfrak{m}$-homomorphism. Thus, if we denote the category of bounded distributive lattices and $R_\mathfrak{m}$-homomorphisms by $\mathbf{D}(\mathfrak{m})$ there is a natural forgetful functor $F: \mathbf{S}(\mathfrak{m}) \rightarrow \mathbf{D}(\mathfrak{m})$. The remainder of this section is devoted to constructing an adjoint to this functor.

Let L be a fixed bounded distributive lattice and let X denote the set of prime ideals of $\mathscr{A}_\mathfrak{m}(L)$. We equip X with the usual Stone topology: U is open in X if and only if $U = X_J = \{x \in X | J \nsubseteq x\}$ for some ideal J of $\mathscr{A}_\mathfrak{m}$.

For any ideal J of L the minimal congruence with J as a congruence class is defined by $(a, b) \in \theta(J)$ if and only if $a \vee j = b \vee j$ for some j in J. The corresponding quotient lattice is denoted by L/J, with elements a/J. Thus if $\alpha: L \rightarrow \mathscr{A}_\mathfrak{m}$ is defined by $a\alpha = a^{\perp\perp}$, we may define a pre-sheaf $\mathscr{F}$ over X by setting $\mathscr{F}(U) = L/(J^\perp)\alpha^{-1}$, where $U = X_J$. It follows that the corresponding sheaf (S, π, X) is a sheaf of bounded distributive lattices for which each stalk is dense; in fact $S_x = L/x\alpha^{-1}$. Further, for all a in L, the map $[a]: X \rightarrow S$, given by $x[a] = a/x\alpha^{-1}$, is a global section and $\{U[a] | U \text{ is open in } X, a \in L\}$ is a basis for the topology on S.

LEMMA 2. *For each σ in Γ there exists $a_1, \ldots, a_n$ in L and $e_1, \ldots, e_n$ in $\mathscr{C}(\Gamma)$ such that $e_i \wedge e_j = [0]$ $(i \neq j)$, $\bigvee_{i=1}^n e_i = [1]$ and $\sigma = \bigvee_{i=1}^n [a_i] \wedge e_i$.*

Proof. For each x in X, $x\sigma = x[a]$ for some a in L. Thus, by Lemma 1, there is a clopen set U_x such that $\sigma = [a]$ in U_x, since X has a basis of clopen sets. By the compactness of X there is a finite set $a_1, \ldots, a_n$ of elements of L and a finite cover of X by disjoint clopen sets $U_1, \ldots, U_n$ such that $\sigma = [a_i]$ in U_i. The result follows with $e_i = \chi_{U_i}$.

Remark. For the case where $\mathfrak{m}$ is finite it is easily seen that the elements of Γ have the following explicit form:

$$\sigma = \bigvee_{i=1}^{n} [a_i] \wedge \bigvee_{j=1}^{l} [b_{ij}]^{**} \wedge [c_{ij}]^{*} \qquad (a_i, b_{ij}, c_{ij} \in L).$$

LEMMA 3. *Γ is an $\mathfrak{m}$-Stone lattice.*

Proof. By Theorem 2(iii) it only remains to prove that $supp(\sigma)$ is clopen in X for all σ in Γ. Applying Lemma 2 we have $supp(\sigma) = \bigcup_{i=1}^{n}(U_i \cap X_{a_i{}^{\perp\perp}})$ $Xa_i{}^{\perp\perp}$, which is clopen. Alternatively, we may use Lemma 2 to prove that S is a Hausdorff space and then apply the corollary to Theorem 2.

The map $\gamma : L \rightarrow \Gamma$, given by $a\gamma = [a]$, is a well defined homomorphism; in fact

LEMMA 4. *$\gamma : L \rightarrow \Gamma$ is an $R_{\mathfrak{m}}$-monomorphism.*

Proof. If $[a] = [b]$, then $a/x\alpha^{-1} = b/x\alpha^{-1}$ for all x in X. Now

$$\bigcap\{x\alpha^{-1}|x \in X\} = (\bigcap\{x|x \in X\})\alpha^{-1} = \{0\}\alpha^{-1} = \{0\},$$

and hence $a = b$. Thus γ is a monomorphism.

Since Γ is an $\mathfrak{m}$-Stone lattice $\mathscr{A}_{\mathfrak{m}}(\Gamma) = \Gamma^{\perp\perp}$. Thus we may define $\bar{\gamma}: \mathscr{A}_{\mathfrak{m}}(L) \rightarrow \mathscr{A}_{\mathfrak{m}}(\Gamma)$ by $A\bar{\gamma} = (\chi_Y)^{\perp\perp}$, where $Y = X_A$. $\bar{\gamma}$ is clearly a Boolean $\mathfrak{m}$-homomorphism.

We are now in a position to prove that the embedding of L into Γ is the required adjoint functor.

THEOREM 3. *For any bounded distributive lattice L there is an $\mathfrak{m}$-Stone lattice Γ and an $R_{\mathfrak{m}}$-monomorphism $\gamma: L \rightarrow \Gamma$ with the following property: for any $R_{\mathfrak{m}}$-homomorphism $\phi: L \rightarrow S$ of L into an $\mathfrak{m}$-Stone lattice S there is a unique $\mathfrak{m}$-Stone homomorphism $\phi' : \Gamma \rightarrow S$ such that $\gamma \circ \phi' = \phi$. Further, the pair (γ, Γ) is unique up to isomorphism over L.*

Proof. We show that the pair (γ, Γ) constructed above satisfies the property; then standard category arguments establish the uniqueness. We define $\phi' : \Gamma \rightarrow S$ be defining $\phi' : L\gamma \rightarrow S$ and $\phi' : \mathscr{C}(\Gamma) \rightarrow \mathscr{C}(S)$, and then appeal to Lemma 3. If $a \in L$ we put $(a\gamma)\phi' = a\phi$. Since S is an $\mathfrak{m}$-Stone lattice we may define $\beta : \mathscr{A}_{\mathfrak{m}}(S) = S^{\perp\perp} \rightarrow \mathscr{C}(S)$ by $(s^{\perp\perp})\beta = s^{**}$, a Boolean isomorphism. Now if $e \in \mathscr{C}(\Gamma)$ we have $e = \chi_Y$, where $Y = X_A$ for some A in $\mathscr{A}_{\mathfrak{m}}(L)$, thus we may put $e\phi' = A\bar{\phi}\beta$. Finally, if

$$\sigma = \bigvee_{i=1}^{n} a_i\gamma \wedge e_i,$$

then

$$\sigma\phi' = \bigvee_{i=1}^{n} a_i\phi \wedge e_i\phi'.$$

ϕ' is well defined and is an $\mathfrak{m}$-Stone homomorphism since ϕ is an $R_{\mathfrak{m}}$-homomorphism.

Remark. It is interesting to note that if L satisfies the condition $\mathfrak{m}(*)$ then $\mathscr{A}_{\mathfrak{m}} = L^{\perp\perp}$ and X is homeomorphic to the set $\mathscr{M}$ of minimal prime ideals of L equipped with the Stone topology, and thus $\Gamma(\mathscr{M}, S)$ is an $\mathfrak{m}$-Stone extension of L. The particular case of $\mathfrak{m}$ finite, where $\mathfrak{m}(*)$ is equivalent to the compactness of $\mathscr{M}$ [**7**, p. 290, Proposition 3.2], is a direct analogue of Kist's result for Baer rings [**3**, p. 157, Theorem 5.3]. The extension of Kist's result to $\mathfrak{m}$-Baer rings has been carried out independently by T. P. Speed [**8**] and D. E. Peercy [**5**], in the latter case using methods similar to those used here.

References

1. C. C. Chen and G. Grätzer, *Stone lattices. I: construction theorems*, Can. J. Math. *21* (1969), 884–894.

2. O. Frink, *Pseudo-complements in semi-lattices*, Duke Math. J. *29* (1962), 505–514.

3. J. Kist, *Compact spaces of minimal prime ideals*, Math. Z. *111* (1969), 151–158.

4. B. Mitchell, *Theory of categories* (Academic Press, New York, 1965).

5. D. E. Peercy, *Complexes and the complete Baer extension of a commutative ring* (to appear).

6. T. P. Speed, *A note on commutative semigroups*, J. Austral. Math. Soc. *8* (1968), 731–736.

7. ——— *Some remarks on a class of distributive lattices*, J. Austral. Math. Soc. *9* (1969), 289–296.

8. ——— *A note on commutative Baer rings* III (to appear).

9. G. Swan, *The theory of sheaves* (University of Chicago Press, Chicago, 1964).

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