Space-filling Tetrahedra in Euclidean Space.

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In the answer to the book-work question, set in a recent examination to investigate the volume of a pyramid, one candidate stated that the three tetrahedra into which a triangular prism can be divided are *congruent*, instead of only equal in volume. It was an interesting question to determine the conditions in order that the three tetrahedra should be congruent, and this led to the wider problem—to determine what tetrahedra can fill up space by repetitions. An exhaustive examination * of this required one to keep an open mind as regards whether space is euclidean, elliptic, or hyperbolic, and then to pick out the forms which exist in euclidean space.

A more detailed description of the four euclidean forms may be of interest, though for the proof that these are the only possible euclidean space-filling types reference must be made to the fuller investigation.

2. Let us consider first the triangular prism (Fig. 1), whose vertices we shall denote by $ABCA'B'C'$; AA', BB', CC' being the parallel edges. In order to divide the prism into three tetrahedra the three parallelograms must each be divided by a diagonal in such a way that for each of the two triangular faces the three vertices must be joined to one other point. This requires that the three diagonals should form a single broken line. Starting with any vertex A, draw the diagonals AB', $B'C$, CA'. We have then three tetrahedra (all named with the same orientation)

$$AA'B'C,\quad BB'CA,\quad CC'A'B'.$$

* See the author's paper: "Division of space by congruent triangles and tetrahedra," Edinburgh, *Proc. Roy. Soc.*, vol. 43 (1923), pp. 85–116.

We shall use the following notation for the lengths of the edges:

$$BC = B'C' = a, \quad B\,C = a', \quad AA' = BB' = CC' = d.$$
$$CA = C'A' = b, \quad CA' = b',$$
$$AB = A'B' = c, \quad AB' = c'.$$

These seven lengths are connected by the identical relation

$$a^2 + b'^2 + c'^2 = a'^2 + b^2 + c^2 + d^2.$$

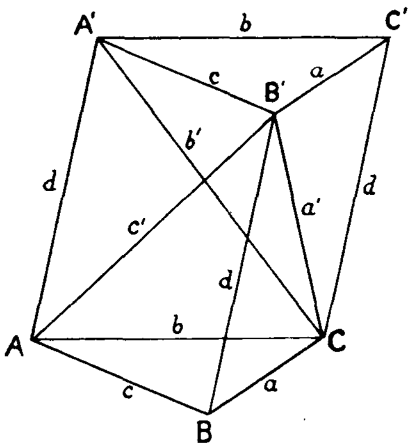

Fig. 1

3. Suppose now that the tetrahedron $AA'B'C$ is congruent to the tetrahedron $BB'CA$, corresponding vertices being in a definite order. There are 24 orders, 12 giving congruence and 12 symmetry. Correlation in any one order implies certain equalities among the edges. Thus, $AA'B'C$ with $BB'CA$ implies $a = b' = c'$ and $a' = b = c$. By an examination of all the possible orders we find that the only possible cases in which the three tetrahedra can be congruent or symmetrical are

(1) $a = b' = c', \; a' = b = c = d.$

$AA'B'C$ is congruent to $\left.\begin{array}{l} BB'CA,\ CABB',\ BACB',\ ABB'C \\ B'BAC,\ ACB'B,\ B'CAB,\ CB'BA \end{array}\right\}$

The tetrahedra are symmetrical,* and each is congruent to each of the others in 8 different ways. (Fig. 2.)

* The necessary and sufficient condition that a tetrahedron should be symmetrical is that a pair of adjacent edges should be equal, and the two edges opposite to them also equal.

(2) $a = b' = c'$, $a' = b = c$.

$$AA'B'C \text{ is congruent to } \left.\begin{matrix} BB'CA \\ ACB'B \end{matrix}\right\} \quad \left.\begin{matrix} CC'A'B' \\ C'CB'A' \end{matrix}\right\}$$

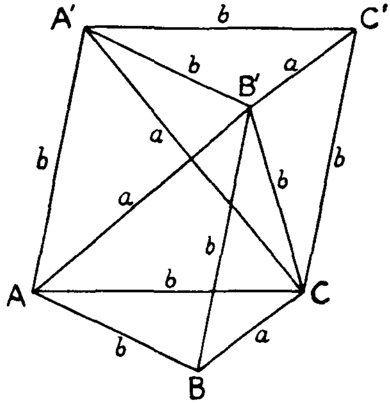

Fig. 2

The tetrahedra are asymmetric, and each is congruent to each of the others in two different ways. (Fig. 3.)

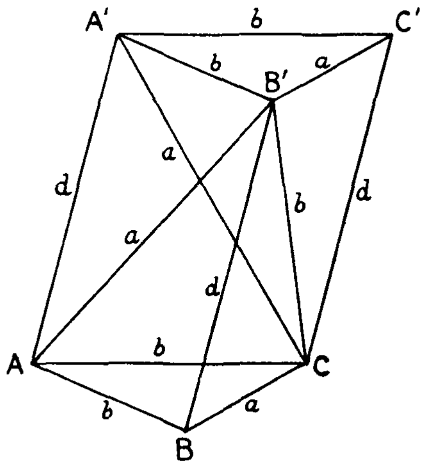

Fig. 3

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(3) $a = b' = c', \ a' = b = d.$

$$AA'B'C \text{ is congruent to } \left.\begin{matrix} B'BAC \\ CABB' \end{matrix}\right\} \quad \left.\begin{matrix} CB'A'C' \\ C'A'B'C \end{matrix}\right\}$$

The tetrahedra are asymmetric; $AA'B'C$ is congruent to $B'BAC$ in two ways, and asymmetrical to $CC'A'B'$ in two ways.

If the tetrahedron $CC'B'A'$ is removed, and its mirror image $CC'A'B' \equiv AA'C_1C$ is placed with the face $CC'B'$ on the face $AA'C$ of the mutilated prism, the resulting prism $B'BCA'AC_1$ is divided in the manner of case (2). (Fig. 4.)

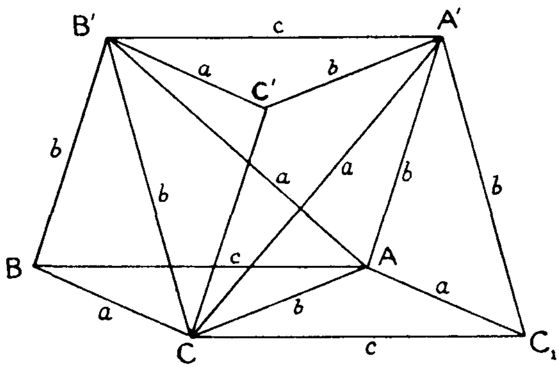

Fig. 4

(4) $a = b' = c', \ a' = c = d.$

As this is got from (3) by interchanging b, c and b', c' it is of the same type as (3).

(5) $a = b' = c', \ b = c = d.$

In this case, if the tetrahedron $BB'CA \equiv B'B_1C'A'$ is moved so that ABC is placed on $A'B'C'$ the resulting triangular prism $C'B_1A'CB'A$ is divided in the manner of case (3). (Fig. 5.)

There are thus four different types of triangular prisms divisible into three congruent or symmetrical tetrahedra: Nos. (3) and (5), in which one of the tetrahedra is symmetrical to the other two, No. (3) being distinguished by its triangular faces being scalene; No. (2), in which the three tetrahedra are all congruent, but still asymmetric; and No. (1), more highly specialized, in which the three tetrahedra are all symmetrical.

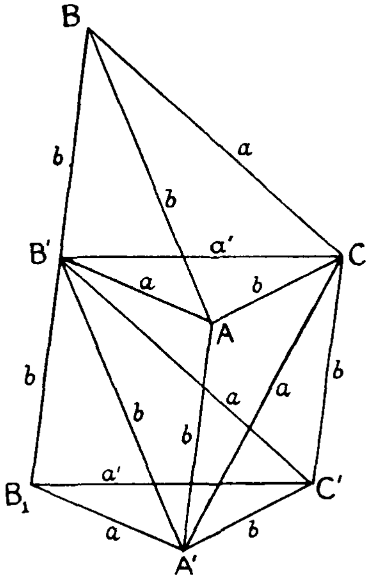

Fig. 5

4. In non-euclidean space corresponding prisms do not exist. Let us try to construct a triangular prism; we start with the triangle ABC and draw an edge $BB' = d$. Then in the planes ABB' and CBB' draw AA' and CC' respectively such that $AA' = BB' = CC'$, $A'B' = AB$ and $B'C' = BC$. With the same notation as before we find then the relation

$$(\cos a + \cos b')(1 + \cos c') = (\cos a' + \cos b)(\cos c + \cos d).$$

But on interchanging the two ends ABC, $A'B'C'$ we interchange b, c and b', c'; hence this relation should be symmetrical in b, c and b', c'. Therefore $b = c$ and $b' = c'$. Similarly $a = b = c$ and $a' = b' = c'$, and it follows that the prism must be symmetrical about the plane which cuts its three equal edges AA', BB', CC' at right angles. The identical relation then reduces to

$$1 + \cos a' = \cos a + \cos d.$$

But this prism cannot now be divided into congruent tetrahedra, for if $a = a'$, $d = 0$.

5. Consider the transverse section of prism No. (2). The distances between the parallel edges AA', BB', CC' are in each case equal to the altitude of a triangle abd perpendicular to the side d. Hence the transverse section is an equilateral triangle. For the

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tetrahedron $BB'CA$ the dihedral angle BB' is 90° since this is the corresponding edge of the tetrahedron $AA'B'C$; from symmetry the dihedral angle BC is also 90°. Then we find that the dihedral angle AB is equal to the dihedral angle $B'C = \sin^{-1} b/a$, while that at $AC = 180° - 2\sin^{-1} b/a$.

For No. (1) the identical relation gives $3a^2 = 4b^2$, hence this prism, and the resulting tetrahedron, are of a definite shape. The tetrahedron has one pair of opposite edges of length 2, and the other four edges of length $\sqrt{3}$; the corresponding dihedral angles are equal to 90° and 60° respectively. The transverse section of the prism is thus an equilateral triangle.

6. The prism No. (2), which includes No. (1), can be constructed as follows. Starting with an equilateral triangular prism (Fig. 6), take a transverse section AMN. Choose a length q,

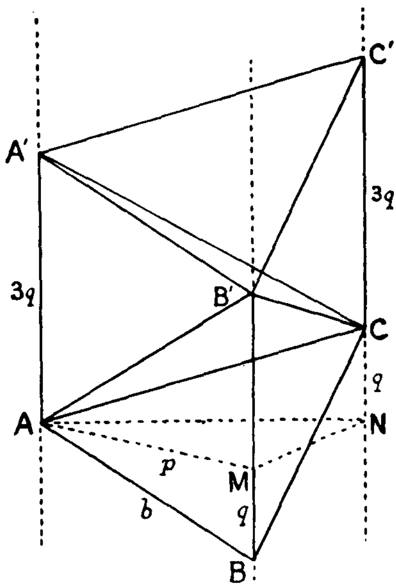

Fig. 6

and make $MB = NC = q$, measuring these in opposite senses. Let $AM = MN = NA = p$. Take $AA' = BB' = CC' = 3q$.

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Then

$$AB^2 = AC^2 = B'C^2 = p^2 + q^2 = b^2 ,$$
$$BC^2 = AB'^2 = CA'^2 = p^2 + 4q^2 = a^2,$$

while $$3q = d.$$

Thus we have the relation

$$3(a^2 - b^2) = d^2.$$

For No. (1) $p = 2\sqrt{2} . q$, and $AB = 3q$.

7. Since the triangular prisms are evidently space-filling, the tetrahedra into which they are dissected are also space-filling; but since the adjacent prisms, of the six which can be fitted round an edge, are mirror images, we have in the case of No. (2) a mixture of tetrahedra and their mirror images. No. (1), which is composed of symmetrical tetrahedra, alone affords a true space-filling tetrahedron (Fig. 7).

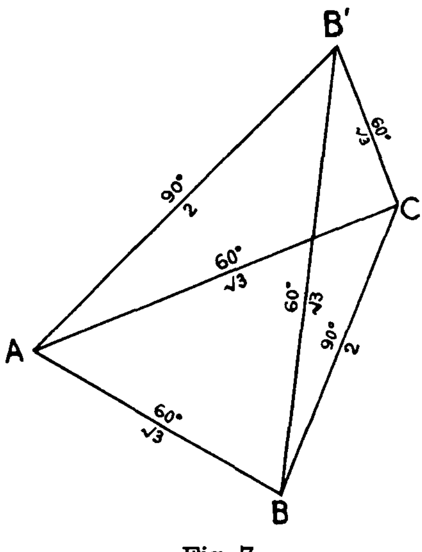

Fig. 7

This tetrahedron can be fitted together into other space-filling solids. From the values of the dihedral angles we find the values

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of the solid angles. The total solid angle at a point being denoted by ϖ, and the solid angle of a trihedral angle with dihedral angles α, β, γ being $\varpi(\alpha+\beta+\gamma-180)/720$, the solid angle at each of the vertices of the tetrahedron is $\varpi/24$. Two tetrahedra placed together form a pyramid whose base is a quadrilateral of angle $\cos^{-1}\frac{1}{3}$; 12 of these with vertices together form a rhombic dodecahedron.

Again, dividing the tetrahedron by a plane of symmetry through AB' we obtain a tetrahedron $ABDB'$ (Fig. 8) with dihedral angles: $AB'=45°$, $AD=BD=B'D=90°$, $AB=BB'=60°$; and solid angles $A=B'=\varpi/48$, $B=\varpi/24$, $D=\varpi/8$. Four of these

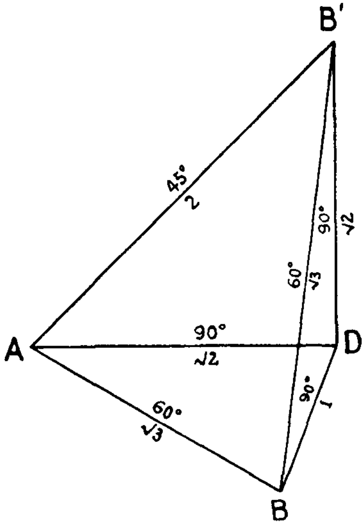

Fig. 8

placed round the edge BD form a pyramid on a square base, and six of these with their vertices together form a cube.

Two of the latter tetrahedra placed with the faces BDB' together form another tetrahedron $ABB'E$ (Fig. 9), of which 12 at the vertex B form the cube, and 24 at the vertex B' form the rhombic dodecahedron.

Finally, returning to the original tetrahedron $ABCB'$, whose faces are all congruent, find the centre S of the circumscribed sphere, so that $SA = SB = SC = SB' = \frac{1}{2}\sqrt{5}$, then we have four

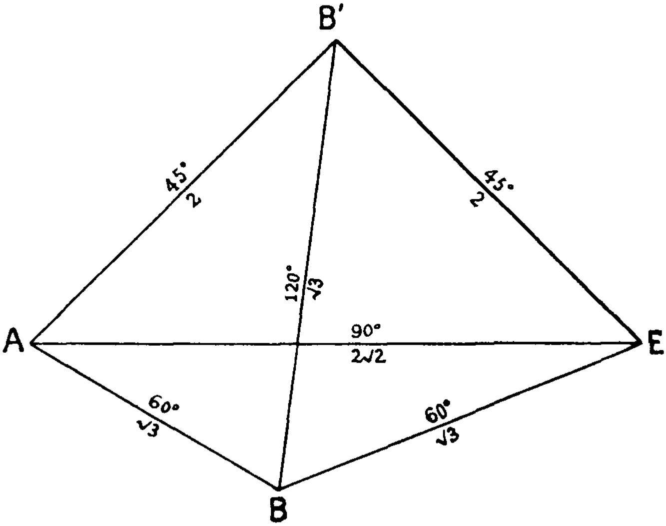

Fig 9

congruent tetrahedra such as $ABCS$ (Fig. 10), which are obviously space-filling; the dihedral angles at SB and $SC = 90° + \frac{1}{2}\cos^{-1}\frac{2}{3}$ while that at $SA = 180° - \cos^{-1}\frac{2}{3}$.

In addition to these four, no space-filling tetrahedra exist in euclidean space.

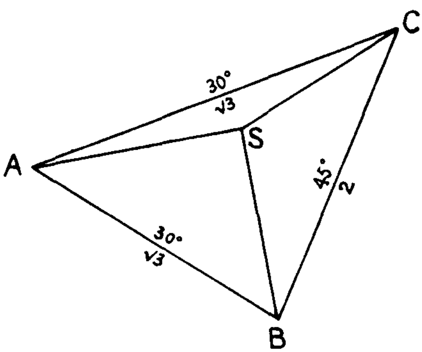

Fig. 10

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