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CONFORMALLY FLAT HYPERSURFACES WITH CONSTANT GAUSS-KRONECKER CURVATURE

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We consider 3-dimensional conformally flat hypersurfaces of $\mathbb{E}^4$ with constant Gauss-Kronecker curvature. We prove that those with three different principal curvatures must necessarily have zero Gauss-Kronecker curvature.

1. INTRODUCTION

A Riemannian manifold (M^n, g) is called conformally flat if every point has a neighbourhood which is conformal to an open set in the Euclidean space $\mathbb{E}^n$. In contrast to the dimensions $n = 2$ and $n \geqslant 4$, the condition for conformal flatness of 3-dimensional manifolds occupies a special place. For 2-dimensional manifolds, the existence of isothermal coordinates shows that every surface is conformally flat. For manifolds of dimension $n \geqslant 4$, the necessary and sufficient condition for conformal flatness is given by the vanishing of the Weyl-conformal curvature tensor, which involves second order derivatives of the metric tensor. In dimension $n = 3$, however, the criterium for conformal flatness is that the Brinkman tensor is a Codazzi tensor; this condition involves third order derivatives of the metric.

In particular for hypersurfaces M^n of a Euclidean space $\mathbb{E}^{n+1}$, we have in dimensions $n \geqslant 4$ a classical result by Cartan-Schouten. The induced metric of a hypersurface $M^n \subset \mathbb{E}^{n+1}$ $(n \geqslant 4)$ is conformally flat if and only if at least $n-1$ of the principal curvatures coincide at each point. Whence in dimensions $n \geqslant 4$, a conformally flat hypersurface can have at most two different principal curvatures at each point. The theorem of Cartan-Schouten was the basis for many results on conformally flat hypersurfaces $M^n \subset \mathbb{E}^{n+1}$, with dimensions $n \geqslant 4$; see for example, **[2, 4, 9]**.

In dimension $n = 3$, the result of Cartan-Schouten no longer holds, and there can be conformally flat hypersurfaces $M^3 \subset \mathbb{E}^4$ with three different principal curvatures at a point. Indeed, **[8]** has given examples of conformally flat hypersurfaces with exactly three different principal curvatures. Recently, **[5]** described more examples of such conformally flat hypersurfaces. In spite of this interesting phenomenon with three different principal curvatures, there are not so many particular results for 3-dimensional conformally flat

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hypersurfaces of $\mathbb{E}^4$. This is perhaps mainly due to the fact that the condition with the Brinkman tensor keeps its nature of a set of coupled partial differential equations of third order.

Recently, however, [5] proved the following structural theorem: for a hypersurface M^3 of $\mathbb{E}^4$ with three different principal curvatures to be conformally flat, it must allow the existence of a Guichard coordinate system (see Section 2). This is a necessary condition. Whether this condition is also sufficient, is still an open problem; at least no counterexamples are known. In order to gain more insight into this question, it would surely be valuable to have more explicit results on conformally flat hypersurfaces of $\mathbb{E}^4$ with three different principal curvatures.

In this paper we consider conformally flat hypersurfaces of $\mathbb{E}^4$ with constant Gauss-Kronecker curvature and prove the following

THEOREM. *For a conformally flat hypersurface M^3 of $\mathbb{E}^4$ with constant Gauss-Kronecker curvature τ and three different principal curvatures, the value of this constant τ must be zero.*

2. PRELIMINARIES.

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞. Denote by ∇, R, S, and κ, the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Ricci tensor, and the scalar curvature of (M^n, g), respectively. (M^n, g) is said to be conformally flat if there exists a function u such that $g = e^u \widetilde{g}$, where $\widetilde{g}$ is a locally flat metric on $\mathbb{E}^n$. In dimensions $n \geqslant 4$, a necessary and sufficient condition for a Riemannian manifold to be conformally flat, is that the Weyl conformal curvature tensor C vanishes: $C = 0$.

For 3-dimensional manifolds, the necessary and sufficient condition for conformal flatness is that the Brinkman tensor T is a Codazzi tensor, or equivalently, that the Bach tensor B vanishes. The Brinkman tensor T is defined (for an n-dimensional manifold) as

$$T(X,Y) = \frac{1}{(n-2)}\left(S(X,Y) - \frac{\kappa}{2(n-1)}g(X,Y)\right). \tag{1}$$

The Bach tensor B is then given by

$$B(X,Y,Z) = (\nabla_X T)(Y,Z) - (\nabla_Y T)(X,Z). \tag{2}$$

Finally, we also recall the Koszul formula,

$$\begin{aligned} 2\langle\nabla_X Y, Z\rangle &= X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle \\ &\quad -\big\langle X,[Y,Z]\big\rangle + \big\langle Y,[Z,X]\big\rangle + \big\langle Z,[X,Y]\big\rangle. \end{aligned} \tag{3}$$

Let now M^3 be a hypersurface of the Euclidean space $\mathbb{E}^4$. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M^3 and $\mathbb{E}^4$ respectively. For any vector fields X, Y tangent to M^3, the formula of Gauss is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi\,, \tag{4}$$

where h is the scalar-valued second fundamental form, and ξ a unit normal vector. Denote by A the shape operator of ξ, then the formula of Weingarten is given by

$$\widetilde{\nabla}_X \xi = -A(X)\,, \tag{5}$$

where $\big\langle A(X), Y\big\rangle = h(X,Y)$. The equation of Codazzi is given by

$$(\nabla_X A)Y = (\nabla_Y A)X\,. \tag{6}$$

The Gauss equation reads

$$R(X,Y)Z = A(X)\big\langle A(Y), Z\big\rangle - A(Y)\big\langle A(X), Z\big\rangle\,. \tag{7}$$

A system of local coordinates (x^1, x^2, x^3) for a 3-dimensional hypersurface $M^3 \subset \mathbb{E}^4$ with induced metric g is said to be a Guichard coordinate system if, with respect to $\{x^i\}_{i=1}^3$ the metric takes the following form:

$$g = {l_1}^2\big(x^i\big)\big(dx^1\big)^2 + {l_2}^2\big(x^i\big)\big(dx^2\big)^2 + {l_3}^2\big(x^i\big)\big(dx^3\big)^2\,, \tag{8}$$

with

$$ {l_1}^2 + {l_2}^2 - {l_3}^2 = 0\,; \tag{9}$$

moreover the coordinate lines have to be curvature lines. With the notation $E_i := \dfrac{\partial}{\partial x^i}$, one thus has that

$$\langle E_i, E_j\rangle = \delta_{ij}{l_i}^2\big(x^i\big)\,. \tag{10}$$

So, $\{E^i\}_{i=1}^3$ is an orthogonal local frame which is not orthonormal.

[5] proved that a 3-dimensional conformally flat hypersurface M^3 of $\mathbb{E}^4$ with three different principal curvatures must allow a Guichard coordinate system. Whilst this results in a necessary condition for a hypersurface M^3 of $\mathbb{E}^4$ to be conformally flat, the converse is still an open problem. The question whether this condition may also be a sufficient one, is partially inspired by the observation that the relation (9) may be seen as some generalisation of isothermal coordinates in two dimensions.

For later use, we first derive a property for conformally flat hypersurfaces of $\mathbb{E}^4$ with three different principal curvatures. Consider a local orthonormal frame $\{e_i\}_{i=1}^3$ consisting of eigenvectors of the shape operator A, and thus diagonalising A; denote by $\omega_i^k(e_j)$ the components of the corresponding Levi-Civita connection, thus $\nabla_{e_i} e_j = \omega_i^k(e_j)e_k$. Write $\Omega_i^k(e_j)$ for the components of the Levi-Civita connection, with respect to the natural orthogonal basis $\{E_i\}_{i=1}^3$ corresponding to the Guichard coordinate system, thus $\nabla_{E_i}E_j = \Omega_i^k(E_j)E_k$. Since the integral curves of the local frames $\{e_i\}_{i=1}^3$ and $\{E_i\}_{i=1}^3$

coincide, one sees that $E_i = l_i e_i$ $(i = 1, 2, 3)$. The transformation rule between the two sets of connection coefficients takes the following form:

$$\omega_i^k(e_j) = \frac{l_k}{l_i l_j}\left(-\frac{E_i(l_j)}{l_j}\delta_j^k + \Omega_i^k(E_j)\right). \tag{11}$$

Since the derivations E_i $(i = 1, 2, 3)$ all commute, the components $\Omega_i^k(E_j) = 0$, for i, j, k all different, and whence from (11) also

$$\omega_i^k(e_j) = 0\,, \qquad (i \neq j, j \neq k, k \neq i)\,. \tag{12}$$

3. Hypersurfaces with constant Gauss-Kronecker curvature

Since we consider C^∞ manifolds and work locally, we can confine ourselves to points with a neighbourhood on which all three principal curvatures λ_1, λ_2, and λ_3 are strictly different. Thus, we assume that on a neighbourhood U of a point p of M^3 we have that

$$\lambda_1 - \lambda_2 \neq 0\,,\ \lambda_2 - \lambda_3 \neq 0\,,\ \lambda_3 - \lambda_1 \neq 0\,. \tag{13}$$

By assumption, between λ_1, λ_2, and λ_3, the following relation holds, with constant τ:

$$\lambda_1\lambda_2\lambda_3 = \tau\,. \tag{14}$$

The Codazzi equations (6) for $\big\langle(\nabla_{e_1}A)e_2, e_1\big\rangle$, and $\big\langle(\nabla_{e_1}A)e_2, e_2\big\rangle$, readily give that

$$\omega_1^1(e_2) = \frac{e_2(\lambda_1)}{\lambda_2 - \lambda_1}\,, \quad \omega_2^2(e_1) = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\,. \tag{15}$$

Analogously, The Codazzi equations (6) for $\big\langle(\nabla_{e_2}A)e_3, e_2\big\rangle$, $\big\langle(\nabla_{e_2}A)e_3, e_3\big\rangle$, and $\big\langle(\nabla_{e_3}A)e_1, e_3\big\rangle$, $\big\langle(\nabla_{e_3}A)e_1, e_1\big\rangle$, respectively, give similar expressions for the remaining connection coefficients, which also follow from (15) under cyclic permutation of the indices $(1 \to 2 \to 3 \to 1)$.

The conditions following from the vanishing of the Bach tensor (2) for the choices of $(X, Y, Z) = (e_1, e_1, e_2)$, (e_2, e_2, e_3), and (e_3, e_3, e_1), respectively, together with e_j $(j = 1, 2, 3)$ applied on the relation (14), form an underdetermined set of algebraic equations for the $e_i(\lambda_j)$ $(1 \leqslant i, j \leqslant 3)$.
For example, concerning the derivatives $e_1(\lambda_i)$ $(i = 1, 2, 3)$, we get the following subsystem

$$\begin{aligned} -(\lambda_2 - \lambda_3)e_1(\lambda_1) + (\lambda_3 - \lambda_1)e_1(\lambda_2) + (\lambda_1 - \lambda_2)e_1(\lambda_3) &= 0\,, \\ \lambda_2\lambda_3 e_1(\lambda_1) + \lambda_1\lambda_3 e_1(\lambda_2) + \lambda_1\lambda_2 e_1(\lambda_3) &= 0\,. \end{aligned}$$

However, introducing the unknown functions A_i $(i = 1, 2, 3)$, allows us to write the derivatives of the λ_i $(i = 1, 2, 3)$ as

$$e_j(\lambda_i) = A_j(\lambda_i)^2(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)^2 \quad \text{for } j \neq i\,, \tag{16}$$

with $k \neq i$ and $k \neq j$, and

$$e_i(\lambda_i) = -A_i\lambda_i(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)(\lambda_i\lambda_j + \lambda_i\lambda_k - 2\lambda_j\lambda_k)\,, \tag{17}$$

with $j \neq i$, $k \neq i$, and $j \neq k$. The A_i $(i = 1,2,3)$ are thus functions which are not yet determined. In view of (16), (15) now takes the form

$$\omega_1^1(e_2) = A_2(\lambda_1)^2(\lambda_2 - \lambda_3)^2\,, \quad \omega_2^2(e_1) = A_1(\lambda_2)^2(\lambda_1 - \lambda_3)^2\,. \tag{18}$$

or, in general, with $i = 1,2,3$

$$\omega_i^i(e_j) = A_j(\lambda_i)^2(\lambda_j - \lambda_k)^2\,, \quad \text{for } j \neq i\,, \tag{19}$$

and with $k \neq i$, and $k \neq j$. Next, the Gauss equations (7) give the following information on the derivatives of the A_i $(i = 1,2,3)$ with respect to the e_j $(j = 1,2,3)$:

$$e_j(A_i) = A_iA_j\lambda_i(\lambda_j - \lambda_k)^2(3\lambda_i + 2\lambda_j)\,, \quad \text{for } j \neq i\,, \tag{20}$$

and with $k \neq i$, and $k \neq j$; and

$$\begin{aligned} e_1(A_1) = \frac{1}{\tau}\Bigg(&\frac{\lambda_1^2(\lambda_2+\lambda_3)}{2(\lambda_3-\lambda_1)(\lambda_1-\lambda_2)} \\ &+(A_1)^2\tau\Big(\lambda_1^3\lambda_2 + \lambda_1^2\lambda_2^2 + \lambda_1^3\lambda_3 + \lambda_1^2\lambda_3^2 + 3\lambda_2^2\lambda_3^2 - 3\lambda_1\tau - 2\lambda_2\tau - 2\lambda_3\tau\Big) \\ &-(A_2)^2\frac{(\lambda_3-\lambda_2)^2}{(\lambda_3-\lambda_1)}\lambda_1^2\Big(4\lambda_1^2\lambda_3^2 - 2\lambda_1\tau - \lambda_2\tau - \lambda_3\tau\Big) \\ &+(A_3)^2\frac{(\lambda_2-\lambda_3)^2}{(\lambda_1-\lambda_2)}\lambda_1^2\Big(4\lambda_1^2\lambda_2^2 - 2\lambda_1\tau - \lambda_2\tau - \lambda_3\tau\Big)\Bigg)\,, \end{aligned} \tag{21}$$

and analogous expressions for $e_2(A_2)$ and $e_3(A_3)$ under cyclic permutation of the indices $(1 \to 2 \to 3 \to 1)$. We have written λ_i^p for $(\lambda_i)^p$ $(i = 1,2,3; p \in \mathbb{N})$; we shall use the same convention in the sequel for reasons of notational brevity.

The above expressions (21) and the analogous ones, are only valid under the condition that $\tau \neq 0$. We show however that the assumption that $\tau \neq 0$ in a neighbourhood U runs into contradiction. The condition $\tau \neq 0$ implies of course that none of the λ_i $(i = 1,2,3)$ can be zero on U.

Now, we consider the system (16)-(17) and (20)-(21) of coupled partial differential equations for the λ_i and A_i $(i = 1,2,3)$. The compatibility conditions for this set give the following 6 equations: $(i = 1,2,3)$

$$\begin{aligned} 0 = \frac{A_i(\lambda_j-\lambda_i)}{2(\lambda_i-\lambda_k)(\lambda_j-\lambda_k)\tau^2}\Big(\lambda_i^3\lambda_j^3\tau^3\Big(&-\lambda_i^5\lambda_j^3 - 3\lambda_i^4\lambda_j^4 - \lambda_i^4\lambda_j\tau - 9\lambda_i^3\lambda_j^2\tau \\ &+10\lambda_i^2\lambda_j^3\tau - 3\lambda_i^2\tau^2 + 10\lambda_i\lambda_j\tau^2 - 3\lambda_j^2\tau^2\Big) \\ +4(A_i)^2\lambda_j^2\tau^4(\lambda_j-\lambda_i)^2\Big(\tau - \lambda_i^2\lambda_j\Big)^2 \end{aligned}$$

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$$
\begin{aligned}
&\qquad\qquad \Big(-\lambda_i^5\lambda_j^2 - \lambda_i^4\tau - 2\lambda_i^2\lambda_j^2\tau - 2\lambda_i\tau^2 + 6\lambda_j\tau^2\Big)\\
&+4(A_j)^2\lambda_i^2\tau^4(\lambda_i-\lambda_j)^2\Big(\tau-\lambda_i\lambda_j^2\Big)^2\\
&\qquad\qquad \Big(-\lambda_i^4\lambda_j^3 + 2\lambda_i^3\lambda_j\tau - 3\lambda_i^2\lambda_j^2\tau - 2\lambda_i\lambda_j^3\tau + 6\lambda_i\tau^2 - 2\lambda_j\tau^2\Big)\\
&+4(A_k)^2\lambda_i\lambda_j\tau^3\Big(\tau-\lambda_i^2\lambda_j\Big)^2\Big(\tau-\lambda_i\lambda_j^2\Big)^2\\
&\qquad\qquad \Big(2\lambda_i^4\lambda_j^2 + 6\lambda_i^3\lambda_j^3 - \lambda_i^3\tau - 3\lambda_i^2\lambda_j\tau - 2\lambda_i\lambda_j^2\tau - 2\tau^2\Big)\Big)\,,
\end{aligned}
\tag{22}
$$

for $j \neq i$, and with $k \neq i$, and $k \neq j$. We remark that at least one A_i ($i \in \{1,2,3\}$) has to be different from zero. Indeed, if all A_i were zero in a neighbourhood, then in view of (16)-(17) all principal curvatures would have to be constant. Hence the hypersurface would be isoparametric, and could have not more than two different principal curvatures at every point. This however contradicts our assumption.
Therefore, we can assume that for example,

$$
A_1 \neq 0\,. \tag{23}
$$

Then, from (22), we see that

$$
\begin{aligned}
0 = {}& 4(A_1)^2(\lambda_1-\lambda_2)^2\lambda_2(\lambda_1-\lambda_3)^2\lambda_3\Big(\lambda_1^3\lambda_2 + \lambda_1^3\lambda_3 + 2\lambda_1\lambda_2^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3^2 - 6\lambda_2^2\lambda_3^2\Big)\\
&+4(A_2)^2\lambda_1^2(\lambda_1-\lambda_2)^2(\lambda_2-\lambda_3)^2\lambda_3\Big(\lambda_1^2\lambda_2 - 2\lambda_1^2\lambda_3 + 3\lambda_1\lambda_2\lambda_3 + 2\lambda_2^2\lambda_3 - 6\lambda_1\lambda_3^2 + 2\lambda_2\lambda_3^2\Big)\\
&-4(A_3)^2\lambda_1^2\lambda_2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2\Big(2\lambda_1^2\lambda_2 + 6\lambda_1\lambda_2^2 - \lambda_1^2\lambda_3 - 3\lambda_1\lambda_2\lambda_3 - 2\lambda_2^2\lambda_3 - 2\lambda_2\lambda_3^2\Big)\\
&+\lambda_1^3\lambda_2 + 3\lambda_1^2\lambda_2^2 + \lambda_1^3\lambda_3 + 9\lambda_1^2\lambda_2\lambda_3 - 10\lambda_1\lambda_2^2\lambda_3 + 3\lambda_1^2\lambda_3^2 - 10\lambda_1\lambda_2\lambda_3^2 + 3\lambda_2^2\lambda_3^2\,.
\end{aligned}
\tag{24}
$$

Taking the derivative of (24) with respect to e_1, in view of (16)-(17), (20)-(21), and upon cancellation of the nonzero factors (13), (23), and $\tau \neq 0$, we get the following additional condition

$$
\begin{aligned}
0 = {}& 2(A_1)^2(\lambda_1-\lambda_2)^2\lambda_2(\lambda_1-\lambda_3)^2\lambda_3\,\Big(3\lambda_1^4\lambda_2^2 + 4\lambda_1^4\lambda_2\lambda_3 - 7\lambda_1^3\lambda_2^2\lambda_3 + 12\lambda_1^2\lambda_2^3\lambda_3\\
&\qquad +3\lambda_1^4\lambda_3^2 - 7\lambda_1^3\lambda_2\lambda_3^2 + 20\lambda_1^2\lambda_2^2\lambda_3^2 - 68\lambda_1\lambda_2^3\lambda_3^2 + 12\lambda_1^2\lambda_2\lambda_3^3 - 68\lambda_1\lambda_2^2\lambda_3^3 + 96\lambda_2^3\lambda_3^3\Big)\\
&+2(A_2)^2\lambda_1^2(\lambda_1-\lambda_2)^2(\lambda_2-\lambda_3)^2\lambda_3\,\Big(3\lambda_1^3\lambda_2^2 - 5\lambda_1^3\lambda_2\lambda_3 + 2\lambda_1^2\lambda_2^2\lambda_3 + 12\lambda_1\lambda_2^3\lambda_3\\
&\qquad -\lambda_1^2\lambda_2\lambda_3^2 - 19\lambda_1\lambda_2^2\lambda_3^2 - 32\lambda_2^3\lambda_3^2 - 36\lambda_1^2\lambda_3^3 + 108\lambda_1\lambda_2\lambda_3^3 - 32\lambda_2^2\lambda_3^3\Big)\\
&-2(A_3)^2\lambda_1^2\lambda_2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2\,\Big(36\lambda_1^2\lambda_2^3 + 5\lambda_1^3\lambda_2\lambda_3 + \lambda_1^2\lambda_2^2\lambda_3 - 108\lambda_1\lambda_2^3\lambda_3\\
&\qquad -3\lambda_1^3\lambda_3^2 - 2\lambda_1^2\lambda_2\lambda_3^2 + 19\lambda_1\lambda_2^2\lambda_3^2 + 32\lambda_2^3\lambda_3^2 - 12\lambda_1\lambda_2\lambda_3^3 + 32\lambda_2^2\lambda_3^3\Big)\\
&+9\lambda_1^3\lambda_2^3 - \lambda_1^4\lambda_2\lambda_3 + 25\lambda_1^3\lambda_2^2\lambda_3 - 54\lambda_1^2\lambda_2^3\lambda_3 + 25\lambda_1^3\lambda_2\lambda_3^2\\
&\qquad -113\lambda_1^2\lambda_2^2\lambda_3^2 + 89\lambda_1\lambda_2^3\lambda_3^2 + 9\lambda_1^3\lambda_3^3 - 54\lambda_1^2\lambda_2\lambda_3^3 + 89\lambda_1\lambda_2^2\lambda_3^2 - 24\lambda_2^3\lambda_3^3\,.
\end{aligned}
\tag{25}
$$

Taking again the derivative of (25) with respect to e_1, in view of (16)-(17), (20)-(21), and after simplification with nonvanishing factors, we get yet another necessary condition

$$\begin{aligned}
0 = {} & 2(A_1)^2(\lambda_1-\lambda_2)^2\lambda_2(\lambda_1-\lambda_3)^2\lambda_3\Big(12\lambda_1^5\lambda_2^3+18\lambda_1^5\lambda_2^2\lambda_3-59\lambda_1^4\lambda_2^3\lambda_3+84\lambda_1^3\lambda_2\lambda_3^2\\
&-88\lambda_1^4\lambda_2^2\lambda_3^2+257\lambda_1^3\lambda_2^3\lambda_3^2-712\lambda_1^2\lambda_2^4\lambda_3^2+12\lambda_1^5\lambda_3^3-59\lambda_1^4\lambda_2\lambda_3^3+257\lambda_1^3\lambda_2^2\lambda_3^3\\
&-1232\lambda_1^2\lambda_2^3\lambda_3^3+2020\lambda_1\lambda_2^4\lambda_3^3+84\lambda_1^3\lambda_2\lambda_3^4-712\lambda_1^2\lambda_2^2\lambda_3^4+2020\lambda_1\lambda_2^3\lambda_3^4-1920\lambda_2^4\lambda_3^4\Big)\\
&+2(A_2)^2\lambda_1^2(\lambda_1-\lambda_2)^2(\lambda_2-\lambda_3)^2\lambda_3\Big(12\lambda_1^4\lambda_2^3-18\lambda_1^4\lambda_2^2\lambda_3-23\lambda_1^3\lambda_2^3\lambda_3+84\lambda_1^2\lambda_2^4\lambda_3\\
&+35\lambda_1^3\lambda_2^2\lambda_3^2-100\lambda_1^2\lambda_2^3\lambda_3^2-460\lambda_1\lambda_2^4\lambda_3^2+24\lambda_1^4\lambda_3^3-296\lambda_1^3\lambda_2\lambda_3^3+725\lambda_1^2\lambda_2^2\lambda_3^3\\
&-95\lambda_1\lambda_2^3\lambda_3^3+640\lambda_2^4\lambda_3^3-252\lambda_1^3\lambda_3^4+1464\lambda_1^2\lambda_2\lambda_3^4-2380\lambda_1\lambda_2^2\lambda_3^4+640\lambda_2^3\lambda_3^4\Big)\\
&+2(A_3)^2\lambda_1^2\lambda_2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2\Big(24\lambda_1^4\lambda_2^3-252\lambda_1^3\lambda_2^4-296\lambda_1^3\lambda_2^3\lambda_3+1464\lambda_1^2\lambda_2^4\lambda_3\\
&-18\lambda_1^4\lambda_2\lambda_3^2+35\lambda_1^3\lambda_2^2\lambda_3^2+725\lambda_1^2\lambda_2^3\lambda_3^2-2380\lambda_1\lambda_2^4\lambda_3^2+12\lambda_1^4\lambda_3^3-23\lambda_1^3\lambda_2\lambda_3^3\\
&-100\lambda_1^2\lambda_2^2\lambda_3^3-95\lambda_1\lambda_2^3\lambda_3^3+640\lambda_2^4\lambda_3^3+84\lambda_1^2\lambda_2\lambda_3^4-460\lambda_1\lambda_2^2\lambda_3^4+640\lambda_2^3\lambda_3^4\Big)\\
&-6\lambda_1^5\lambda_2^3+63\lambda_1^4\lambda_2^4-18\lambda_1^5\lambda_2^2\lambda_3+200\lambda_1^4\lambda_2^3\lambda_3-555\lambda_1^3\lambda_2^4\lambda_3-18\lambda_1^5\lambda_2\lambda_3^2+286\lambda_1^4\lambda_2^2\lambda_3^2\\
&-1454\lambda_1^3\lambda_2^3\lambda_3^2+1693\lambda_1^2\lambda_2^4\lambda_3^2-6\lambda_1^5\lambda_3^3+200\lambda_1^4\lambda_2\lambda_3^3-1454\lambda_1^3\lambda_2^2\lambda_3^3+3278\lambda_1^2\lambda_2^3\lambda_3^3\\
&-1945\lambda_1\lambda_2^4\lambda_3^3+63\lambda_1^4\lambda_3^4-555\lambda_1^3\lambda_2\lambda_3^4+1693\lambda_1^2\lambda_2^2\lambda_3^4-1945\lambda_1\lambda_2^3\lambda_3^4+480\lambda_2^4\lambda_3^4\,.
\end{aligned} \tag{26}$$

Since $A_1\neq 0$ and $\lambda_1-\lambda_2\neq 0$, $\lambda_2-\lambda_3\neq 0$, $\lambda_3-\lambda_1\neq 0$, we get an inhomogeneous system of 3 linear equations in $(A_1)^2$, $(A_2)^2$, and $(A_3)^2$. One can check that the determinant of the coefficient matrix of this system of 3 linear equations is zero, taken into account that $\tau=\lambda_1\lambda_2\lambda_3$. The solvability condition yields the surprisingly simple equation

$$\lambda_1^3\lambda_2-6\lambda_1^2\lambda_2^2+\lambda_1^3\lambda_3-12\lambda_1^2\lambda_2\lambda_3+26\lambda_1\lambda_2^2\lambda_3-6\lambda_1^2\lambda_3^2+26\lambda_1\lambda_2\lambda_3^2-30\lambda_2^2\lambda_3^2=0\,, \tag{27}$$

or, equivalently,

$$0=30\lambda_2^5\lambda_3^5-26\lambda_2^4\lambda_3^3\tau-26\lambda_2^3\lambda_3^4\tau+6\lambda_2^3\lambda_3\tau^2+12\lambda_2^2\lambda_3^2\tau^2+6\lambda_2\lambda_3^3\tau^2-\lambda_2\tau^3-\lambda_3\tau^3 \tag{28}$$

when written as a function of λ_2 and λ_3 only. Taking the derivative of (28) with respect to e_1, in view of the relation $\tau=\lambda_1\lambda_2\lambda_3$, leads to a second, independent algebraic relation between λ_2 and λ_3, also with constant coefficients

$$\begin{aligned}
0 = {} & -120\lambda_2^7\lambda_3^7+86\lambda_2^6\lambda_3^5\tau+86\lambda_2^5\lambda_3^6\tau-14\lambda_2^5\lambda_3^3\tau^2+24\lambda_2^4\lambda_3^4\tau^2-14\lambda_2^3\lambda_3^5\\
&\tau^2-29\lambda_2^3\lambda_3^2\tau^3-29\lambda_2^2\lambda_3^3\tau^3+2\lambda_2^2\tau^4+6\lambda_2\lambda_3\tau^4+2\lambda_2^2\tau^4\,.
\end{aligned} \tag{29}$$

By elimination of either λ_2 or λ_3 between (28) and (29), we see that both of them satisfy the same algebraic equation with constant coefficients, involving the constant τ:

$$\begin{aligned}
0=F(X,\tau) = {} & 696960X^{18}-4346880X^{15}\tau-326752X^{12}\tau^2\\
&+16015X^9\tau^3+11204X^6\tau^4+3040X^3\tau^5-352\tau^6\,;
\end{aligned} \tag{30}$$

that is, we have that both $F(\lambda_2,\tau)=0$ and $F(\lambda_3,\tau)=0$. Without having to solve this algebraic equation (30) explicitly, this shows that λ_2, and λ_3, and hence also λ_1 (by

$\tau = \lambda_1\lambda_2\lambda_3$), should all be constant. However, in such case, the hypersurface would be isoparametric. But it is well-known that an isoparametric hypersurface of a Euclidean space can have at most two different pricipal curvatures, which contradicts our assumption.

Summarising, we have shown that $\tau \neq 0$ for a conformally flat hypersurface of $\mathbb{E}^4$ with three different principal curvatures and constant Gauss-Kronecker curvature τ, runs into contradiction. We conclude that τ must be zero.

This proves the following

THEOREM. *A conformally flat hypersurface M^3 of $\mathbb{E}^4$ with constant Gauss-Kronecker curvature τ and having three different principal curvatures at every point must necessarily have $\tau = 0$.*

With $\tau = 0$ as a necessary condition, from relation (14) it follows that at least one of the λ_i must be zero. Since the three principal curvatures are assumed different (13), we conclude that exactly one of them is equal to zero. Thus, for example,

$$\lambda_1 = 0\,, \quad \lambda_2 \neq 0\,, \quad \lambda_3 \neq 0\,, \quad \lambda_2 \neq \lambda_3\,. \tag{31}$$

REMARK. To finish, we still have to prove the existence of nontrivial conformally flat hypersurfaces of $\mathbb{E}^4$ with three different principal curvatures and having constant Gauss-Kronecker curvature $\tau = 0$. In order to do so, we construct an example following the same scheme; however we have to start anew from the beginning. In view of (31) we thus take $\lambda_1 = 0$. Then, the same reasoning which led to the expressions (16) and (17), now gives for the derivatives of λ_i ($i = 2, 3$):

$$e_1(\lambda_i) = A_1(\lambda_i)^2\lambda_j\,, \qquad j \neq i\,, \quad 2 \leqslant i, j \leqslant 3\,, \tag{32}$$

$$e_j(\lambda_i) = -A_j\lambda_i(\lambda_2 - \lambda_3)\,, \qquad j \neq i\,, \quad 2 \leqslant i, j \leqslant 3\,, \tag{33}$$

$$e_i(\lambda_i) = A_i\lambda_i(\lambda_2 - \lambda_3)\,, \qquad 2 \leqslant i \leqslant 3\,. \tag{34}$$

Then, proceeding similarly, the only nonzero connection coefficients are the following

$$\omega_2^2(e_1) = \omega_3^3(e_1) = -A_1\lambda_2\lambda_3\,, \tag{35}$$

$$\omega_2^2(e_3) = A_3\lambda_2\,, \tag{36}$$

$$\omega_3^3(e_2) = -A_2\lambda_3\,. \tag{37}$$

At this point, we make an Ansatz, and look for solutions to the system with $A_2 = A_3 = 0$. Under these assumptions, the Gauss equations reduce to

$$(A_1)^2 = -\frac{1}{\lambda_2\lambda_3}\,, \tag{38}$$

$$e_1(A_1) = -(A_1)^2\lambda_2\lambda_3\,. \tag{39}$$

We now choose $\lambda_3 = -\lambda_2$; then (38) can be solved by

$$A_1 = \frac{1}{\lambda_2}, \tag{40}$$

and all other remaining equations which are nontrivial, amount to the same equation

$$e_1(\lambda_2) = -(\lambda_2)^2. \tag{41}$$

The only nonzero connection coefficients which are left, are

$$\omega_2^2(e_1) = \omega_3^3(e_1) = \lambda_2, \tag{42}$$

and those related to them by symmetry. In order to find an explicit solution, we proceed as follows.

$$\frac{e_1(l_2)}{l_2} = \frac{e_1(l_3)}{l_3} = \lambda_2, \tag{43}$$

and, in view of (9), we also have that for $i = 1, 2, 3$

$$e_j(l_i) = 0, \quad \text{for } j = 2, 3. \tag{44}$$

With (44), (9) shows that the l_i ($i = 1, 2, 3$) can be written as follows

$$l_1 = l_1\left(x^1\right), \tag{45}$$

$$l_2 = l_1\left(x^1\right)\mathrm{sh}C, \tag{46}$$

$$l_3 = l_1\left(x^1\right)\mathrm{ch}C, \tag{47}$$

with C a constant. Finally, comparing $\omega_3^3(e_1)$ via (15) and (19), taking into account (43), one can check that a solution is given for example by the following expressions

$$l_1 = e^{x^1}, \tag{48}$$

$$\lambda_2 = e^{-x^1}. \tag{49}$$

REMARK. In [**3**] we considered conformally flat hypersurfaces M^3 of $\mathbb{E}^4$ with constant mean curvature. If M^3 has three different principal curvatures, we proved that the hypersurface must be minimal. In this different context, we arrived basically at the same example as derived here, and thus showed how it fits in the setting there. Indeed, one can verify that the example has both zero mean curvature and zero Gauss-Kronecker curvature. Thus, in view of the theorems here and in [**3**], it is no surprise that for a conformally flat hypersurface M^3 of $\mathbb{E}^4$ with three different principal curvatures, and with constant mean curvature H and constant Gauss-Kronecker curvature τ, both values of H and τ must be zero. It is perhaps more surprising that they exist at all. The example is however still in agreement with a result by Kohlmann, following which a convex hypersurface M^n of a real space form $\widetilde{N}^{n+1}(c)$, with two constant generalised curvatures H_r and H_s, where $r \in \{1, 2\}$ and $s \in \{n-1, n\}$, is isoparametric. Our example is indeed nonconvex. For a discussion and survey of results of this type in the comparable case of a hypersurface M^3 in the ambient space $S^4(1)$ of constant positive sectional curvature, we can refer to [**1**], and references therein.

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