1

Introduction

ABSTRACT

To offer a perspective, the genesis of Earth is very briefly reviewed. Since Earth rotates on its axis, its atmosphere and oceans behave as rotating fluids to leading order. This greatly helps unify their representation. The mathematical equations governing the flow of each of these two geophysical fluids then define an associated dynamical core. To obtain a realistic atmospheric or oceanic model for prediction purposes, the governing equations of the corresponding dynamical core need to be supplemented with forcing terms that represent other important physical processes (e.g. solar radiative heating, which drives atmospheric and oceanic circulations). Since gravity is such a dominant physical force, its representation is identified as a key factor for accurate, predictive, atmospheric, and oceanic modelling. Earth's shape is determined by its rotation rate and by gravity; it is not spherical, but spheroidal. Consequently, gravity varies as a function of latitude, but this variation is traditionally neglected in atmospheric and oceanic models. Although Earth's atmosphere and oceans behave to leading order as fluids, they nevertheless behave somewhat differently in detail. This is because their fluid composition is inherently different, with different properties, particularly with regard to thermodynamics. The fluid composition of Earth's atmosphere and oceans is briefly reviewed, both separately and comparatively. Both fluids are stratified, but in different ways according to their composition. This review helps relate later mathematical developments to physical reality. Finally, a summary guide to the organisation of the chapters is given.

1.1 A VERY BRIEF HISTORY OF TIME

Once upon a time there was – according to current scientific theory – a **BIG BANG**. A very big and very hot bang. Instantaneously, unimaginable amounts of hydrogen, helium, and other gases started whizzing around at breakneck speeds within rapidly expanding space. Over billions of years:

- These gases cooled and condensed under the influence of gravity, first into gas clouds, and then into stars.
- Thermonuclear reactions took place.
- Different forms of matter resulted.
- Yet more stars were created, with planets and their moons orbiting around them.

Fast forward more billions of years, and there was a far more modest bang. A celestial body crashed into a minor planet, in orbit around a minor star, of a minor constellation of stars, in a minor galaxy of the universe; and a moon was born.

Fast forward again and we are where we are now, 13.8 billion years (give or take) after the very big, very hot bang. The Moon orbits the rotating planet, Earth, which in turn orbits the Sun. This

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concludes our very brief history of time.[1] For a very readable, far more comprehensive history of time, see Hawking (2011).

This minor, rotating planet has both an atmosphere and oceans. The purpose of this book is to mathematically describe them in a fairly rigorous and physically realistic manner, based on their representation as fluids. This then provides the firm foundation for the construction of realistic, quantitative, atmospheric, and oceanic models for weather and climate prediction.

1.2 DYNAMICAL CORES

Earth's atmosphere and its oceans, *to leading order*, behave as fluids. From a modelling perspective, this is of crucial importance, since it allows us to translate the *physics* governing the motion of a fluid into *mathematics*. The resulting geophysical-fluid-dynamical, partial differential equations are the governing equations for the *dynamical core* of an atmospheric or oceanic model. In reality, these equations have to be supplemented by (often complicated) mathematical representations of other important physical phenomena (e.g. radiative heating by the Sun, which drives the global circulations of Earth's atmosphere and oceans). These representations then appear as *forcing terms* for the governing equations of a dynamical core.

To illustrate this, the governing equations of an atmospheric or oceanic model may be formally expressed in much simplified form as

The Dynamical Core Equations

$$\frac{\partial X}{\partial t} + Y(X) = F(X), \tag{1.1}$$

where t is time and:

- X is the vector of *dependent forecast variables* (such as velocity components, pressure, density, internal energy, water substance, salinity).
- Vector $Y(X)$ represents the *dynamical coupling* of these forecast variables in the absence of forcings other than gravitational attraction.
- $F(X)$ is the vector of all other *forcing functions* (such as solar radiative heating and friction).[2]

The left-hand side of (1.1) then represents the governing (geophysical-fluid-dynamical) equations of a dynamical core, and the right-hand side the forcing terms needed to turn a dynamical core into a realistic, quantitative, atmospheric, or oceanic forecast model.

Numerical approximation of the governing mathematical equations (including forcing terms, $F(X)$), coupled with use of many observations to determine an *initial state* of the atmosphere and/or of an ocean, then allows (approximate) *future states* to be determined using supercomputers. For numerical weather forecasting, Roulstone and Norbury (2013) and Lynch (2014) give very readable accounts of this process, intertwined with its historical development. Similar considerations also apply to oceanic forecasting.

The dynamical core is, as the name suggests, the cornerstone of an atmospheric or oceanic model; everything is built upon or around it. It is therefore very important that its governing equations not be oversimplified, otherwise an undesirable loss of fidelity with respect to reality results. This book focuses attention on this by:

[1] Well, you were advised that this history would be *very* brief … unlike the remainder of this book, you may say.

[2] Gravity is treated as part of the dynamical core because it has a known form, and because it guides the choice of coordinate systems. All other forcings are considered not to belong to the dynamical core. These forcings are important both practically and conceptually, and their accurate formulation represents a major scientific challenge; but the focus here is on the equally important and challenging problem of accurately formulating dynamical cores.

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Part I Describing how to minimally approximate these governing equations, without prejudicing practical viability.

Part II Providing a means to understand the significance and potential impact of making further approximations, with emphasis on the importance of maintaining analogues in any approximated equation set of physical conservation laws embodied in the original set.

Part III Developing some exact, steady and unsteady, non-linear solutions of governing dynamical-core equation sets for testing purposes.

This book also places these important aspects into a broader context.

1.3 GRAVITY

Key to achievement of the three goals just presented is recognition that rotating planets are more accurately represented as oblate spheroids than spheres. A planet's shape is due to the combined influence of rotation about its axis and gravity; see Chapters 7 and 8. The faster a planet rotates, the more oblate it is. A deformable planet can only be spherical if it does not rotate. But Earth *is* deformable and it *does* rotate.

Atmospheric and oceanic dynamical cores traditionally assume the *spherical-geopotential approximation*; see Sections 7.1.5 and 8.6 for details. This practice dates back to the very beginnings of numerical modelling when:

- Computer power and modelling know-how were very limited indeed.
- Spatio-temporal resolution was very low.
- The representation of physical forcings was very crude.
- Observational data were sparse.
- Initial conditions had very large errors.

It was then clear that any error incurred by the spherical-geopotential approximation was negligible compared to other errors. Many decades later, with the advent of ever-more-powerful computers coupled with enormous improvements in data availability and its assimilation, numerical methods, and the representation of physical forcings, this is no longer evident. Looking to the future, the only convincing way to assess the validity of the spherical-geopotential approximation is to perform careful, controlled experiments using a model with a more accurate, *spheroidal* representation and an appropriate coordinate system. But first the governing equations of such a model have to be properly formulated!

It is relatively straightforward to describe the spherical-geopotential approximation and how it is applied in practice. However, it is surprisingly difficult to *rigorously* and convincingly derive it; see van der Toorn and Zimmerman (2008)'s critique of arguments advanced by previous authors for this. The underlying difficulty is that if a deformable planet is truly spherical and rotating, then the corresponding geopotential surfaces cannot be spherical, and vice versa. Also, to respect physical conservation principles, gravity cannot vary meridionally (i.e. as a function of latitude) with the spherical-geopotential approximation. However, it has been known for centuries that it does. Furthermore, artificial satellites have, over the past several decades, become a crucially important source of observational data for atmospheric and oceanic prediction systems. Locations of satellite observing platforms are reported (using the Global Positioning System, or GPS for short) with respect to a reference *ellipsoid*, and *not* to a spherical one; see Section 12.2. This then leads to *inconsistencies* in the use of satellite data.

Gravity is a dominant physical force for Earth's atmospheric and oceanic circulations. For the reasons just given, it seems perverse to represent Earth's geopotentials as *spheres*; mathematical simplicity and convenience should not outweigh physical fidelity. This practice, with the risk of creating cumulative, systematic, forecast errors, is shown to be unnecessary. A suitable coordinate system for formulating the governing equations of the dynamical cores of mildly oblate planets

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Gas	Chemical symbol	Atomicity	Mass fraction with respect to dry air
Nitrogen	N_2	Diatomic	0.755 184 73
Oxygen	O_2	Diatomic	0.231 318 60
Argon	Ar	Monatomic	0.012 870 36
Carbon dioxide	CO_2	Triatomic	0.000 607 75
Neon	Ne	Monatomic	0.000 012 68
Krypton	Kr	Monatomic	0.000 003 18
Methane	CH_4	Polyatomic	0.000 000 83
Helium	He	Monatomic	0.000 000 72
Nitrous oxide	N_2O	Triatomic	0.000 000 46
Xenon	Xe	Monatomic	0.000 000 45
Carbon monoxide	CO	Diatomic	0.000 000 19
Hydrogen	H_2	Diatomic	0.000 000 03

Table 1.1 Composition of Earth's dry atmosphere by mass fraction of dry air. Source: Feistel et al. (2010a). By definition, molecules of monatomic, diatomic, and triatomic gases have one, two, and three atoms, respectively. Methane is a polyatomic gas with five atoms.

in *ellipsoidal* geometry is developed in Chapter 12. Adopting this ellipsoidal coordinate system should result in only a relatively modest increase in the complexity and computational cost of a forecast model compared to using *spherical* coordinates. Appropriately taking an asymptotic limit of this more general, ellipsoidal formulation, recovers the familiar set of governing equations in spherical geometry as a special case. As a side benefit, this sheds light on the validity and interpretation of the classical spherical-geopotential approximation.

1.4 FLUID COMPOSITION OF EARTH'S ATMOSPHERE AND OCEANS

To state the obvious, Earth's atmosphere is primarily *gaseous*, whilst its oceans are primarily *liquid*. This then influences the precise form of their respective governing equations. Their similarities unify many aspects, whereas their differences necessitate separate treatments of other aspects, particularly regarding equations of thermodynamic state. Before going into further details (in the following chapters), and to prepare the way for this, a brief overview is now given of the composition of Earth's atmosphere and of its oceans. This helps to relate later mathematical developments to physical reality.

1.4.1 The Atmosphere

Dry Gases

Earth's atmosphere is composed primarily of a mixture of dry gases; see Table 1.1. Just two dry (diatomic) gases (nitrogen and oxygen) comprise almost 99% of the total mass of dry gas in the atmosphere. The mass of carbon dioxide in Earth's atmosphere has gradually increased from pre-industrial levels (and currently continues to increase); this mass has an annual fluctuation, primarily due to the Northern Hemisphere's agricultural growing season.

Each dry gas (to an excellent approximation) individually behaves as an ideal gas. For most purposes, the mixture of dry gases can be considered to be of constant composition.[3] Because an individual dry gas behaves as an ideal gas, and because the mixture is of constant composition,[4]

[3] In other words, the individual masses of these gases, for any sample, are everywhere in the same proportion to one another.

[4] Depending on the application, there are exceptions to this rule. These include ozone (e.g. in the stratosphere over polar regions) and near-surface noxious gases (e.g. emissions from vehicles and factories). This is because some gases can

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the mixture behaves as if it were a *single* ideal gas (see Section 2.3.3). This greatly simplifies matters. Instead of having to represent a dozen or more dry gases and their mutual interactions, only a single, equivalent, composite, (dry) ideal gas needs representation.

Water Vapour

A very important constituent of Earth's atmosphere is water vapour, the gaseous state of water substance (H_2O). Water vapour, a triatomic gas, occupies about 0.4% of the mass of the entire atmosphere and typically 1–4% of the mass of air near Earth's surface. Unlike the dry gas constituents of Earth's atmosphere, water vapour varies considerably both in space and time.

It can change from its vapour state to both its liquid state (e.g. cloud droplets, and rain droplets) and its frozen state (e.g. ice crystals, hail, and frost). When it does so, it can strongly interact with its environment (e.g. by heating the atmosphere via the release of latent heat, and by moisture loss from the atmosphere through precipitation). At Earth's surface, water vapour is also both continuously lost (e.g. via precipitation and condensation) and replenished (e.g. via evaporation over land and water, and via evapotranspiration by plants and trees). These various complex processes are part of the *hydrological cycle* – that is, the cycling of water substance (in its various forms):

- Within the atmosphere.
- At Earth's surface.
- Below its surface (e.g. the root systems of plants and trees, aquifers, and underground streams).

Furthermore, water vapour is a so-called *greenhouse gas* (in fact, the most important one by far, ahead of carbon dioxide, methane, and ozone), and it thereby considerably influences Earth's weather and climate through its radiative properties.[5] It also interacts chemically with various atmospheric constituents (e.g. sulphur dioxide and nitrogen oxides interact with water substance to produce acid rain).

Because of its many different properties, water vapour in physically realistic, quantitative weather and climate prediction models needs to be represented separately from dry gases. That said, for a lot of theoretical work, aimed at improving understanding by studying simplified models, it can often be ignored!

Various Liquids and Solids

Earth's atmosphere not only contains dry gases and water vapour, but also various liquids and solids. These are particularly important for representing clouds, and the physical processes taking place within them. In physically realistic, quantitative, weather and climate prediction models, aerosols,[6] cloud liquid water, and cloud frozen water are typically represented. In air-quality models, dozens (or more) of chemically active species may be needed.

chemically and photochemically (via solar radiation) interact with other atmospheric constituents. Furthermore, dry gases can interact with vegetation at Earth's surface, thereby changing the amounts of dry gas present in the atmosphere. For example, plants process oxygen and carbon dioxide via photosynthesis. If any such phenomenon is important for a particular application (e.g. modelling the destruction of stratospheric ozone over the Poles in wintertime, or for air-quality studies, or for surface emissions), then any such gas should not form part of the mixture of dry ideal gases, but instead be represented separately; see the '$+\cdots$' terms of (3.1) in Chapter 3.

[5] In the absence of water vapour in the atmosphere, the temperature of Earth's atmosphere would be considerably colder (by an estimated 20° Celsius) and life, as we know it, would be considerably different (and possibly non-existent, albeit life forms have been found in the deep ocean at temperatures near 0° Celsius).

[6] Aerosols are small particles, suspended in the air, with diameters typically smaller than a micrometre. They include dust, pollen, sea-salt particles, volcanic ash, and by-products of combustion. Although very small, they provide a nucleus for condensation to take place and thereby facilitate cloud and fog formation, and chemical reactions associated with air pollution and ozone depletion. They also influence radiative transfer within the atmosphere. Small though they are, their properties and vast number make them important.

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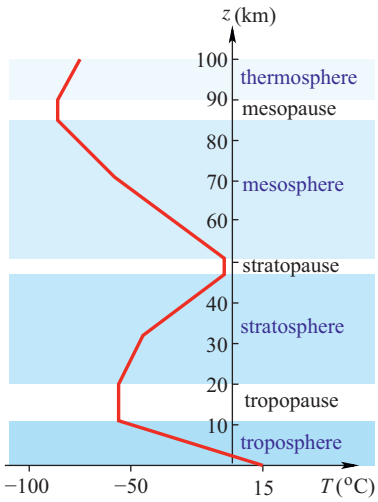

Figure 1.1 Temperature profile for the International Standard Atmosphere (1976), extended to include the mesopause and the lowest part of the thermosphere. Abscissa is temperature T (° C); ordinate is altitude z (km). Lowest four layers ('spheres') of the atmosphere (troposphere, stratosphere, mesosphere, and thermosphere) are shaded; locations of the tropopause, stratopause, and mesopause (isothermal) layers separating them are also depicted.

Stratification

To leading order, observations show that Earth's atmosphere can be considered to be a stratified fluid with several fairly distinct layers. Pressure (p) and density (ρ) in a column of air generally decrease monotonically as a function of increasing altitude (z).[7] Temperature (T), however, does not. It is the vertical variation of T that facilitates classification of the atmosphere into layers; see Fig. 1.1 for a schematic of the lowest layers. This figure is based on the International Standard Atmosphere (1976) for a representative mid-latitude column of (assumed dry) air extending upwards from Earth's surface. In reality, significant departures from this standard atmosphere are observed, even in mid-latitudes, but much more so in polar and tropical regions.

Based on time and space averages of observed temperature profiles in mid-latitudes, the International Standard Atmosphere (1976) defines a piecewise-linear, vertical temperature profile as a function of altitude (z). This is depicted in red in Fig. 1.1. Four (shaded) 'spheres' are shown (troposphere, stratosphere, mesosphere, and thermosphere), with a fifth 'sphere' (the exosphere) above them not shown.[8] For each of these 'spheres', T (by construction) either decreases or increases at a constant rate (measured in °C per km), termed the *temperature lapse rate*. Interleaved with these 'spheres' are 'pauses' (tropopause, stratopause, mesopause, thermopause), where T is isothermal (i.e. the temperature is constant, and consequently the temperature lapse rate is zero).

Assuming a stationary, hydrostatic atmosphere composed of ideal gases at constant composition, pressure and density profiles corresponding to a given temperature profile may be deduced. See Section 13.9 regarding the methodology for this.

[7] This decrease is approximately exponential – see Section 13.9.1.
[8] Since Earth is spheroidal, *troposphere* might whimsically be relabelled *tropospheroid*, and so on.

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Solute	Ion	Mass fraction with respect to total sea salt
Chloride	Cl^-	0.550 3396
Sodium	Na^+	0.306 5958
Sulphate	SO_4^{--}	0.077 1319
Magnesium	Mg^{++}	0.036 5055
Calcium	Ca^{++}	0.011 7186
Potassium	K^+	0.011 3495
Bicarbonate	$HC0_3^-$	0.002 9805
Bromide	Br^-	0.001 9134

Table 1.2 Composition of Earth's oceans by mass fraction of total sea salt. Source: Feistel et al. (2010a).

1.4.2 The Oceans

Pure Liquid Water

Seawater is a dilute solution of salts. The oceanic analogue of dry air is pure liquid water – the water content of the oceans in the absence of any dissolved salts (also referred to as solutes) or other substances. Seawater and pure liquid water are much denser than dry and moist air. For example, the density of seawater at the atmosphere/ocean interface is about 800 times larger than that of the overlying air.

Salinity

Solutes collectively comprise approximately 3.5% of the total mass of the oceans. See Table 1.2 for the composition of Earth's oceans by mass fraction of total sea salt. Similarly to dry gases in Earth's atmosphere, the solutes in the oceans can also usually be considered to be of constant composition. This means that instead of having to represent many different solutes and their mutual interactions, only a single, equivalent, composite quantity, namely *salinity*, needs representation.

For many years salinity was defined to be *Practical Salinity* (denoted S_P). This definition is based on measured electrical conductivity; Practical Salinity is nevertheless dimensionless. In 2010 a new standard for the properties of seawater was developed and adopted by the international oceanographic community. It is called the Thermodynamic Equation of Seawater 2010 (or TEOS-10, for short); see IOC et al. (2010) for (lots of) details. In particular, a new measure of salinity, termed *Absolute Salinity* (S_A), has been defined. This new measure represents the mass fraction of solutes with respect to the combined mass of pure liquid water plus solutes. Since it is a true mass fraction, Absolute Salinity is also dimensionless. However, it is usually expressed in units of $\mathrm{g\,kg^{-1}}$ (i.e. as parts per thousand).

Stratification

Density (ρ) in an ocean is (with a few exceptions) generally constant to within 2 per cent. It has the value $\rho_0 = 1\,035\ \mathrm{kg\ m^{-3}}$. This quasi-constancy reflects the fact that whereas air is highly compressible, liquid water is almost incompressible. That ρ is nearly constant can be exploited to determine, to a very good approximation, how pressure (p) in an ocean varies with respect to ocean depth ($-z$).

For a motionless (static) ocean, a vertical column of water satisfies the hydrostatic-balance equation,

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g, \tag{1.2}$$

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where g is gravity.[9] For given ρ and g, (1.2) is a first-order differential equation that can be integrated downwards from a given pressure, say $p(z = 0) = p_0 = 1.01325 \times 10^5$ Pa $\equiv$ 1 atm $\equiv$ 1.01325 bar $\equiv$ 1013.25 hPa $\equiv$ 1013.25 mb (i.e. the value of 'standard sea-level pressure', a representative value at an ocean's upper surface).[10] Making the approximations $\rho \approx \rho_0 = 1035\text{kgm}^{-3}$ and $g \approx g_0 = 9.80665\text{ms}^{-2}$ (i.e. the value of 'standard gravity') in (1.2), and integrating downwards, then yields

The Approximate Pressure within a Water Column

$$p \approx p_0 - \rho_0 g_0 z. \tag{1.3}$$

Depth ($\equiv -z$) in (1.3) is measured *downwards* from an ocean's surface at $z = 0$. Thus pressure increases approximately linearly downwards (with respect to depth) from an ocean's surface at a rate of $\rho_0 g_0 = 1.015 \times 10^4$ kg m^{-2} s^{-2} $= 1.015 \times 10^4$ Pa m^{-1}, or approximately 1 atm $\equiv$ 1.01325×10^5 Pa (approximate pressure at an ocean's upper surface) for every 10 m of descent. This makes it easy to roughly estimate the pressure at a particular ocean depth; just divide the depth, measured in metres, by ten and then add unity to obtain how many times greater the pressure is than that at an ocean's upper surface. For example, at the bottom of the Mariana Trench (the lowest point of Earth's oceans) in the Western Pacific Ocean, the depth is about 11 km = 11 000 m. Thus the pressure there is approximately 1 100 times greater than that at the ocean's upper surface. And life still exists down there – wow!

Although density is approximately constant throughout the ocean, *density variation* nevertheless has an important influence on the *thermohaline* circulation of Earth's oceans.[11] Cold, dense water sinks in polar regions (this is termed *downwelling*) and feeds a deep-ocean return flow of the poleward transport of heat from the tropics by upper-ocean currents, such as the West Atlantic Gulf Stream. This circulation is part of what is termed the *ocean conveyor belt*. This circulates a huge amount of heat within Earth's oceans on much longer timescales than those for the atmospheric circulation. It is why oceanic circulation is so important for climate modelling and prediction.

For the atmosphere, the vertical variation of temperature (T) facilitates classification of the atmosphere into layers; see Section 1.4.1. This is also true for the oceans, but in a different way. For the oceans there are three fairly distinct layers:

- The mixed (or surface) layer, with small, vertical temperature gradient.
- The thermocline, with large gradient.
- The deep ocean, with small gradient.

This stratification is depicted in Fig. 1.2 for a representative mid-latitude temperature profile. For the atmosphere (see Fig. 1.1), it is the *sign* of the vertical temperature gradient that characterises

[9] Equation (1.2) follows from staticity in the vertical component of the momentum equation. Physically, the vertical component of the pressure-gradient force then exactly balances gravity.

[10] Many units for pressure have been used in meteorology and oceanography over the years, with standard SI (Système International) units now in frequent use. But old habits die hard, and there are many old publications that are still relevant today, so it is good to be aware of other units.

[11] *Thermo* in 'thermohaline' refers to temperature, and *haline* to salt content, both of which jointly determine seawater density. The lower the temperature and the higher the salinity (e.g. due to an excess of surface evaporation versus precipitation over a region), the higher is the density. Conversely, the higher the temperature (e.g. due to solar heating) and the lower the salinity (e.g. due to dilution by fresh water from river outflow, or melting of icebergs), the lower is the density.

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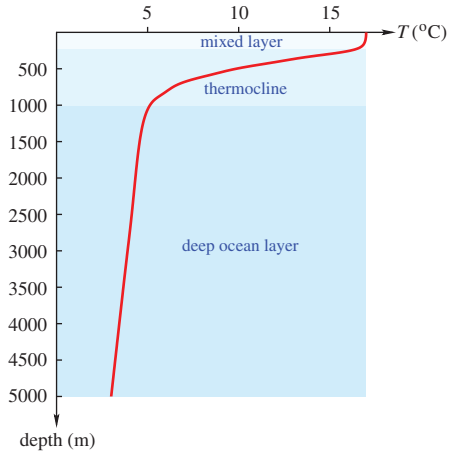

Figure 1.2 A representative, mid-latitude, oceanic temperature profile T (° C), plotted as a function of depth (m) measured downwards from an ocean's upper surface. Three layers are depicted. From top to bottom; mixed layer (light shading), thermocline (medium shading), and deep ocean (dark shading).

the layers. For the oceans (see Fig. 1.2), it is instead the *magnitude* of the vertical gradient, with two low-gradient layers (mixed and deep-ocean) separated by a large-gradient one (the thermocline).

Incoming solar radiation during daylight hours heats the near-surface of an ocean, primarily in its upper few centimetres; this is governed by the radiative properties of the ocean and the air above it. Wave activity (often wind-driven at an ocean's surface) and turbulent mixing then redistribute this heat fairly uniformly in the *mixed layer*. The depth of this layer depends upon a number of factors, particularly latitude and season. For the tropics and mid-latitudes, this depth is typically 100–200 m, but may be barely discernible, and even absent, at high latitudes. The temperature of this layer is generally higher in the tropics and subtropics (where solar heating is strong) than in mid- and high-latitudes (where it is much weaker). It is also somewhat colder during night-time due to the absence of incoming, short-wave, solar radiative heating, and the presence of outgoing, long-wave radiative cooling. In wintertime, the depth of the mixed layer is generally greater than in summertime, albeit it changes relatively little in the tropics, where the seasonal variation of T is relatively small. Mass, momentum, and energy are exchanged between the mixed layer and the atmosphere.

The *thermocline* is a transition layer between the mixed layer and the deep-ocean layer.[12] Its distinguishing feature is a strong temperature gradient between the top and bottom of the layer. This is generally strongest in the tropics and subtropics, reducing to non-existent in polar regions in wintertime. Its depth is highly variable, but typically of order 1 km. Associated with the thermocline (for temperature) are the *pycnocline* (for density) and *halocline* (for dissolved salt). The thermocline and pycnocline are generally quite well defined and correlated with one another. The halocline is generally less well-defined and only weakly correlated with the thermocline and pycnocline. A large density gradient can originate from a large temperature gradient, a very large salinity gradient, or a combination of the two.

The *deep-ocean layer* extends from the bottom of the thermocline down to an ocean's bathymetry. Conditions are generally fairly calm within this layer, with gradually reducing temperature

[12] '*Cline*' in oceanography denotes a relatively shallow layer of seawater with a strong vertical gradient of an associated quantity. Thus, *thermo*cline denotes a strong vertical gradient of temperature; *pycno*cline, a strong vertical gradient of density; and *halo*cline, a strong vertical gradient of dissolved salt.

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and gradually increasing density as a function of depth. It comprises the vast majority of the total volume of the oceans; of order 90%. Because of its distance from an ocean's upper surface, the daily and seasonal variation of the deep-ocean layer is negligible.

1.5 ORGANISATION OF CHAPTERS

Part I

This part describes how to obtain a general set of governing equations, in a unified manner, for the dynamical core of a quantitative atmospheric or oceanic prediction model. To this end, *approximations are made only when necessary and justified*.

By the end of Chapter 4, a unified set of governing equations (in vector form) has been given for both the atmosphere and the oceans, under the assumptions that both the geopotential (Φ) and the internal energy ($\mathscr{E}$) are of known functional form. Nothing is said therein about a suitable coordinate system.

To prepare the way for this eventuality, definitions and relevant properties of various orthogonal-curvilinear coordinate systems are documented in Chapter 5. To accommodate the later description of some popular, approximate, governing equation sets (e.g. the hydrostatic primitive equations), shallow versions of these coordinates also are included.[13]

By the end of Chapter 6, the governing equations (in vector form) of Chapter 4 for the motion of geophysical fluids have been re-expressed in general axial-orthogonal-curvilinear coordinates. These equations are also given explicitly for the special cases of spherical-polar and cylindrical-polar coordinates. However, functional forms for Φ for the atmosphere and the oceans, and $\mathscr{E}$ for the oceans, remain to be specified. (Specification of $\mathscr{E}$ for a dry atmosphere, and also for a simple moist atmosphere, has already been taken care of in Chapters 2 and 3, respectively, since this is both analytically tractable and instructive.)

In Chapters 7 and 8, suitable functional forms for Φ are systematically developed from first principles. Doing so defines the mean shape of the planet (classically called its *figure*) – which is *ellipsoidal* rather than spherical – and the planet's gravitational attraction, both above *and* below its geoid. These functional forms are deliberately chosen to be consistent with the Reference *Ellipsoid* used for the Global Positioning System.

At this juncture, we have the basic ingredients for a realistic, *atmospheric,* dynamical core. This is not the case, however, for a realistic, *oceanic,* dynamical core since we still do not have a suitable functional form for $\mathscr{E}$, not even for a pure-water ocean with no salt.[14] To address this deficiency requires a more comprehensive and far more complicated treatment of thermodynamics than the relatively simple one – described in Chapters 2 and 3 – that suffices for the atmosphere. This is given in Chapters 9–11, with the bonus of also describing phase changes of water substance between its vapour, liquid, and frozen forms. This is an important aspect of both atmospheric and oceanic models.

By the end of Chapter 12, a self-consistent, geopotential coordinate system for a *smooth*, ellipsoidal (as opposed to spherical) planet has been developed. Planets are not, however, generally smooth; an atmosphere or an ocean usually has an underlying orography or bathymetry, respectively. To address this, generalised vertical coordinates – with suitable lower and upper boundary conditions – are developed in Chapter 13. This includes terrain-following coordinates as a special case. These provide a natural representation of orography and bathymetry in atmospheric and oceanic models, respectively.

Chapter 14, on variational methods and Hamilton's principle of stationary action, introduces an alternative methodology for deriving governing equation sets. This often-overlooked subject

[13] Shallow coordinates are a non-Euclidean geometric distortion of corresponding, deep, unapproximated Euclidean coordinates.

[14] Whereas an incompressible pure-water ocean can be very useful in theoretical studies designed to aid conceptual development and interpretation of model output, it is insufficiently accurate for realistic quantitative modelling.

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helps to set the scene for Part II on dynamical consistency. Chapter 15 answers the question: what do various conservation principles look like:

- In vector form and in axial-orthogonal-curvilinear coordinates?
- In the presence of internal and external (conservative and non-conservative) forcings?

This prepares the way for constructing diagnostic budget relations for monitoring model integrations. It too helps to set the scene for Part II on dynamical consistency.

Part II

This part provides a means to understand the significance and potential impact of approximating the general set of governing dynamical-core equations, with emphasis on the importance of maintaining analogues in any approximated equation set of physical conservation laws embodied in the general, unapproximated set. Chapters in Part II are ordered from least approximated (3D) equation sets (when compared to the most general one developed in earlier chapters) to most approximated (2D) ones. Chapters 18 and 19 on 2D shallow-water equation sets and the 2D barotropic-potential-vorticity (BPV) equation then naturally follow on from Chapters 16 and 17 on dynamical consistency in 3D.

Part III

In this part, some exact, steady and unsteady, non-linear solutions of governing dynamical-core equation sets are developed for testing dynamical cores, once constructed. Chapters in Part III are ordered from the simplest exact solutions to the most complex ones. Some steady non-linear solutions are developed in Chapters 20 and 21. Some unsteady ones are similarly developed in Chapters 22 and 23.

An Appendix on Vectors

Vectors are used in many chapters, particularly in the early ones. For convenience, and for reference purposes, a comprehensive set of vector identities is assembled in the Appendix at the end of this book.

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