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Multidimensional Exponential Inequalities with Weights

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Abstract. We establish sufficient conditions on the weight functions u and v for the validity of the multidimensional weighted inequality

$$\Big(\int_E \Phi(T_k f(x))^q u(x)\,dx\Big)^{1/q} \le C\Big(\int_E \Phi(f(x))^p v(x)\,dx\Big)^{1/p},$$

where $0 < p, q < \infty$, Φ is a logarithmically convex function, and T_k is an integral operator over star-shaped regions. The condition is also necessary for the exponential integral inequality. Moreover, the estimation of C is given and we apply the obtained results to generalize some multidimensional Levin–Cochran-Lee type inequalities.

1 Introduction

We investigate the weighted modular inequality of the form

$$\Big(\int_E \Phi(T_k f(x))^q u(x)\,dx\Big)^{1/q} \le C\Big(\int_E \Phi(f(x))^p v(x)\,dx\Big)^{1/p}, \tag{1.1}$$

where $0 < p, q < \infty$, u and v are weight functions, Φ is logarithmically convex, and T_k is the integral operator defined by

$$T_k f(x) := \int_{S_x} k(x,t) f(t)\,dt, \quad x \in E,$$

which averages functions over dilations of a fixed star-shaped region S in $\mathbb{R}^n$ (the terms S, S_x, and E are defined below). The kernel k is a positive function defined on $\Omega = \{(x,t) \in E \times E : t \in S_x\}$. A weight function is a measurable function which is positive and finite almost everywhere on E. The function Φ is said to be logarithmically convex on an open interval $I \subseteq (-\infty, \infty)$ if Φ is defined and positive on I such that $\log \Phi$ is convex on I. We also assume that Φ takes its limits, finite or infinite, at the ends of I. In particular, if $\Phi(x) = e^x$ and we replace f by $\log f$, then (1.1) can be reduced to

$$\Big(\int_E (G_k f(x))^q u(x)\,dx\Big)^{1/q} \le C\Big(\int_E f(x)^p v(x)\,dx\Big)^{1/p}, \tag{1.2}$$

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where $f \geq 0$ and G_k is the geometric mean operator defined by

$$G_k f(x) := \exp\Big(\int_{S_x} k(x,t) \log f(t)\, dt \Big).$$

In the one-dimensional case with $S_x = (0, x]$, inequality (1.1) has been considered by Levinson [18] for $k(x,t) = r(t)/(\int_0^x r(t)dt)$, $p = q = 1$, and $u(x) = v(x) = 1$ with $C = e$, and by Heinig [9, Theorem 2.2(ii)] for $k(x,t) = 1/x$, $p = q = 1$, and $v(x) = x^\alpha \int_x^\infty t^{-\alpha-1} u(t)dt$, $\alpha > 0$ with $C = e^\alpha$. Inequality (1.2) has also been investigated by many authors (see [2,4,7,8,10–13,15–17,19–21,23,24] and the references therein). The higher dimensional theory of (1.1) and (1.2) for $k(x,t) = \alpha |S_x|^{-\alpha} |S_t|^{\alpha-1}$ is discussed by Heinig [9], Drábek–Heinig–Kufner [5], Jain–Persson–Wedestig [14] for $\alpha = 1$ and by Čižmešija–Pečarić–Perić [3] for $\alpha > 0$. In these papers, $E = \mathbb{R}^n$, S_x is the ball $B(|x|)$ in $\mathbb{R}^n$ centered at the origin and of radius $|x|$, and $|S_x|$ is the volume of S_x. Gupta *et al.* [6] also considered (1.2) for the case when $\alpha = 1$, E is a spherical cone in $\mathbb{R}^n$, and S_x is the part of E such that the length of every element in S_x is less than $|x|$.

We call a region S *smoothly star-shaped* if there exists a nonnegative, piece-wise-C^1 function ψ defined on the unit sphere in $\mathbb{R}^n$ with $S = \{x \in \mathbb{R}^n \backslash \{0\} : |x| \leq \psi(x/|x|)\}$. Throughout this paper, we denote $E = \bigcup_{\alpha>0} \alpha S$, where $S \subseteq \mathbb{R}^n$ is a smoothly star-shaped region. For nonzero $x \in E$, there is a least positive dilation $\alpha_x S$ that contains x. We write $S_x = \alpha_x S$. Let $B = \{x \in \mathbb{R}^n \backslash \{0\} : |x| = \psi(x/|x|)\}$ and note that $x/\alpha_x \in B$ so that x is on the boundary of S_x. For nonzero $x, t \in E$, we make the changes of variables $x = s\sigma$ and $t = y\tau$, where $s, y \in (0, \infty)$ and $\sigma, \tau \in B$. Then $\alpha_x = s$, and for any measurable g, we have

$$\begin{aligned} \int_{S_x} g(t)\, dt &= \int_0^s \int_B g(y\tau) y^{n-1}\, d\tau dy; \\ \int_{E \backslash S_x} g(t)\, dt &= \int_s^\infty \int_B g(y\tau) y^{n-1}\, d\tau dy. \end{aligned} \tag{1.3}$$

The volume of S_x, denoted by $|S_x|$, is then $|S_x| = \int_{S_x} dt = s^n |B|/n$.

In this paper, we consider $k \colon \Omega \mapsto (0, \infty)$ and k satisfies the following conditions.

(K1) $\int_{S_x} k(x,t)\, dt = 1$ for all nonzero $x \in E$.

(K2) For any $\epsilon > 0$, there exists $M(\epsilon) > 0$ such that

$$\exp\Big(\int_{S_x} k(x,t) \log[k(x,t)^{-1} |S_t|^{\epsilon-1}]\, dt \Big) \geq M(\epsilon) |S_x|^\epsilon \text{ for all nonzero } x \in E.$$

Our main object is to find a condition on weight functions u, v so that (1.1) holds with a finite constant C independent of f. In particular, a characterization is established for (1.2) to hold. The estimation of C is also given. Furthermore, we discuss some applications of our main results to the case $k(x,t) = |S_x|^{-1} \ell(|S_t|/|S_x|)$, which includes $k(x,t) = \alpha |S_x|^{-\alpha} |S_t|^{\alpha-1}$ and $\alpha |S_x|^{-\alpha} (|S_x| - |S_t|)^{\alpha-1}$ for $\alpha > 0$. Our results are generalizations of works of [3, 5, 6, 14].

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We assume that all functions involved in this paper are measurable on their domains. For $0 < p < \infty$ and $\eta\colon E \mapsto [0, \infty]$, define

$$L^+_{p,\eta} := \Big\{ f\colon E \mapsto [0,\infty] : \|f\|_{p,\eta} = \Big(\int_E f(x)^p \eta(x)\,dx\Big)^{1/p} < \infty \Big\}.$$

If $\eta \equiv 1$, we write L^+_p instead of $L^+_{p,\eta}$. For $0 < z < \infty$, we define z^* by $1/z + 1/z^* = 1$. We also take $\exp(-\infty) = 0$, $\log 0 = -\infty$, and $0 \cdot \infty = 0$.

2 Main Results

To prove the main results, we need the following Theorem A, which was proved by G. Sinnamon [26, Theorem 2.1]. The upper estimation of C in (2.2) for $p \le q$ is based on the results [26, Theorem 2.2] and [22, Lemma 3.2].

Theorem A **(Sinnamon)** *Let $0 < q < \infty$, $1 < p < \infty$, and ρ, η are nonnegative functions on E. Then*

$$\Big(\int_E \Big(\int_{S_x} f(t)\,dt\Big)^q \rho(x)dx\Big)^{1/q} \le C\Big(\int_E f(x)^p \eta(x)dx\Big)^{1/p} \tag{2.1}$$

holds for all $f \in L^+_{p,\eta}$ if and only if $A < \infty$, where

$$A = \begin{cases} \sup\limits_{z\in E\setminus\{0\}} \Big(\int_{S_z} \eta(x)^{1-p^*}\,dx\Big)^{1/p^*} \Big(\int_{E\setminus S_z} \rho(x)dx\Big)^{1/q} & \text{if } p \le q,\\ \Big\{\int_E \Big(\int_{S_z} \eta(t)^{1-p^*}\,dt\Big)^{r/p^*} \Big(\int_{E\setminus S_z} \rho(t)\,dt\Big)^{r/p} \rho(z)\,dz\Big\}^{1/r} & \text{if } q < p, \end{cases}$$

and $1/r = 1/q - 1/p$. Moreover, the best constant C in (2.1) satisfies

$$\begin{cases} A \le C \le (1 + q/p^*)^{1/q}(1 + p^*/q)^{1/p^*} A & \text{if } p \le q,\\ q^{1/p}(p^*)^{1/p^*}(1 - q/p)A \le C \le p^{1/p}(p^*)^{1/p^*}(r/q)^{1/r}A & \text{if } q < p. \end{cases} \tag{2.2}$$

Let $0 < p, q < \infty$, $k\colon \Omega \mapsto (0,\infty)$, u, v be weight functions on E, and condition (2.3) hold.

$$\int_{S_x} k(x,t)\log(1/v(t))\,dt \text{ is well defined and finite for all nonzero } x \in E. \tag{2.3}$$

Define $w(x) = G_k(1/v)(x)^{q/p}u(x)$. For $p \le q$, we define

$$A_\delta := \sup_{z\in E\setminus\{0\}} |S_z|^{(\delta-1)/p}\Big(\int_{E\setminus S_z} |S_t|^{-\delta q/p} w(t)\,dt\Big)^{1/q},$$

and if $q < p$, define

$$A_\delta := \Big\{\int_E \Big(\int_{E\setminus S_z} |S_t|^{-\delta q/p} w(t)\,dt\Big)^{q/(p-q)} |S_z|^{q(\delta q - p)/(p^2 - pq)} w(z)dz\Big\}^{(p-q)/(pq)}.$$

Our main result is the following.

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Theorem 2.1 *Let $0 < p, q < \infty$, u, v be weight functions, Φ be logarithmically convex on an open interval I, $k\colon \Omega \mapsto (0,\infty)$, and let k satisfy* (K1), (K2), *and* (2.3). *Suppose $A_\delta < \infty$ for some $\delta > 1$. If the range of values of f lies in the closure of I, $T_k f(x)$ exists for all nonzero $x \in E$, and $\Phi(f) \in L^+_{p,v}$, then* (1.1) *holds with*

$$C \le U_\delta A_\delta, \tag{2.4}$$

where

$$U_\delta = \begin{cases} \inf_{s>1}\Big(\dfrac{p+(s-1)q}{p}\Big)^{1/q}\Big(\dfrac{p+(s-1)q}{(\delta-1)q}\Big)^{(s-1)/p} M(\delta/s)^{-s/p} & \text{if } p \le q, \\ \inf_{s>1}\Big(\dfrac{p}{p-q}\Big)^{1/q-1/p} s^{1/p}\Big(\dfrac{s}{\delta-1}\Big)^{(s-1)/p} M(\delta/s)^{-s/p} & \text{if } q < p. \end{cases} \tag{2.5}$$

Before proving Theorem 2.1, we first deal with the existence of $G_k\Phi(f)(x)$.

Lemma 2.2 *Let p, v, k be given as in Theorem* 2.1. *Then for all $h \in L^+_{p,v}$, $G_k h(x)$ exists and is finite for all nonzero $x \in E$.*

Proof Let x be a nonzero element in E. We first prove that if $h \in L^+_1$, then $G_k h(x)$ exists. Suppose $\int_E h(t)dt < \infty$. Then $\int_{S_x} k(x,t)k(x,t)^{-1}h(t)dt = \int_{S_x} h(t)dt < \infty$. By [7, Theorem 187], $\int_{S_x} k(x,t)\log[k(x,t)^{-1}h(t)]\,dt$ is well defined and

$$\exp\Big(\int_{S_x} k(x,t)\log[k(x,t)^{-1}h(t)]\,dt\Big) = \lim_{r\to 0^+}\Big\{\int_{S_x} k(x,t)(k(x,t)^{-1}h(t))^r\,dt\Big\}^{1/r}$$

exists and is finite. Since condition (K2) ensures that $\int_{S_x} k(x,t)\log k(x,t)dt$ is finite, we have

$$\int_{S_x} k(x,t)\log h(t)\,dt = \int_{S_x} k(x,t)\log k(x,t)\,dt + \int_{S_x} k(x,t)\log[k(x,t)^{-1}h(t)]\,dt.$$

Therefore

$$G_k h(x) = \exp\Big(\int_{S_x} k(x,t)\log k(x,t)\,dt\Big)\,\exp\Big(\int_{S_x} k(x,t)\log[k(x,t)^{-1}h(t)]\,dt\Big)$$

exists and is finite. For $h \in L^+_{p,v}$, let $\tilde{h} = h^p v$ and hence $\tilde{h} \in L^+_1$. Since $p\log h(t) = \log\tilde{h}(t) + \log(1/v(t))$ and $G_k\tilde{h}(x)$, $G_k(1/v)(x)$ both exist and are finite, we have $G_k h(x) = G_k\tilde{h}(x)^{1/p} G_k(1/v)(x)^{1/p}$ exists and is finite. ■

Proof of Theorem 2.1 By Lemma 2.2, $G_k\Phi(f)(x)$ exists and is finite for all nonzero $x \in E$. Since Φ is logarithmically convex on I, Jensen's inequality implies that $\Phi(T_k f(x)) \le G_k\Phi(f)(x)$. For any $s > 1$, let $h^s = \Phi(f)^p v$. Then $h \in L^+_s$. By a similar argument to that given in the proof of Lemma 2.2, we see that $G_k\Phi(f)(x) = G_k h(x)^{s/p} G_k(1/v)(x)^{1/p}$. Therefore,

$$\Big(\int_E \Phi(T_k f(x))^q u(x)\,dx\Big)^{1/q} \le \Big(\int_E (G_k h(x))^{sq/p} w(x)\,dx\Big)^{1/q}, \tag{2.6}$$

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where $w(x) = G_k(1/v)(x)^{q/p}u(x)$. Suppose $A_\delta < \infty$ for some $\delta > 1$. Hölder's inequality implies that

$$\int_{S_x} |S_t|^{\delta/s-1} h(t)\,dt \le \Big(\int_{S_x} |S_t|^{(\delta-s)(s^*-1)}\,dt\Big)^{1/s^*} \Big(\int_{S_x} h(t)^s\,dt\Big)^{1/s}.$$

For non-zero $t \in E$, we write $t = y\tau$, where $y \in (0,\infty)$ and $\tau \in B$. By choosing $g(t) = |S_{y\tau}|^{(\delta-s)(s^*-1)} = (y^n|B|/n)^{(\delta-s)(s^*-1)}$ in (1.3), we have

$$\int_{S_x} |S_t|^{(\delta-s)(s^*-1)}\,dt = \Big(\frac{s-1}{\delta-1}\Big)|S_x|^{(\delta-1)/(s-1)}.$$

This shows that $\int_{S_x} |S_t|^{\delta/s-1}h(t)\,dt < \infty$ and hence

$$\exp\Big(\int_{S_x} k(x,t)\log[k(x,t)^{-1}|S_t|^{\delta/s-1}h(t)]\,dt\Big)$$

is finite. By Jensen's inequality and (K2), we have

$$\begin{aligned} G_k h(x) &\le \exp\Big(-\int_{S_x} k(x,t)\log[k(x,t)^{-1}|S_t|^{\delta/s-1}]\,dt\Big)\int_{S_x} |S_t|^{\delta/s-1}h(t)\,dt \\ &\le M(\delta/s)^{-1}|S_x|^{-\delta/s}\int_{S_x}|S_t|^{\delta/s-1}h(t)\,dt. \end{aligned}$$

Hence the integral in the right-hand side of (2.6) is less than

$$M(\delta/s)^{-sq/p}\int_E\Big(\int_{S_x}|S_t|^{\delta/s-1}h(t)\,dt\Big)^{sq/p}|S_x|^{-\delta q/p}w(x)dx.$$

Replace $p, q, \rho(x), \eta(x)$, and $f(t)$ in Theorem A by $s, sq/p, |S_x|^{-\delta q/p}w(x), |S_x|^{s-\delta}$, and $|S_t|^{\delta/s-1}h(t)$, respectively. Then

$$\Big(\int_E (G_k h(x))^{sq/p}w(x)\,dx\Big)^{p/(sq)} \le C^{p/s}\Big(\int_E h(x)^s\,dx\Big)^{1/s} \tag{2.7}$$

holds with

$$C \le \begin{cases} \Big(\dfrac{p+(s-1)q}{p}\Big)^{1/q}\Big(\dfrac{p+(s-1)q}{(\delta-1)q}\Big)^{(s-1)/p} M(\delta/s)^{-s/p}A_\delta & (p \le q), \\ \Big(\dfrac{p}{p-q}\Big)^{1/q-1/p} s^{1/p}\Big(\dfrac{s}{\delta-1}\Big)^{(s-1)/p} M(\delta/s)^{-s/p}A_\delta & (q < p). \end{cases} \tag{2.8}$$

Putting (2.6) and (2.7) together yields (1.1) with C satisfying (2.8). Since (2.8) is true for arbitrary $s > 1$, we have (2.4) and (2.5). ■

If Φ is strictly monotone, then Φ^{-1} exists. Replacing f by $\Phi^{-1}(f)$ in (1.1), where $f \in L^+_{p,v}$, we obtain the inequality of the form

$$\Big(\int_E \Phi(T_k\Phi^{-1}(f)(x))^q u(x)\,dx\Big)^{1/q} \le C\Big(\int_E f(x)^p v(x)\,dx\Big)^{1/p}. \tag{2.9}$$

In the case $\Phi(x) = e^x$, $I = (-\infty,\infty)$ and $\Phi^{-1}(x) = \log x$. If $f \in L^+_{p,v}$, then by Lemma 2.2, $\Phi(T_k\Phi^{-1}(f)(x)) = G_k f(x)$ exists and is finite for all non-zero $x \in E$. Inequality (2.9) then can be reduced to (1.2). Theorem 2.1 shows that $A_\delta < \infty$ for some $\delta > 1$ is a sufficient condition for (1.2) to hold for all $f \in L^+_{p,v}$. Theorem 2.3 proves that this condition is also necessary.

Theorem 2.3 *Let $0 < p, q < \infty$, k, u, and v be given as in Theorem 2.1. Then (1.2) holds for all $f \in L^+_{p,v}$ if and only if $A_\delta < \infty$ for all $\delta > 1$. Moreover,*

$$\sup_{\delta>1} L_\delta A_\delta \le C \le \inf_{\delta>1} U_\delta A_\delta, \tag{2.10}$$

where U_δ is given by (2.5) and

$$L_\delta = \begin{cases} \Big(\dfrac{\delta-1}{\delta}\Big)^{1/p} & \text{if } p \le q,\\[2ex] \Big(\dfrac{\delta q - q}{p}\Big)^{1/p} \min\Big(d_1^{\frac{\delta q-p}{p(p-q)}},\, d_2^{\frac{\delta q-p}{p(p-q)}}\Big) & \text{if } q < p.\end{cases} \tag{2.11}$$

Here d_1, d_2 are positive constants that satisfy $d_1|S_x| \le \exp(\int_{S_x} k(x,t)\log|S_t|\,dt) \le d_2|S_x|$ for all nonzero $x \in E$.

Proof If $A_\delta < \infty$ for all $\delta > 1$, then by Theorem 2.1 and (2.9) with $\Phi(x) = e^x$, inequality (1.2) holds for all $f \in L^+_{p,v}$ and the estimation of C satisfies (2.4)–(2.5) for all $\delta > 1$. This gives us the upper estimation of C in (2.10). Suppose that (1.2) holds for all $f \in L^+_{p,v}$. Let $h = f^p v$. Then

$$\Big(\int_E (G_k h(x))^{q/p} w(x)\,dx\Big)^{1/q} \le C\Big(\int_E h(x)\,dx\Big)^{1/p} \tag{2.12}$$

holds for all $h \in L^+_1$, where $w(x) = G_k(1/v)(x)^{q/p}u(x)$ and C is the same as in (1.2). We first consider the case $p \le q$. Let $\delta > 1$, ξ is a nonzero element in E, and

$$h(t) = \chi_{S_\xi}(t)|S_\xi|^{-1} + \chi_{E\setminus S_\xi}(t)|S_\xi|^{\delta-1}|S_t|^{-\delta}.$$

Then we have

$$\begin{aligned}\int_E h(x)\,dx &= 1 + |S_\xi|^{\delta-1}\int_{E\setminus S_\xi} |S_t|^{-\delta}\,dt\\ &= 1 + |S_\xi|^{\delta-1}\Big(\frac{|B|}{n}\Big)^{-\delta}\int_{\alpha_\xi}^{\infty}\int_B y^{-n\delta+n-1}d\tau\,dy = \frac{\delta}{\delta-1}.\end{aligned} \tag{2.13}$$

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On the other hand, for non-zero $x \in E \backslash S_\xi$ we have

$$\begin{aligned}\int_{S_x} k(x,t)\log h(t)dt &= -\log|S_\xi| + \delta\int_{S_x\backslash S_\xi} k(x,t)\log\Big[\frac{|S_\xi|}{|S_t|}\Big]\,dt\\ &\geq -\log|S_\xi| + \delta\Big(\int_{S_x\backslash S_\xi} k(x,t)dt\Big)\,\log\Big[\frac{|S_\xi|}{|S_x|}\Big]\\ &\geq \log[|S_\xi|^{\delta-1}|S_x|^{-\delta}],\end{aligned}$$

and this implies $G_k h(x) \geq |S_\xi|^{\delta-1}|S_x|^{-\delta}$. Hence

$$\int_E (G_k h(x))^{q/p} w(x)\,dx \geq |S_\xi|^{(\delta-1)q/p}\int_{E\backslash S_\xi}|S_x|^{-\delta q/p} w(x)\,dx. \tag{2.14}$$

By (2.12), (2.13), and (2.14), we have

$$C\Big(\frac{\delta}{\delta-1}\Big)^{1/p} \geq |S_\xi|^{(\delta-1)/p}\Big(\int_{E\backslash S_\xi}|S_x|^{-\delta q/p} w(x)\,dx\Big)^{1/q}. \tag{2.15}$$

Since (2.15) holds for all nonzero $\xi \in E$,

$$C \geq \Big(\frac{\delta-1}{\delta}\Big)^{1/p} A_\delta. \tag{2.16}$$

Inequality (2.16) is true for all $\delta > 1$, so we have the lower estimation given in (2.10) and (2.11).

Consider the case $q < p$. For $m \in \mathbb{N}$, let $x_m \in mB$ and we simply write S_m for S_{x_m}. Define

$$w_m(x) = [\min(w(x), m)]\chi_{S_m}(x) + [\min(w(x), |S_x|^{-2q/r})]\chi_{E\backslash S_m}(x),$$

where $1/r = 1/q - 1/p$. For $\delta > 1$, define

$$h_m(x) = |S_x|^{(\delta q-p)/(p-q)}\Big(\int_{E\backslash S_x}|S_t|^{-\delta q/p} w_m(t)\,dt\Big)^{p/(p-q)}.$$

We first show that $h_m \in L_1^+$. By (1.3) with $g(t) = |S_t|^{-\delta q/p} w_m(t)$, we have

$$\int_E h_m(x)\,dx = \Big(\frac{|B|}{n}\Big)^{-p/(p-q)}|B|\int_0^\infty\Big(\int_s^\infty g(y)\,dy\Big)^{p/(p-q)} s^{nq(\delta-1)/(p-q)-1}\,ds, \tag{2.17}$$

where $g(y) = \int_B y^{-\delta nq/p+n-1} w_m(y\tau)d\tau$. The dual Hardy inequality and Hölder's inequality show that for some finite constants c and d,

$$\begin{aligned}\int_E h_m(x)dx &\le c\int_0^\infty g(y)^{p/(p-q)}y^{q(n\delta-n+1)/(p-q)}dy\\ &= c\int_0^\infty \Big(\int_B w_m(y\tau)d\tau\Big)^{p/(p-q)} y^{n-1}dy\\ &\le c\int_0^\infty \Big(\int_B w_m(y\tau)^{p/(p-q)}d\tau\Big)|B|^{q/(p-q)}y^{n-1}dy\\ &= d\int_E w_m(t)^{p/(p-q)}dt\\ &\le d\int_{S_m} m^{p/(p-q)}dt + d\int_{E\setminus S_m}|S_t|^{-2}dt < \infty.\end{aligned}$$

Hence $G_k h_m(x)$ exists and is finite for all non-zero $x \in E$. Replace h by h_m in (2.12). Since $w_m \le w$, we have

$$\Big(\int_E (G_k h_m(x))^{q/p} w_m(x)\,dx\Big)^{1/q} \le C\Big(\int_E h_m(x)\,dx\Big)^{1/p}. \tag{2.18}$$

Condition (K2) implies that $d_1|S_x| \le \exp(\int_{S_x} k(x,t)\log|S_t|\,dt) \le d_2|S_x|$ for some positive constants d_1 and d_2. Therefore

$$\exp\Big(\int_{S_x} k(x,t)\log[|S_t|^{(\delta q-p)/(p-q)}]\,dt\Big) \ge \tilde{d}^p|S_x|^{(\delta q-p)/(p-q)},$$

where $\tilde{d} = \min\big(d_1^{\frac{\delta q-p}{p(p-q)}}, d_2^{\frac{\delta q-p}{p(p-q)}}\big)$. This implies

$$G_k h_m(x) \ge \tilde{d}^p|S_x|^{(\delta q-p)/(p-q)}\Big(\int_{E\setminus S_x}|S_t|^{-\delta q/p}w_m(t)\,dt\Big)^{p/(p-q)},$$

and hence

$$\Big(\int_E (G_k h_m(x))^{q/p}w_m(x)\,dx\Big)^{1/q} \ge \tilde{d}B_{\delta,m}^{1/q},$$

where

$$B_{\delta,m} = \int_E\Big(\int_{E\setminus S_x}|S_t|^{-\delta q/p}w_m(t)\,dt\Big)^{q/(p-q)}|S_x|^{(\delta q^2-pq)/(p^2-pq)}w_m(x)\,dx.$$

On the other hand, by [25, Lemma 1] we have

$$\begin{aligned}&\int_0^\infty\Big(\int_s^\infty g(y)\,dy\Big)^{p/(p-q)}s^{nq(\delta-1)/(p-q)-1}\,ds\\ &\qquad = \frac{p}{nq(\delta-1)}\int_0^\infty\Big(\int_s^\infty g(y)\,dy\Big)^{q/(p-q)}g(s)s^{nq(\delta-1)/(p-q)}\,ds.\end{aligned}$$

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Hence (2.17) implies

$$\int_E h_m(x)\,dx = \Big(\frac{|B|}{n}\Big)^{-p/(p-q)} \frac{p|B|}{nq(\delta-1)} \times \int_0^\infty \int_B \Big(\int_s^\infty \int_B y^{-\delta nq/p+n-1} w_m(y\tau)d\tau dy\Big)^{q/(p-q)} \times s^{n(\delta q^2-p^2)/(p^2-pq)+2n-1} w_m(s\sigma)d\sigma ds$$
$$= \frac{p}{q(\delta-1)} B_{\delta,m}.$$

Therefore (2.18) implies $C \geq \tilde{d}((\delta q - q)/p)^{1/p} B_{\delta,m}^{(p-q)/(pq)}$. Let $m \to \infty$. Since $w_m \to w$, we have $C \geq ((\delta q - q)/p)^{1/p}\tilde{d}A_\delta$. This holds for all $\delta > 1$, so we have the lower estimation given in (2.10) and (2.11). This completes the proof. ∎

3 Applications

Suppose that $\ell\colon (0,1) \mapsto (0,\infty)$ satisfies the following.

(KH1) $\int_0^1 \ell(t)dt = 1$.

(KH2) $M_1 = \exp(\int_0^1 \ell(t)\log \ell(t)dt) < \infty$.

(KH3) $M_2 = \exp(\int_0^1 \ell(t)\log t dt) > 0$.

We apply Theorem 2.3 to the case $k(x,t) = |S_x|^{-1}\ell(|S_t|/|S_x|)$. For such a case,

$$\int_{S_x} k(x,t)\,dt = \frac{1}{|S_x|}\int_0^{\alpha_x}\int_B \ell\Big(\frac{y^n}{\alpha_x^n}\Big) y^{n-1}d\tau\,dy = \int_0^1 \ell(u)\,du = 1$$

and

$$\int_{S_x} k(x,t)\log[k(x,t)^{-1}|S_t|^{\epsilon-1}]\,dt$$
$$= \frac{1}{|S_x|}\int_0^{\alpha_x}\int_B \ell\Big(\frac{y^n}{\alpha_x^n}\Big)\log\Big[|S_x|\ell^{-1}\Big(\frac{y^n}{\alpha_x^n}\Big)\Big(\frac{y^n|B|}{n}\Big)^{\epsilon-1}\Big] y^{n-1}d\tau dy$$
$$= \int_0^1 \ell(z)\log\Big[|S_x|\ell^{-1}(z)\Big(\frac{\alpha_x^n z|B|}{n}\Big)^{\epsilon-1}\Big]\,dz = \log[|S_x|^\epsilon M_1^{-1}M_2^{\epsilon-1}].$$

Hence (K1)–(K2) are satisfied with $M(\epsilon) = M_1^{-1}M_2^{\epsilon-1}$. Similarly, $d_1 = d_2 = M_2$ in (2.11). The following Theorem 3.1 can be obtained by Theorem 2.3.

Theorem 3.1 *Let $0 < p,q < \infty$, u and v be given as in Theorem 2.1, and let $\ell\colon (0,1) \mapsto (0,\infty)$ satisfy (KH1)–(KH3). Define $k\colon \Omega \mapsto (0,\infty)$ by $k(x,t) = |S_x|^{-1}\ell(|S_t|/|S_x|)$. Suppose that (2.3) holds. Then (1.2) holds for all $f \in L^+_{p,v}$ if and only if $A_\delta < \infty$ for all $\delta > 1$. The estimation of C can be obtained by (2.10), (2.5), and (2.11) with*

$$M(\delta/s) = M_1^{-1}M_2^{\delta/s-1}, \quad d_1 = d_2 = M_2. \tag{3.1}$$

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By taking limits $s \to 1$ in (2.5), the upper estimation of C satisfies

$$C \le \begin{cases} \inf_{\delta>1} M_1^{1/p} M_2^{(1-\delta)/p} A_\delta & \text{if } p \le q, \\ \inf_{\delta>1} (\frac{p}{p-q})^{1/q-1/p} M_1^{1/p} M_2^{(1-\delta)/p} A_\delta & \text{if } q < p. \end{cases} \tag{3.2}$$

Consider the particular case $u(x) = |S_x|^a$ and $v(x) = |S_x|^b$. Then

$$w(x) = G_k(1/v)(x)^{q/p} u(x) = M_2^{-bq/p} |S_x|^{a-(bq/p)}.$$

For $q < p$, $A_\delta = \infty$ for all $\delta > 1$. If $p \le q$ and $(a+1)/q = (b+1)/p$, then

$$\begin{aligned} A_\delta &= M_2^{-b/p} \sup_{z \in E \setminus \{0\}} |S_z|^{(\delta-1)/p} \Big(\int_{E \setminus S_z} |S_t|^{a-(b+\delta)q/p} dt \Big)^{1/q} \\ &= M_2^{-b/p} \sup_{s>0} \Big(\frac{s^n |B|}{n} \Big)^{(\delta-1)/p} \Big(\int_s^\infty \int_B \Big(\frac{y^n |B|}{n} \Big)^{a-(b+\delta)q/p} y^{n-1} d\tau dy \Big)^{1/q} \\ &= M_2^{-b/p} n^{1/q} \sup_{s>0} s^{n(\delta-1)/p} \Big(\int_s^\infty y^{nq(1-\delta)/p-1} dy \Big)^{1/q} = M_2^{-b/p} \Big(\frac{p}{\delta q - q} \Big)^{1/q}. \end{aligned}$$

By (3.2) we have

$$C \le M_1^{1/p} M_2^{-b/p} \inf_{\delta>1} M_2^{(1-\delta)/p} \Big(\frac{p}{\delta q - q} \Big)^{1/q} = M_1^{1/p} M_2^{-b/p} (-e \log M_2)^{1/q}.$$

Therefore,

$$\Big(\int_E (G_k f(x))^q |S_x|^a \, dx \Big)^{1/q} \le M_1^{1/p} M_2^{-b/p} (-e \log M_2)^{1/q} \Big(\int_E f(x)^p |S_x|^b \, dx \Big)^{1/p}. \tag{3.3}$$

The following corollary considers the case $\ell(t) = \alpha t^{\alpha-1}$, where $\alpha > 0$. For such a case,

$$M_1 = \alpha e^{1/\alpha - 1}, \quad M_2 = e^{-1/\alpha}. \tag{3.4}$$

Corollary 3.2 *Let $0 < p, q < \infty$, and $\alpha > 0$. Define $k\colon \Omega \mapsto (0, \infty)$ by $k(x,t) = \alpha |S_t|^{\alpha-1} / |S_x|^\alpha$. Suppose that u, v are weight functions, and* (2.3) *holds. Then*

$$\Big(\int_E \Big\{ \exp \Big(\frac{\alpha}{|S_x|^\alpha} \int_{S_x} |S_t|^{\alpha-1} \log f(t) \, dt \Big) \Big\}^q u(x) dx \Big)^{1/q} \le C \Big(\int_E f(x)^p v(x) \, dx \Big)^{1/p} \tag{3.5}$$

holds for all $f \in L_{p,v}^+$ if and only if $A_\delta < \infty$ for all $\delta > 1$. The estimation of C can be obtained by (2.10), (2.5), *and* (2.11) *with* (3.1) *and* (3.4).

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Consider the particular case $\alpha = 1$. In [5, Theorem 4.1], Drábek-Heinig-Kufner proved (3.5) for the case $p = q = 1$, $E = \mathbb{R}^n$, and $S_x = B(|x|)$. They showed that (3.5) holds if and only if $A_2 < \infty$. Hence Corollary 3.2 is a generalization of [5, Theorem 4.1]. Another type of characterizations for the case that $0 < p \le q < \infty$ and E is a spherical cone in $\mathbb{R}^n$ can also be found in [6, Theorem 3.1]. If $p \le q$, $u(x) = |S_x|^a$, $v(x) = |S_x|^b$, and $(a+1)/q = (b+1)/p$, then by (3.3),

$$\Big(\int_E \Big\{ \exp\Big(\frac{\alpha}{|S_x|^\alpha} \int_{S_x} |S_t|^{\alpha-1} \log f(t)dt \Big) \Big\}^q |S_x|^a dx \Big)^{1/q} \le \alpha^{1/p-1/q} e^{1/q+(b-\alpha+1)/(\alpha p)} \Big(\int_E f(x)^p |S_x|^b dx \Big)^{1/p}. \tag{3.6}$$

Since $e^{1/q-1/p} \le (p/q)^{1/q}$ for $p \le q$, the constant given in (3.6) is better than that given in [6, Proposition 3.6] and [14, Theorem 2]. If $p = q = 1$, $a = b$, $E = \mathbb{R}^n$, and $S_x = B(|x|)$, then (3.6) reduces to [3, (23)].

We can also apply Theorem 3.1 to the case $\ell(t) = \alpha(1-t)^{\alpha-1}$, where $\alpha > 0$. In this case,

$$M_1 = \alpha e^{1/\alpha - 1}, \quad M_2 = e^{-\gamma - \Gamma'(\alpha+1)/\Gamma(\alpha+1)}, \tag{3.7}$$

where γ is the Euler constant and $\Gamma(x)$ is the Gamma function. The constant M_2 can be obtained by the following equalities

$$\log M_2 = \alpha \int_0^1 z^{\alpha-1} \log(1-z)dz = -\alpha \int_0^1 \sum_{n=1}^\infty \frac{z^{n+\alpha-1}}{n} dz = -\gamma - \frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)}.$$

The last equality is based on [1, Theorem 1.2.5]. We have the following corollary.

Corollary 3.3 *Let $0 < p, q < \infty$ and $\alpha > 0$. Define $k\colon \Omega \mapsto (0,\infty)$ by $k(x,t) = \alpha(|S_x| - |S_t|)^{\alpha-1}/|S_x|^\alpha$. Suppose that u, v are weight functions, and (2.3) holds. Then*

$$\Big(\int_E \Big\{ \exp\Big(\frac{\alpha}{|S_x|^\alpha} \int_{S_x} (|S_x| - |S_t|)^{\alpha-1} \log f(t)dt \Big) \Big\}^q u(x) dx \Big)^{1/q} \le C \Big(\int_E f(x)^p v(x) dx \Big)^{1/p}$$

holds for all $f \in L^+_{p,v}$ if and only if $A_\delta < \infty$ for all $\delta > 1$. The estimation of C can be obtained by (2.10), (2.5), *and* (2.11) *with* (3.1) *and* (3.7).

If $p \le q$, $u(x) = |S_x|^a$, $v(x) = |S_x|^b$, and $(a+1)/q = (b+1)/p$, then by (3.3),

$$\Big(\int_E \Big\{ \exp\Big(\frac{\alpha}{|S_x|^\alpha} \int_{S_x} (|S_x| - |S_t|)^{\alpha-1} \log f(t)dt \Big) \Big\}^q |S_x|^a dx \Big)^{1/q} \le C \Big(\int_E f(x)^p |S_x|^b dx \Big)^{1/p},$$

where

$$C = \alpha^{1/p} e^{1/q+(1-\alpha+\alpha\gamma b)/(\alpha p)+b\Gamma'(\alpha+1)/(p\Gamma(\alpha+1))} \Big(\gamma + \frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)}\Big)^{1/q}.$$

Remark In our Theorem 2.1 and Theorem 2.3, we suppose that the kernel k satisfies (K1) and (K2). In the following, we replace (K2) by the condition (K2*).

(K2*) There exists $M > 0$ such that $\exp\big(\int_{S_x} k(x,t)\log[k(x,t)^{-1}]\,dt\big) \geq M|S_x|$ for all non-zero $x \in E$.

According to the proof of Theorem 2.1, we see that Theorem 2.1 still holds with (2.4) being replaced by the following estimation.

$$C \leq \begin{cases} \Big(\dfrac{p+(\delta-1)q}{p}\Big)^{1/q}\Big(\dfrac{p+(\delta-1)q}{(\delta-1)q}\Big)^{(\delta-1)/p} M^{-\delta/p}A_\delta & (p \leq q), \\ \Big(\dfrac{p}{p-q}\Big)^{1/q-1/p}\delta^{1/p}\Big(\dfrac{\delta}{\delta-1}\Big)^{(\delta-1)/p} M^{-\delta/p}A_\delta & (q < p). \end{cases}$$

Similarly, according to the proof of Theorem 2.3, we obtain a characterization for (1.2) to hold for all $f \in L^+_{p,v}$, which is given as follows.

(i) In the case $p \leq q$, $A_\delta < \infty$ for all $\delta > 1$ and

(3.8)
$$\sup_{\delta>1}\Big(\frac{\delta-1}{\delta}\Big)^{1/p}A_\delta \leq C \leq \inf_{\delta>1}\Big(\frac{p+(\delta-1)q}{p}\Big)^{1/q}\Big(\frac{p+(\delta-1)q}{(\delta-1)q}\Big)^{(\delta-1)/p}M^{-\delta/p}A_\delta.$$

(ii) In the case $q < p$, $A_{p/q} < \infty$ and

$$\Big(\frac{p-q}{p}\Big)^{1/p}A_{p/q} \leq C \leq \Big(\frac{p}{p-q}\Big)^{2(1/q-1/p)}\Big(\frac{p}{q}\Big)^{1/p}M^{-1/q}A_{p/q}. \tag{3.9}$$

We now apply the above results to the case $k(x,t) = |S_x|^{-1}\ell(|S_t|/|S_x|)$, where $\ell\colon (0,1) \mapsto (0,\infty)$. Then we see that the condition (KH3) in Theorem 3.1 can be removed and the estimation of C can be obtained by (3.8) and (3.9) with $M = M_1^{-1}$.

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