Glasgow Math. J. **51** (2009) 55–70. © 2008 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089508004539. Printed in the United Kingdom

ON LITTLEWOOD-PALEY FUNCTIONS ASSOCIATED WITH BESSEL OPERATORS

J. J. BETANCOR

Departamento de Análisis Matemático, Universidad de La Laguna, 38271 – La Laguna, Tenerife, Islas Canarias, España
e-mail: jbetanco@ull.es

J. C. FARIÑA

Departamento de Análisis Matemático, Universidad de La Laguna, 38271 – La Laguna, Tenerife, Islas Canarias, España
e-mail: jcfarina@ull.es

and A. SANABRIA

Departamento de Análisis Matemático, Universidad de La Laguna, 38271 – La Laguna, Tenerife, Islas Canarias, España
e-mail: asgarcia@ull.es

(Received 25 July 2007; revise 1 May 2008; accepted 25 September 2008)

Abstract. In this paper, we study L_p-boundedness properties for higher order Littlewood-Paley g-functions in the Bessel setting. We use the Calderón-Zygmund theory in a homogeneous-type space (in the sense of Coifman and Weiss) $((0, \infty), d, \gamma_\alpha)$, where d represents the usual metric on $(0, \infty)$ and γ_α denotes the doubling measure on $(0, \infty)$ with respect to d defined by $d\gamma_\alpha(x) = x^{2\alpha+1}dx$, with $\alpha > -1/2$.

2000 *Mathematics Subject Classification.* 42C05 (primary), 42C15 (secondary).

1. Introduction. In the past years, the Calderón-Zygmund theory has been used by several authors for the study of operators that occur in harmonic analysis in different settings (see, for instance, [**2–4**, **9**, **20**, **21**, **24**]). In this paper, we study operators that appear in the harmonic analysis related to Bessel differential operators viewed as Calderón-Zygmund integral operators in the spaces $((0, \infty), d, \gamma_\alpha)$, where d denotes the standard Euclidean metric, and for $\alpha > -1/2$, $d\gamma_\alpha(x) = x^{2\alpha+1}dx$. These spaces are homogeneous in the sense of Coifman and Weiss [**6**].

Suppose that $(X, \ \rho, \ \gamma)$ is a homogeneous-type space, that is, ρ is a quasimetric on X and γ is a positive measure on X verifying:

(i) $\gamma(B(x, r)) < \infty, \ x \in X$ and $r > 0$.
(ii) There exists $C > 0$ for which $\gamma(B(x, 2r)) \leq C\gamma(B(x, r)), \ x \in X$ and $r > 0$,

where $B(x, r) = \{y \in X : \rho(x, y) < r\}$ for every $x \in X$ and $r > 0$.

Let $\mathbb{B}$ be a Banach space. We say that a $\mathbb{B}$-valued function K defined on $X \times X \setminus D_X$, where $D_X = \{(x, y) \in X \times X : x = y\}$ is a standard kernel when there exist $\epsilon, \ C > 0$ such that

(i) $\|K(x, y)\|_{\mathbb{B}} \leq \frac{C}{\gamma(B(x,\rho(x,y)))}, \ x, y \in X, \ x \neq y.$

(ii) $\|K(x, y) - K(z, y)\|_{\mathbb{B}} + \|K(y, x) - K(y, z)\|_{\mathbb{B}} \leq \frac{\rho(x,z)}{\rho(x,y)} \frac{C}{\gamma(B(x,\rho(x,y)))}$, $x, y, z \in X$, $x \neq y$, $\rho(x, z) \leq \epsilon\rho(x, y)$.

As usual, the integration of $\mathbb{B}$-valued functions is understood in Bochner's sense. Definitions and main properties of the Lebesgue spaces of $\mathbb{B}$-valued functions $L^p_{\mathbb{B}}(X, \gamma)$, $1 \leq p \leq \infty$, can be encountered in [**7**]. In the sequel, we write $L^p(X, \gamma)$ to denote $L^p_{\mathbb{C}}(X, \gamma)$. A Calderón-Zygmund operator associated with a standard $\mathbb{B}$-valued kernel K is a linear operator T that is bounded from $L^2(X, \gamma)$ into $L^2_{\mathbb{B}}(X, \gamma)$ and such that for every $f \in L^2(X, \gamma)$,

$$Tf(x) = \int_X K(x, y)f(y)d\gamma(y), \qquad x \notin \operatorname{supp} f. \tag{1.1}$$

The main results concerning singular integrals on Banach spaces of valued functions were established in [**16**]. Function spaces (Lebesgue, Hardy, Lipschitz, etc.) defined on homogeneous-type spaces (X, ρ, γ) and operators acting between them were studied by Macías and colleagues [**10–12**]. The corresponding spaces of $\mathbb{B}$-valued functions are defined in a natural way, and they are denoted with a subindex $\mathbb{B}$ ([**16**]).

The boundedness properties of our Calderón-Zygmund operators can be stated as follows. Here, as usual, for every $1 \leq p < \infty$, we denote by $A_p(X, \rho, \gamma)$ the Muckenhoupt class of weights with respect to the measure γ on X endowed with the quasimetric ρ.

THEOREM 1.1. ([**15**, Theorem 5] and [**16**, proposition in Remark 7]) *Let (X, ρ, γ) be a homogeneous-type space and let $\mathbb{B}$ be a Banach space. Assume that T is a Calderón-Zygmund operator associated to a $\mathbb{B}$-valued standard kernel K on X. Then,*

(i) *for every $1 < p < \infty$ and $\omega \in A_p(X, \rho, \gamma)$, T is bounded from $L^p(X, \omega d\gamma)$ into $L^p_{\mathbb{B}}(X, \omega d\gamma)$.*

(ii) *for every $\omega \in A_1(X, \rho, \gamma)$, T is bounded from $L^1(X, \omega d\gamma)$ into $L^{1,\infty}_{\mathbb{B}}(X, \omega d\gamma)$.*

(iii) *T is bounded from $L^\infty_0(X, d\gamma)$ into $BMO_{\mathbb{B}}(X, \rho, d\gamma)$, where $L^\infty_0(X, d\gamma)$ denotes the space that consists of all the functions in $L^\infty(X, d\gamma)$ having compact support.*

(iv) *T is bounded from $H^1(X, \rho, d\gamma)$ into $L^1_{\mathbb{B}}(X, d\gamma)$.*

When $\gamma(X) = \infty$ to replace $L^\infty_0(X, d\gamma)$ by $L^\infty(X, d\gamma)$ in the property (*iii*) presented in Theorem 1.1, it is necessary to give a new definition of the operator T (see [**5**, pp. 117 and 118], [**8**, p. 119] and [**22**, Proposition 2.1]).

In this paper, we consider the homogeneous-type spaces $((0, \infty), d, \gamma_\alpha)$. In this setting, a suitable estimate for the measure $\gamma_\alpha(B(x, |x - y|))$ of the ball $B(x, |x - y|)$, where $x, y \in (0, \infty)$, and the arguments developed by Nowak and Stempak [**14**] lead to the following result that it establishes sufficient conditions in order that a $\mathbb{B}$-valued kernel K is a standard kernel.

THEOREM 1.2. *Let $\mathbb{B}$ be a Banach space and $\alpha > -1/2$. Assume that K is a $\mathbb{B}$-valued function defined on $(0, \infty) \times (0, \infty) \setminus D_{(0,\infty)}$. If K satisfies the following two conditions:*

(i) $\|K(x, y)\|_{\mathbb{B}} \leq \dfrac{C}{|x - y| \max\{x^{2\alpha+1}, y^{2\alpha+1}\}}$, $\quad x, y \in (0, \infty), \quad x \neq y$,

(ii) $$\left\|\frac{\partial}{\partial x}K(x, y)\right\|_{\mathbb{B}} + \left\|\frac{\partial}{\partial y}K(x, y)\right\|_{\mathbb{B}} \leq \frac{C}{|x - y|^2 \max\{x^{2\alpha+1}, y^{2\alpha+1}\}}, \quad x, y \in (0, \infty), \quad x \neq y,$$

https://doi.org/10.1017/S0017089508004539 Published online by Cambridge University Press

for a certain $C > 0$, where $\|\cdot\|_{\mathbb{B}}$ denotes the norm in $\mathbb{B}$, then K is a standard kernel in $((0,\infty), d, \gamma_\alpha)$.

As a consequence of Theorem 1.1 and Theorem 1.2, we study boundedness properties of operators appearing in the harmonic analysis associated with Bessel differential operators.

From now on, we assume $\alpha > -1/2$. The Bessel operator Δ_α is defined by

$$\Delta_\alpha = -x^{-2\alpha-1}Dx^{2\alpha+1}D = -D^2 - \frac{2\alpha+1}{x}D,$$

where $D = d/dx$. It is not hard to see that

$$\Delta_\alpha = D^*D, \tag{1.2}$$

where D^* represents the (formal) adjoint of D in $L^2((0,\infty), d\gamma_\alpha)$. In the Bessel setting, the Hankel transform h_α plays the role of the Fourier transform in the Euclidean one. The Hankel transform $h_\alpha f$ of $f \in L^1((0,\infty), d\gamma_\alpha)$ is defined by

$$h_\alpha(f)(y) = \int_0^\infty (xy)^{-\alpha}J_\alpha(xy)f(x)\,d\gamma_\alpha(x), \quad y \in (0,\infty),$$

where J_α denotes the Bessel function of the first kind and order α. The Hankel transform h_α can be extended to an isometry of $L^2((0,\infty), d\gamma_\alpha)$. Moreover, for every $y > 0$, the function $\varphi_y(x) = (xy)^{-\alpha}J_\alpha(xy)$, $x \in (0,\infty)$, is an eigenfunction of Δ_α associated to the eigenvalue y^2.

The harmonic analysis in the Bessel setting was begun by Muckenhoupt and Stein [**13**]. They considered the Poisson kernel $P_\alpha(t,x,y)$ given by

$$P_\alpha(t,x,y) = \int_0^\infty e^{-tz}(xz)^{-\alpha}J_\alpha(xz)(yz)^{-\alpha}J_\alpha(yz)\,d\gamma_\alpha(z), \quad t,x,y \in (0,\infty).$$

Weinstein [**25**] established the following representation formula for the Poisson kernel:

$$P_\alpha(t,x,y) = \frac{2\alpha+1}{\pi}t\int_0^\pi \frac{\sin^{2\alpha}\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\alpha+\frac{3}{2}}}d\theta, \quad t,x,y \in (0,\infty). \tag{1.3}$$

The Poisson integral $P_{\alpha,t}(f)$ of $f \in L^p((0,\infty), d\gamma_\alpha)$, $1 \le p \le \infty$, is defined by

$$P_{\alpha,t}(f)(x) = \int_0^\infty f(y)P_\alpha(t,x,y)d\gamma_\alpha(y), \quad t,x \in (0,\infty). \tag{1.4}$$

Then, $\{P_{\alpha,t}\}_{t>0}$ is a semigroup of contractions in $L^p((0,\infty), d\gamma_\alpha)$, $1 \le p < \infty$.

In [**13**], a notion of conjugation is introduced. Taking as a starting point a suitable Cauchy-Riemann-type equations, Muckenhoupt and Stein defined for every $f \in L^p((0,\infty), d\gamma_\alpha)$, $1 \le p < \infty$, the α-harmonic-conjugated extension $Q_{\alpha,t}(f)$ of f by

$$Q_{\alpha,t}(f)(x) = \int_0^\infty f(y)Q_\alpha(t,x,y)d\gamma_\alpha(y), \quad t,x \in (0,\infty),$$

where

$$Q_\alpha(t,x,y) = -\frac{2\alpha+1}{\pi}\int_0^\pi \frac{(x-y\cos\theta)\sin^{2\alpha}\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\alpha+\frac{3}{2}}}\,d\theta, \qquad t,x,y\in(0,\infty).$$

The limit $R_\alpha f(x) = \lim_{t\to 0^+} Q_{\alpha,t}(f)(x)$ exists for almost every $x\in(0,\infty)$ and for every $f\in L^p((0,\infty),d\gamma_\alpha)$, $1\le p<\infty$. Weighted inequalities for the operator R_α were established by Andersen and Kerman [**1**].

Recently, a conjugacy associated with the Bessel operator $S_\alpha = x^{-\alpha-\frac{1}{2}} Dx^{2\alpha+1}Dx^{-\alpha-\frac{1}{2}}$ has been studied [**2**]. The procedure employed in [**2**] uses the Calderón-Zygmund theory and it is different from the one followed in [**13**]. From results in [**2**], it can be deduced that

$$R_\alpha f = D\Delta_\alpha^{-1/2} f,$$

for every $f\in L^p((0,\infty),d\gamma_\alpha)$, $1\le p<\infty$. Thus, according to (1.2), R_α can be seen as a Riesz transform associated to Δ_α in the sense of Stein [**18**]. Here, the negative power $\Delta_\alpha^{-1/2}$ of Δ_α admits in terms of the Poisson semigroup the representation

$$\Delta_\alpha^{-1/2} f(x) = \int_0^\infty P_{\alpha,t}(f)(x)\,dt.$$

In this paper, we complete the study of the harmonic analysis associated with Δ_α showing L^p-boundedness properties of the Littlewood-Paley g-functions related to semigroups in the Bessel setting.

Let $(k,m)\in\mathbb{N}\times\mathbb{N}\setminus\{(0,0)\}$. We define the Littlewood-Paley g-function $g_\alpha^{(k,m)}(f)$ of f associated with the Poisson semigroup $\{P_{\alpha,t}\}_{t>0}$ as follows:

$$g_\alpha^{(k,m)}(f)(x) = \left(\int_0^\infty \left| t^{k+m}\frac{\partial^{k+m}}{\partial t^k \partial x^m} P_{\alpha,t}(f)(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}.$$

Stempak [**19**] proved that the $g_\alpha^{(k,0)}$-function is bounded on $L^p((0,\infty),d\gamma_\alpha)$ provided that $1<p<\infty$. He adapted a well-known pattern (see [**18**]) to the Bessel setting. We establish that $g_\alpha^{(k,m)}$ is the $\mathbb{B}$-norm of a Calderón-Zygmund operator associated to a $\mathbb{B}$-valued standard kernel for the Banach space $\mathbb{B} = L^2((0,\infty), t^{2(m+k)-1}dt)$.

We consider the operator $G_\alpha^{(k,m)}$ defined by

$$f \longrightarrow G_\alpha^{(k,m)}(f)(x) = \left(\frac{\partial^{m+k}}{\partial x^m \partial t^k} P_{\alpha,t}(f)(x)\right)_{t>0}. \tag{1.5}$$

Note that $g_\alpha^{(k,m)}(f)(x) = \|G_\alpha^{(k,m)}(f)(x)\|_{\mathbb{B}}$.

The main achievement of this paper is the following result.

THEOREM 1.3. *Let* $(k,m)\in\mathbb{N}\times\mathbb{N}\setminus\{(0,0)\}$. *The* $\mathbb{B}$-*valued operator* $f\to G_\alpha^{(k,m)}(f)$ *defined by* (1.5) *maps boundedly* $L^2((0,\infty),d\gamma_\alpha)$ *into* $L^2_{\mathbb{B}}((0,\infty),d\gamma_\alpha)$, *and it is the Calderón-Zygmund operator associated, in the sense of* (1.1), *with the* $\mathbb{B}$-*valued kernel*

$$K_\alpha^{(k,m)}(x,y) = \left(\frac{\partial^{m+k}}{\partial t^k \partial x^m} P_\alpha(t,x,y)\right)_{t>0}, \qquad x,y\in(0,\infty) \tag{1.6}$$

https://doi.org/10.1017/S0017089508004539 Published online by Cambridge University Press

that satisfies

$$\text{(i)}\ \left\|K_\alpha^{(k,m)}(x,y)\right\|_{\mathbb{B}} \leq \frac{C}{|x-y|\max\{x^{2\alpha+1},y^{2\alpha+1}\}},\quad x,y\in(0,\infty),\quad x\neq y$$

$$\text{(ii)}\ \left\|\frac{\partial}{\partial x}K_\alpha^{(k,m)}(x,y)\right\|_{\mathbb{B}} + \left\|\frac{\partial}{\partial y}K_\alpha^{(k,m)}(x,y)\right\|_{\mathbb{B}} \leq \frac{C}{|x-y|^2\max\{x^{2\alpha+1},y^{2\alpha+1}\}},$$
$$x,y\in(0,\infty),\quad x\neq y. \qquad (1.7)$$

Note that the derivatives in Theorem 1.2 must be understood in the corresponding Banach space. However, by taking into account the integral Minkowski inequality, to see that the $\mathbb{B}$-valued kernel $K_\alpha^{(k,m)}$ defined in Theorem 1.3 is a standard kernel, it is sufficient to prove (i) and (ii) in Theorem 1.3 considering pointwise derivatives.

Theorem 1.3 allows us to obtain, as a consequence of Theorem 1.1, the boundedness properties for $g_\alpha^{(k,m)}$ that are stated:

THEOREM 1.4. *Let* $(k,m)\in\mathbb{N}\times\mathbb{N}\setminus\{(0,0)\}$. *Then,*

(i) *for every* $1<p<\infty$ *and* $\omega\in A_p((0,\infty),d,d\gamma_\alpha)$, $g_\alpha^{(k,m)}$ *is bounded on* $L^p((0,\infty),\omega(x)d\gamma_\alpha)$.

(ii) *for every* $\omega\in A_1((0,\infty),d,d\gamma_\alpha)$, $g_\alpha^{(k,m)}$ *is bounded from* $L^1((0,\infty),\omega(x)d\gamma_\alpha)$ *into* $L^{1,\infty}((0,\infty),\omega(x)d\gamma_\alpha)$.

(iii) $g_\alpha^{(k,m)}$ *is bounded from* $L_0^\infty((0,\infty),d\gamma_\alpha)$ *into* $BMO((0,\infty),d,d\gamma_\alpha)$.

(iv) $g_\alpha^{(k,m)}$ *is bounded from* $H^1((0,\infty),d,d\gamma_\alpha)$ *into* $L^1((0,\infty),d\gamma_\alpha)$.

The arguments developed here also allow us to generalize results given in [**13**] by proving L^p-boundedness properties for the higher order Riesz transforms that are associated with Bessel operators.

Also, a similar procedure can be used to obtain L^p-boundedness properties for g-functions associated with the heat semigroup for the Bessel operator Δ_α. Although the results in the Poisson case (Theorem 1.4) can be deduced from the corresponding ones for the heat case, we prefer to present here the complete proof of Theorem 1.4 because both cases (heat and Poisson) can be proved by analogous procedures, but the manipulations are much more involved for the heat semigroup.

Throughout this paper, C always denotes a positive constant that is not the same in each occurrence.

2. Proof of Theorem 1.3.

The proof of Theorem 1.3 naturally splits into three parts: L^2-boundedness of $G_\alpha^{(k,m)}$, association of the kernel $K_\alpha^{(k,m)}$ to the operator $G_\alpha^{(k,m)}$ and standard estimates of $K_\alpha^{(k,m)}(x,y)$. These three parts are proved as Lemma 2.1, Lemma 2.3 and Lemma 2.4, respectively. In Lemma 2.2, the most difficult part of Lemma 2.1 is separated.

Assume that $f\in L^2((0,\infty),d\gamma_\alpha)$. According to [**19**, Lemma 2, p. 23], we can write

$$P_{\alpha,t}(f)(x) = \int_0^\infty (xy)^{-\alpha}J_\alpha(xy)\,e^{-yt}\,h_\alpha(f)(y)\,d\gamma_\alpha(y),\quad x\in(0,\infty).$$

Since $h_\alpha(f)\in L^2((0,\infty),d\gamma_\alpha)$ by using [**24**, p. 46, (6)], we can see that $P_{\alpha,t}(f)(x)$ is infinitely differentiable on $(0,\infty)\times(0,\infty)$.

https://doi.org/10.1017/S0017089508004539 Published online by Cambridge University Press

LEMMA 2.1. *Let* $(k, m) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0)\}$. *Then,* $G_\alpha^{(k,m)}$ *is a bounded operator from* $L^2((0, \infty), d\gamma_\alpha)$ *into* $L^2_{\mathbb{B}}((0, \infty), d\gamma_\alpha)$.

Proof. According to (1.4), we get

$$\frac{\partial^{m+k}}{\partial x^m \partial t^k} P_{\alpha,t}(f)(x) = \int_0^\infty \frac{\partial^{m+k}}{\partial x^m \partial t^k} P_\alpha(t, x, y) f(y)\, d\gamma_\alpha(y), \quad t, x \in (0, \infty). \tag{2.1}$$

To establish that $G_\alpha^{(k,m)}$ defines a bounded operator from $L^2((0, \infty), d\gamma_\alpha)$ into $L^2_{\mathbb{B}}((0, \infty), d\gamma_\alpha)$, we proceed as follows. We split the kernel

$$\frac{\partial^{m+k}}{\partial t^k \partial x^m} P_\alpha(t, x, y) = \frac{1}{\pi}(xy)^{-\alpha-\frac{1}{2}} \frac{\partial^{m+k}}{\partial t^k \partial x^m}\left(\frac{t}{(x-y)^2 + t^2}\right) + h(t, x, y), \quad t, x, y \in (0, \infty) \tag{2.2}$$

and define the operator

$$\mathcal{H}(f)(x) = \int_0^\infty H(x, y) f(y) d\gamma_\alpha(y), \tag{2.3}$$

where $H(x, y) = (\int_0^\infty |h(t, x, y)|^2 t^{2(m+k)-1} dt)^{1/2}$, $x, y \in (0, \infty)$. Then, when we show that $\mathcal{H}$ is a bounded operator on $L^2((0, \infty), d\gamma_\alpha)$, the proof of this lemma is complete.

Indeed, assume that this is the case. By using the integral Minkowski inequality, we can write

$$\begin{aligned}
&\|G_\alpha^{(k,m)}(f)\|^2_{L^2_{\mathbb{B}}((0,\infty),d\gamma_\alpha)} = \int_0^\infty \|G_\alpha^{(k,m)}(f)(x)\|^2_{\mathbb{B}}\, d\gamma_\alpha(x) \\
&= \int_0^\infty \int_0^\infty \left| \frac{\partial^{m+k}}{\partial t^k \partial x^m} P_{\alpha,t}(f)(x)\right|^2 t^{2(m+k)-1} dt\, d\gamma_\alpha(x) \\
&\leq C\left(\int_0^\infty \int_0^\infty \left| \int_0^\infty \frac{\partial^{m+k}}{\partial t^k \partial x^m}\left(\frac{t}{(x-y)^2+t^2}\right) f(y)(xy)^{-\alpha-\frac{1}{2}} d\gamma_\alpha(y)\right|^2 t^{2(m+k)-1} dt\, d\gamma_\alpha(x)\right. \\
&\quad \left. + \int_0^\infty \int_0^\infty \left| \int_0^\infty h(t, x, y) f(y) d\gamma_\alpha(y)\right|^2 t^{2(m+k)-1} dt\, d\gamma_\alpha(x)\right) \\
&\leq C\left(\int_0^\infty \int_0^\infty \left| \int_0^\infty \frac{\partial^{m+k}}{\partial t^k \partial x^m}\left(\frac{t}{(x-y)^2+t^2}\right) y^{\alpha+\frac{1}{2}} f(y) dy\right|^2 t^{2(m+k)-1} dt\, dx\right. \\
&\quad \left. + \int_0^\infty \left(\int_0^\infty |f(y)| \left(\int_0^\infty |h(t, x, y)|^2 t^{2(m+k)-1} dt\right)^{1/2} d\gamma_\alpha(y)\right)^2 d\gamma_\alpha(x)\right) \\
&\leq C\left(\int_0^\infty \left| g^{(k,m)}(f(y) y^{\alpha+\frac{1}{2}})(x)\right|^2 dx + \int_0^\infty |\mathcal{H}(f)(x)|^2 d\gamma_\alpha(x)\right),
\end{aligned}$$

where $g^{(k,m)}$ denotes the Euclidean g-function restricted to $(0, \infty)$, that is,

$$g^{(k,m)}(F)(x) = \left(\int_0^\infty \left| \frac{\partial^{m+k}}{\partial t^k \partial x^m} \int_0^\infty \frac{t}{(x-y)^2 + t^2} F(y) dy\right|^2 t^{2(m+k)-1} dt\right)^{\frac{1}{2}}.$$

It is well known that $g^{(k,m)}$ is bounded on $L^2(\mathbb{R}, dx)$ [**17**, p. 86]. Then, the claim of Lemma 2.1 follows by using the result of Lemma 2.2. $\square$

LEMMA 2.2. *The operator $\mathcal{H}$ defined by* (2.3) *is bounded on* $L^2((0,\infty), d\gamma_\alpha)$.

Proof. Firstly, a few words about the strategy of the proof. We split the function h defined by (2.2) as follows:

$$h = h_1 + h_2,$$

where

$$h_1(t,x,y) = \frac{\partial^{m+k}}{\partial t^k \partial x^m} P_\alpha(t,x,y) - \frac{1}{\pi}\chi_{\{x/b<y<bx\}}(y)(xy)^{-\alpha-\frac{1}{2}} \frac{\partial^{m+k}}{\partial t^k \partial x^m}\left(\frac{t}{(x-y)^2+t^2}\right),$$

with $t, x, y \in (0,\infty)$, for a certain $b > 1$ that is specified later. It is then sufficient to prove that the operators

$$\mathcal{H}_i(f)(x) = \int_0^\infty H_i(x,y) f(y)\, d\gamma_\alpha(y),$$

where

$$H_i(x,y) = \left(\int_0^\infty |h_i(t,x,y)|^2 t^{2(m+k)-1}\, dt\right)^{1/2}, \quad x, y \in (0,\infty), \quad i = 1, 2,$$

are bounded on $L^2((0,\infty), d\gamma_\alpha)$. This is obtained by pointwise estimates of the kernels H_1 and H_2. The operator $\mathcal{H}_1$ is further decomposed as $\mathcal{H}_1 = \mathcal{H}_{1,1} + \mathcal{H}_{1,2} + \mathcal{H}_{1,3}$ according to the splitting of $H_1(x,y)$ as $H_1 = \chi_{\{0<y<x/b\}}H_1 + \chi_{\{x/b<y<bx\}}H_1 + \chi_{\{bx<y<\infty\}}H_1$. The $L^2((0,\infty), d\gamma_\alpha)$-boundedness of the operators $\mathcal{H}_{1,1}$ and $\mathcal{H}_{1,3}$ is deduced from the corresponding boundedness properties of certain Hardy-type operators. We prove that the operator $\mathcal{H}_{1,2}$ is controlled by a positive operator that is bounded in $L^2((0,\infty), d\gamma_\alpha)$.

Treatment of $\mathcal{H}_2$ is similar but much easier since for every $t > 0$, $h_2(t,x,y)$ has support in the region $\{(x,y) : \frac{x}{b} < y < bx\}^c$. This property allows to obtain relevant estimates of $H_2(x,y)$ on $0 < y < x/b$ and $bx < y < \infty$, and again Hardy's inequalities are applied to obtain the claim.

We start with analysing $\mathcal{H}_1$. By using (1.3) and induction on $k+n$, it may be easily checked that for every $t, x, y \in (0,\infty)$,

$$\frac{\partial^{k+m}}{\partial t^k \partial x^m} P_\alpha(t,x,y) = \sum_{j=[\frac{m+1}{2}]}^{m} \sum_{l=0}^{2j-m} \sum_{\beta=[\frac{k}{2}]}^{k} a_{j,l,\beta} S_{j,l,\beta}(t,x,y), \tag{2.4}$$

where $a_{j,l,\beta} \in \mathbb{R}$ and

$$S_{j,l,\beta}(t,x,y) = t^{2\beta-k+1}(x-y)^l y^{2j-m-l} \int_0^\pi \frac{\sin^{2\alpha}\theta\,(1-\cos\theta)^{2j-m-l}}{(t^2+(x-y)^2+2xy(1-\cos\theta))^{\alpha+\frac{3}{2}+j+\beta}}\, d\theta.$$

We write $S_{j,l,\beta} = S^1_{j,l,\beta} + S^2_{j,l,\beta}$, where $S^1_{j,l,\beta}$ and $S^2_{j,l,\beta}$ are defined as $S_{j,l,\beta}$ but taking the integral in $\theta \in (0, \frac{\pi}{2})$ and $\theta \in (\frac{\pi}{2}, \pi)$, respectively.

Firstly, we study $S^2_{j,l,\beta}$. If $0 < \frac{x}{b} < y < bx$, then

$$\begin{aligned}
&\left(\int_0^\infty \left|S^2_{j,l,\beta}(t,x,y)\right|^2 t^{2k+2m-1}dt\right)^{\frac{1}{2}}\\
&\quad \le C|x-y|^l y^{2j-m-l}\left(\left(\int_0^y + \int_y^\infty\right)\frac{t^{2m+4\beta+1}}{(t^2+(x-y)^2+2xy)^{2\alpha+3+2j+2\beta}}dt\right)^{\frac{1}{2}}\\
&\quad \le C|x-y|^l y^{2j-m-l}\left(\frac{y^{2m+4\beta+2}}{(xy)^{2\alpha+3+2j+2\beta}} + y^{-2(2\alpha+2+2j-m)}\right)^{\frac{1}{2}} \le \frac{C}{y^{2\alpha+2}}.
\end{aligned}$$

On the other hand, to analyse $S^i_{j,l,\beta}$, $i = 1, 2$ when $0 < y \le x/b$ or $y \ge bx$, we can write

$$\begin{aligned}
\left(\int_0^\infty \left|S^2_{j,l,\beta}(t,x,y)\right|^2 t^{2k+2m-1}dt\right)^{\frac{1}{2}} &\le C|x-y|^l y^{2j-m-l}\left(\int_0^\infty \frac{t^{2m+4\beta+1}}{(t^2+(x-y)^2)^{2\alpha+3+2j+2\beta}}dt\right)^{\frac{1}{2}}\\
&\le C|x-y|^{2j-m}\left(\int_0^\infty \frac{t^{2m+4\beta+1}}{(t^2+(x-y)^2)^{2\alpha+3+2j+2\beta}}dt\right)^{\frac{1}{2}}\\
&\le C\left(\int_0^\infty \frac{t}{(t^2+(x-y)^2)^{2\alpha+3}}dt\right)^{\frac{1}{2}} \le \frac{C}{|x-y|^{2\alpha+2}};
\end{aligned}$$

the same estimate holds when $S^2_{j,l,\beta}$ is replaced by $S^1_{j,l,\beta}$.

Hence, we conclude that for $i = 1, 2$

$$\left(\int_0^\infty \left|S^i_{j,l,\beta}(t,x,y)\right|^2 t^{2k+2m-1}dt\right)^{\frac{1}{2}} \le C\begin{cases} x^{-2\alpha-2}, & y < \frac{x}{b}\\ y^{-2\alpha-2}, & y > bx\end{cases}. \tag{2.5}$$

Then, at this moment, we have proved that if $0 < x/b < y < bx$, then

$$\left(\int_0^\infty \left|\frac{\partial^{m+k}}{\partial t^k \partial x^m}\int_{\pi/2}^{\pi}\frac{(\sin\theta)^{2\alpha}}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\alpha+3/2}}d\theta\right| t^{2k+2m-1}dt\right)^{\frac{1}{2}} \le C\frac{1}{y^{2\alpha+2}} \tag{2.6}$$

and

$$\left(\int_0^\infty \left|\frac{\partial^{m+k}}{\partial t^k \partial x^m}P_\alpha(t,x,y)\right| t^{2k+2m-1}dt\right)^{\frac{1}{2}} \le C\begin{cases} x^{-2\alpha-2}, & y < \frac{x}{b}\\ y^{-2\alpha-2}, & y > bx\end{cases}. \tag{2.7}$$

Now, we complete the study of $S^1_{j,l,\beta}$. Assume in the sequel that $\frac{x}{b} < y < bx$ and $t > 0$. Since $\sin\theta \sim \theta$ and $1-\cos\theta \sim \frac{\theta^2}{2}$, for $\theta \in [0, \frac{\pi}{2}]$, using the mean value theorem, we get

$$\begin{aligned}
&\left|S^1_{j,l,\beta}(t,x,y) - \frac{t^{2\beta-k+1}}{2^{2j-m-l}}(x-y)^l y^{2j-m-l}\int_0^{\frac{\pi}{2}}\frac{\theta^{2\alpha+2(2j-m-l)}}{(t^2+(x-y)^2+2xy(1-\cos\theta))^{\alpha+\frac{3}{2}+j+\beta}}d\theta\right|\\
&\quad \le C\,t^{2\beta-k+1}|x-y|^l y^{2j-m-l}\int_0^{\frac{\pi}{2}}\frac{\theta^{2\alpha+2(2j-m-l)+2}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}+j+\beta}}d\theta
\end{aligned}$$

$$\leq C\, t^{2\beta-k+1} \frac{|x-y|^l y^{2j-m-l}}{(t^2+(x-y)^2)^{m+l-j+\beta}} (xy)^{-\alpha-2j+m+l-\frac{3}{2}} \int_0^{\frac{\pi}{2}\sqrt{\frac{xy}{(x-y)^2+t^2}}} \frac{z^{2(\alpha+2j-m-l+1)}}{(1+z^2)^{\alpha+\frac{3}{2}+j+\beta}} dz$$
$$\leq C\, \frac{t^{2\beta-k+1}|x-y|^l y^{2j-m-l}}{(t^2+(x-y)^2)^{m+l-j+\beta}} (xy)^{-\alpha-2j+m+l-\frac{3}{2}} \left(\frac{\sqrt{xy}}{\sqrt{xy}+\sqrt{(x-y)^2+t^2}} \right)^{2(\alpha+2j-m-l+1)+1}$$

provided that $(j, l, \beta) \neq (m, 0, 0)$. In the last inequality, we have used the estimation established at the bottom of [**13**, p. 60].

Moreover,

$$\left| S^1_{m,0,0}(t,x,y) - \frac{t^{1-k}}{2^{2m}} y^m \int_0^{\frac{\pi}{2}} \frac{\theta^{2\alpha+2m}}{(t^2+(x-y)^2+2xy(1-\cos\theta))^{\alpha+\frac{3}{2}+m}} d\theta \right|$$
$$\leq C\, t^{1-k} y^m \int_0^{\frac{\pi}{2}} \frac{\theta^{2\alpha+2m+2}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}+m}} d\theta$$
$$\leq C\, t^{1-k} y^m \int_0^{\frac{\pi}{2}} \frac{\theta}{(x-y)^2+xy\theta^2} \frac{\theta^{2\alpha+2m+1}}{(xy\theta^2)^{\alpha+\frac{1}{2}+\frac{m}{2}}(t^2)^{\frac{m}{2}}} d\theta$$
$$\leq C\, t^{1-k-m} y^{-2\alpha-1} \int_0^{\frac{\pi}{2}} \frac{\theta}{(x-y)^2+xy\theta^2} d\theta$$
$$\leq C\, t^{1-k-m} y^{-2\alpha-3} \log\left(1+\frac{\pi^2}{4}\frac{xy}{(x-y)^2}\right).$$

We take $\delta > 0$. We have specified the value of δ later. We have that

$$\int_0^{\delta y} \left| S^1_{j,l,\beta}(t,x,y) - \frac{t^{2\beta-k+1}}{2^{2j-m-l}} (x-y)^l y^{2j-m-l} \right.$$
$$\left. \times \int_0^{\frac{\pi}{2}} \frac{\theta^{2\alpha+2(2j-m-l)}}{(t^2+(x-y)^2+2xy(1-\cos\theta))^{\alpha+\frac{3}{2}+j+\beta}} d\theta \right|^2 t^{2k+2m-1} dt$$
$$\leq C \int_0^{\delta y} \left(\frac{t^{2\beta-k+1}|x-y|^l y^{2j-m-l}}{(t^2+(x-y)^2)^{m+l-j+\beta}(xy)^{\alpha+1+j-\frac{m}{2}-\frac{l}{2}}(t^2+(x-y)^2)^{j-\frac{m}{2}-\frac{l}{2}+\frac{1}{2}}} \right)^2 t^{2k+2m-1} dt$$
$$\leq \frac{C}{y^{4(\alpha+1)}} \int_0^{\delta y} \frac{t^{2m+1}}{((x-y)^2+t^2)^{m+1}} dt \leq \frac{C}{y^{4(\alpha+1)}} \log\left(1+\frac{\delta^2 xy}{(x-y)^2}\right), \; (j,l,\beta) \neq (m,0,0)$$

and

$$\int_0^{\delta y} \left| S^1_{m,0,0}(t,x,y) - \frac{t^{1-k}}{2^{2m}} y^m \int_0^{\frac{\pi}{2}} \frac{\theta^{2\alpha+2m}}{(t^2+(x-y)^2+2xy(1-\cos\theta))^{\alpha+\frac{3}{2}+m}} d\theta \right|^2 t^{2k+2m-1} dt$$
$$\leq \frac{C}{y^{2(2\alpha+3)}} \left(\log\left(1+\frac{\pi^2}{4}\frac{xy}{(x-y)^2}\right)\right)^2 \int_0^{\delta y} t\, dt \leq \frac{C}{y^{4(\alpha+1)}} \left(\log\left(1+\frac{\pi^2}{4}\frac{xy}{(x-y)^2}\right)\right)^2.$$

https://doi.org/10.1017/S0017089508004539 Published online by Cambridge University Press

Using the mean value theorem and proceeding as in the previous estimation, we obtain

$$\int_0^{\delta y} \left| t^{2\beta-k+1}\left(\int_0^{\frac{\pi}{2}} \frac{\theta^{2\alpha+2(2j-m-l)}}{(t^2+(x-y)^2+2xy(1-\cos\theta))^{\alpha+\frac{3}{2}+j+\beta}}d\theta \right.\right.$$
$$\left.\left. - \int_0^{\frac{\pi}{2}} \frac{\theta^{2\alpha+2(2j-m-l)}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}+j+\beta}}d\theta\right)(x-y)^l y^{2j-m-l}\right|^2 t^{2k+2m-1}dt$$
$$\leq C\int_0^{\delta y}\left| t^{2\beta-k+1}(x-y)^l y^{2j-m-l}\int_0^{\frac{\pi}{2}} \frac{\theta^{2\alpha+2(2j-m-l)+4}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}+j+\beta}}d\theta\right|^2 t^{2k+2m-1}dt$$
$$\leq \frac{C}{y^{4(\alpha+1)}}\begin{cases}1, & (i,l,\beta)\neq(m,0,0)\\ \left(\log\left(1+\frac{\pi^2}{4}\frac{xy}{(x-y)^2}\right)\right)^2, & (i,l,\beta)=(m,0,0).\end{cases}$$

By performing differentiation with respect to x and t as in (2.4), we obtain

$$\sum_{j=[\frac{m+1}{2}]}^{m}\sum_{l=0}^{2j-m}\sum_{\beta=[\frac{k}{2}]}^{k} t^{2\beta-m+1}\frac{a_{j,l,\beta}}{2^{2j-m-l}}(x-y)^l y^{2j-m-l}\int_0^{\frac{\pi}{2}}\frac{\theta^{2\alpha+2(2j-m-l)}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}+j+\beta}}d\theta$$
$$= \frac{2\alpha+1}{\pi}\frac{\partial^{m+k}}{\partial t^k\partial x^m}\left(t\int_0^{\frac{\pi}{2}}\frac{\theta^{2\alpha}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}}}d\theta\right)$$
$$= \frac{2\alpha+1}{\pi}\frac{\partial^{m+k}}{\partial t^k\partial x^m}\left(t\left(\int_0^{\infty}-\int_{\frac{\pi}{2}}^{\infty}\right)\frac{\theta^{2\alpha}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}}}d\theta\right)$$
$$= \frac{2\alpha+1}{\pi}\frac{\partial^{m+k}}{\partial t^k\partial x^m}\left(\frac{t(xy)^{-\alpha-\frac{1}{2}}}{(x-y)^2+t^2}\int_0^{\infty}\frac{z^{2\alpha}}{(1+z^2)^{\alpha+\frac{3}{2}}}dz - t\int_{\frac{\pi}{2}}^{\infty}\frac{\theta^{2\alpha}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}}}d\theta\right)$$
$$= \frac{1}{\pi}\frac{\partial^{m+k}}{\partial t^k\partial x^m}\left(\frac{t(xy)^{-\alpha-\frac{1}{2}}}{(x-y)^2+t^2}\right) - \frac{2\alpha+1}{\pi}\frac{\partial^{m+k}}{\partial t^k\partial x^m}\left(t\int_{\frac{\pi}{2}}^{\infty}\frac{\theta^{2\alpha}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}}}d\theta\right).$$

Here $a_{j,l,\beta}$, $\beta=[\frac{k}{2}],\ldots,k, j=[\frac{m+1}{2}],\ldots,m, l=0,\ldots,2j-m$, are as in (2.4).

Differentiating again as in (2.4), we get

$$\frac{2\alpha+1}{\pi}\frac{\partial^{m+k}}{\partial t^k\partial x^m}\left(t\int_{\frac{\pi}{2}}^{\infty}\frac{\theta^{2\alpha}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}}}d\theta\right)$$
$$= \sum_{j=[\frac{m+1}{2}]}^{m}\sum_{l=0}^{2j-m}\sum_{\beta=[\frac{k}{2}]}^{k}\epsilon_{j,l,\beta}t^{2\beta-k+1}\int_{\frac{\pi}{2}}^{\infty}\frac{\theta^{2\alpha+2(2j-m-l)}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}+j+\beta}}d\theta\,(x-y)^l y^{2j-m-l},$$

where $\epsilon_{j,l,\beta}\in\mathbb{R}$ and

$$\frac{1}{\pi}\frac{\partial^{m+k}}{\partial t^k\partial x^m}\left(t\frac{(xy)^{-\alpha-\frac{1}{2}}}{(x-y)^2+t^2}\right) = \frac{1}{\pi}(xy)^{-\alpha-\frac{1}{2}}\frac{\partial^{m+k}}{\partial t^k\partial x^m}\frac{t}{(x-y)^2+t^2}$$
$$+\sum_{l=0}^{m-1}\sum_{s=[\frac{l+1}{2}]}^{l}\sum_{\beta=[\frac{k}{2}]}^{k}\alpha_{l,s,\beta}y^{-\alpha-\frac{1}{2}}x^{-\alpha-\frac{1}{2}-m+l}(x-y)^{2s-l}\frac{t^{2\beta-k+1}}{((x-y)^2+t^2)^{1+s+\beta}}$$

being $\alpha_{l,s,\beta}\in\mathbb{R}$.

https://doi.org/10.1017/S0017089508004539 Published online by Cambridge University Press

We can find $\delta > 0$ and $b > 1$ such that

$$g_{j,l,\beta}(z) \leq g_{j,l,\beta}\left(\frac{\pi}{2}\sqrt{\frac{xy}{(x-y)^2+t^2}}\right), \quad 0 < t < \delta y \text{ and } \frac{x}{b} < y < bx,$$

where $g_{j,l,\beta}(z) = \frac{z^{2(\alpha+2j-m-l)}}{(1+z^2)^{\alpha+\frac{1}{2}+j+\beta}}$, $z \in (0,\infty)$, with $j = [\frac{m+1}{2}], \ldots, m$, $l = 0, \ldots, 2j-m$, $\beta = [\frac{k}{2}], \ldots, k$. In the sequel, δ and b are fixed in this way. Then, we can write

$$\begin{aligned}
&\int_0^{\delta y} \left| t^{2\beta-k+1} \int_{\frac{\pi}{2}}^{\infty} \frac{\theta^{2(\alpha+2j-m-l)}}{(t^2+(x-y)^2+xy\theta^2)^{\alpha+\frac{3}{2}+j+\beta}} d\theta \right|^2 t^{2(m+k)-1} dt\, |x-y|^{2l} y^{2(2j-m-l)} \\
&\leq C \int_0^{\delta y} \left| t^{2\beta-k+1} \frac{(xy)^{-\alpha-2j+m+l-\frac{1}{2}}}{((x-y)^2+t^2)^{m+l-j+\beta+1}} (x-y)^l y^{2j-m-l} \right. \\
&\quad \times \left. \int_{\frac{\pi}{2}\sqrt{\frac{xy}{(x-y)^2+t^2}}}^{\infty} \frac{z^{2(\alpha+2j-m-l)}}{(1+z^2)^{\alpha+\frac{3}{2}+j+\beta}} dz \right|^2 t^{2(m+k)-1} dt \\
&\leq \frac{C}{y^{4(\alpha+1)}} \int_0^{\delta y} \frac{t}{(x-y)^2+t^2} dt \leq \frac{C}{y^{4(\alpha+1)}} \log\left(1+\frac{\delta^2 xy}{(x-y)^2}\right).
\end{aligned}$$

Also, using the estimate at the top of [**13**, p. 61], we obtain

$$\int_0^{\frac{\delta y}{|x-y|}} \frac{z^{2(l+2\beta)+3}}{(1+z^2)^{2(1+s+\beta)}} dz \leq C\left(\frac{\delta y}{\delta y+|x-y|}\right)^{2(l+2\beta+2)} \left(1+\log\left(1+\frac{\delta y}{|x-y|}\right)\right),$$

and proceeding as above, we get

$$\begin{aligned}
&\int_0^{\delta y} \left| \frac{t^{2\beta-k+1} y^{-\alpha-\frac{1}{2}} x^{-\alpha-\frac{1}{2}-m+l}(x-y)^{2s-l}}{((x-y)^2+t^2)^{1+s+\beta}} \right|^2 t^{2(m+k)-1} dt \\
&\leq C y^{-2\alpha-1} x^{-2\alpha-1+2(l-m)} \frac{1}{|x-y|^{2(1+l-m)}} \int_0^{\frac{\delta y}{|x-y|}} \frac{z^{2(m+2\beta)+1}}{(1+z^2)^{2(1+s+\beta)}} dz \\
&\leq C \frac{y^{-2\alpha-1} x^{-2\alpha-1+2(l-m)}}{|x-y|^{2(1+l-m)}} \left(\frac{y}{|x-y|}\right)^{2(m-l-1)} \int_0^{\frac{\delta y}{|x-y|}} \frac{z^{2(l+2\beta)+3}}{(1+z^2)^{2(1+s+\beta)}} dz \\
&\leq C \frac{y^{-2\alpha-1} x^{-2\alpha-1+2(l-m)}}{|x-y|^{2(1+l-m)}} \left(\frac{y}{|x-y|}\right)^{2(m-l-1)} \left(\frac{\delta y}{\delta y+|x-y|}\right)^{2(l+2\beta+2)} \\
&\leq \left(1+\log\left(1+\frac{\delta y}{|x-y|}\right)\right) \leq \frac{C}{y^{4(\alpha+1)}} \left(1+\log\left(1+\frac{\delta\sqrt{xy}}{|x-y|}\right)\right).
\end{aligned}$$

By combining the estimates that we have just proved and (2.6), it follows that

$$\left(\int_0^{\delta y} |h_1(t,x,y)|^2 t^{2(m+k)-1} dt\right)^{\frac{1}{2}} \leq \frac{C}{y^{2(\alpha+1)}} \left(1+\log\left(1+\frac{xy}{|x-y|^2}\right)\right). \tag{2.8}$$

Finally, we deduce

$$\begin{aligned}
&\int_{\delta y}^{\infty} \left|S^1_{j,l,\beta}(t,x,y)\right|^2 t^{2(m+k)-1}\,dt \\
&\leq C\int_{\delta y}^{\infty}\left| t^{2\beta-k+1}\int_0^{\frac{\pi}{2}} \frac{\theta^{2\alpha+2(2j-m-l)}}{((x-y)^2+t^2+xy\theta^2)^{\alpha+\frac{3}{2}+j+\beta}}\,d\theta\,(x-y)^l y^{2j-m-l}\right|^2 t^{2(m+k)-1}\,dt \\
&\leq C\int_{\delta y}^{\infty}\left|\frac{(xy)^{-\alpha-2j+m+l-\frac{1}{2}}}{(t^2+(x-y)^2)^{m+l-j+\beta+1}}\int_0^{\frac{\pi}{2}\sqrt{\frac{xy}{(x-y)^2+t^2}}} \frac{z^{2(\alpha+2j-m-l)}}{(1+z^2)^{\alpha+\frac{3}{2}+j+\beta}}\,dz\right|^2 \\
&\quad\times|x-y|^{2l}y^{2(2j-m-l)}t^{2(m+k)-1}\,dt \\
&\leq \frac{C}{y^{4\alpha+2}}\int_{\delta y}^{\infty}\frac{1}{t^3}\frac{t^{2(m+2\beta+2)}|x-y|^{2l}}{(t^2+(x-y)^2)^{2(m+l-j+\beta+1)}}\left(\frac{\sqrt{xy}}{\sqrt{xy}+\sqrt{(x-y)^2+t^2}}\right)^{4(\alpha+2j-m-l)+2} dt \\
&\leq \frac{C}{y^{4(\alpha+1)}},
\end{aligned}$$

where the third inequality is proved by using the estimate at the bottom of [**13**, p. 60].

Also, by differentiation as in (2.4), it follows that

$$\begin{aligned}
&(xy)^{-(2\alpha+1)}\int_{\delta y}^{\infty}\left|\frac{\partial^{m+k}}{\partial t^k\partial x^m}\frac{t}{(x-y)^2+t^2}\right|^2 t^{2(m+k)-1}\,dt \\
&\quad\leq C(xy)^{-(2\alpha+1)}\sum_{j=[\frac{m+1}{2}]}^{m}\sum_{\beta=[\frac{k}{2}]}^{k}|x-y|^{2(2j-m)}\int_{\delta y}^{\infty}\frac{t^{4\beta+2m+1}}{(|x-y|^2+t^2)^{2j+2\beta+2}}\,dt \\
&\quad\leq C(xy)^{-(2\alpha+1)}\sum_{j=[\frac{m+1}{2}]}^{m}\sum_{\beta=[\frac{k}{2}]}^{k}\int_{\delta y}^{\infty}t^{-3}\,dt\leq\frac{C}{y^{4(\alpha+1)}}.
\end{aligned}$$

Thus, by using (2.6), we prove that

$$\left(\int_{\delta y}^{\infty}|h_1(t,x,y)|^2t^{2(m+k)-1}\,dt\right)^{\frac{1}{2}}\leq\frac{C}{y^{2(\alpha+1)}}. \tag{2.9}$$

From (2.8) and (2.9), we conclude that

$$\begin{aligned}
&\left(\int_0^{\infty}|h_1(t,x,y)|^2t^{2(m+k)-1}\,dt\right)^{\frac{1}{2}} \\
&\quad\leq\frac{C}{y^{2(\alpha+1)}}\left(1+\log\left(1+\frac{xy}{|x-y|^2}\right)\right),\quad 0<x/2<y<2x.
\end{aligned}\tag{2.10}$$

Now, (2.7) and (2.10) lead to

$$\left(\int_0^{\infty}|h_1(t,x,y)|^2t^{2(m+k)-1}\,dt\right)^{\frac{1}{2}}\leq C\begin{cases}x^{-2\alpha-2}, & 0<y<\frac{x}{b}\\ y^{-2\alpha-2}r(x,y), & \frac{x}{b}<y<bx\\ y^{-2\alpha-2}, & bx<y<\infty\end{cases} \tag{2.11}$$

https://doi.org/10.1017/S0017089508004539 Published online by Cambridge University Press

where $r(x,y) = 1 + \log(1 + \frac{xy}{(x-y)^2})$. Hence, according to [**17**, p. 272], the operators $\mathcal{H}_{1,1}(f)(x) = \int_0^{x/b} H_1(x,y)f(y)d\gamma_\alpha(y)$ and $\mathcal{H}_{1,3}(f)(x) = \int_{bx}^\infty H_1(x,y)f(y)d\gamma_\alpha(y)$ are bounded in $L^2((0,\infty), d\gamma_\alpha)$.

Also, the operator $\mathcal{H}_{1,2}(f)(x) = \int_{x/b}^{bx} H_1(x,y)f(y)d\gamma_\alpha(y)$ is bounded in $L^2((0,\infty), d\gamma_\alpha)$. Indeed, note that

$$\int_{x/b}^{bx} y^{-2\alpha-2} r(x,y)d\gamma_\alpha(y) = \int_{1/b}^{b} \frac{1}{u}\left(1 + \log\left(1 + \frac{u}{|1-u|^2}\right)\right)du \neq 0, \quad x \in (0,\infty).$$

Then, Jensen's inequality leads to

$$\begin{aligned}
\|\mathcal{H}_{1,2}(f)\|^2_{L^2((0,\infty),d\gamma_\alpha)} &= \int_0^\infty \left| \int_{x/b}^{bx} H_1(x,y)f(y)d\gamma_\alpha(y)\right|^2 d\gamma_\alpha(x)\\
&\leq C\int_0^\infty \left(\int_{x/b}^{bx} |f(y)|y^{-2\alpha-2}r(x,y)d\gamma_\alpha(y)\right)^2 d\gamma_\alpha(x)\\
&\leq C\int_0^\infty \int_{x/b}^{bx} |f(y)|^2 y^{-2\alpha-2}r(x,y)d\gamma_\alpha(y)d\gamma_\alpha(x)\\
&\leq C\int_0^\infty |f(y)|^2 \int_{y/b}^{by} x^{-2\alpha-2}r(x,y)d\gamma_\alpha(x)d\gamma_\alpha(y)\\
&\leq C\|f\|^2_{L^2((0,\infty),d\gamma_\alpha)}
\end{aligned}$$

for every $f \in L^2((0,\infty), d\gamma_\alpha)$.

This finishes the proof of boundedness of $\mathcal{H}_1$ in $L^2((0,\infty), d\gamma_\alpha)$.

To treat $\mathcal{H}_2$ since $h_2(t,x,y) = -\frac{1}{\pi}\chi_{\{\frac{x}{b}<y<bx\}^c}(y)(xy)^{-\alpha-1/2}\frac{\partial^{m+k}}{\partial t^k \partial x^m}(\frac{t}{(x-y)^2+t^2})$, we write as in (2.4)

$$h_2(t,x,y) = -\chi_{\{\frac{x}{b}<y<bx\}^c}(y)(xy)^{-\alpha-\frac{1}{2}} \sum_{j=[\frac{m+1}{2}]}^{m} \sum_{\beta=[\frac{k}{2}]}^{k} c_{j,\beta}\, t^{2\beta-k+1}\frac{(x-y)^{2j-m}}{(t^2+(x-y)^2)^{j+\beta+1}},$$
$$t,x,y \in (0,\infty),$$

where $c_{j,\beta} \in \mathbb{R}$.

Then,

$$\begin{aligned}
H_2(x,y) &= \left(\int_0^\infty |h_2(t,x,y)|^2 t^{2(k+m)-1}dt\right)^{\frac{1}{2}}\\
&\leq C\chi_{\{\frac{x}{b}<y<bx\}^c}(y)(xy)^{-\alpha-\frac{1}{2}} \sum_{j=[\frac{m+1}{2}]}^{m} \sum_{\beta=[\frac{k}{2}]}^{k} \left(\int_0^\infty \frac{t^{2m+4\beta+1}|x-y|^{4j-2m}}{(t^2+(x-y)^2)^{2j+2\beta+2}}dt\right)^{\frac{1}{2}}\\
&\leq C\chi_{\{\frac{x}{b}<y<bx\}^c}(y)(xy)^{-\alpha-\frac{1}{2}}\left(\int_0^\infty (t+|x-y|)^{-3}dt\right)^{\frac{1}{2}}\\
&\leq C(xy)^{-\alpha-\frac{1}{2}}\begin{cases} 1/x, & y < \frac{x}{b}\\ 1/y, & y > bx\end{cases}.
\end{aligned}$$

https://doi.org/10.1017/S0017089508004539 Published online by Cambridge University Press

Then, according to [**17**, p. 272], we can deduce that the operator $\mathcal{H}_2(f)(x) = \int_0^\infty H_2(x,y)f(y)d\gamma_\alpha(y)$ is bounded on $L^2((0,\infty), d\gamma_\alpha)$. By combining these results, we prove that the operator $\mathcal{H}$ is bounded on $L^2((0,\infty), d\gamma_\alpha)$. □

LEMMA 2.3. *Suppose that $f \in L^2((0,\infty), d\gamma_\alpha)$. Then,*

$$G_\alpha^{(k,m)}(f)(x) = \int_0^\infty K_\alpha^{(k,m)}(x,y)f(y)d\gamma_\alpha(y), \quad a.e. \;\; x \notin \operatorname{supp} f,$$

where $K_\alpha^{(k,m)}$ is given by (1.6).

Proof. Denote by $A = \operatorname{supp} f$. Let $g \in L^2((0,\infty), d\gamma_\alpha)$, and assume that $g \in L^2_{\mathbb{B}}(A^c, d\gamma_\alpha)$ is smooth and has compact support contained in A^c. Our objective is to show that

$$\left\langle G_\alpha^{(k,m)}(f), g\right\rangle_{L^2_{\mathbb{B}}(A^c, d\gamma_\alpha)} = \left\langle \int_0^\infty K_\alpha^{(k,m)}(x,y)f(y)d\gamma_\alpha(y), g(x)\right\rangle_{L^2_{\mathbb{B}}(A^c, d\gamma_\alpha)}. \tag{2.12}$$

To see (2.12), we note that

$$\begin{aligned}\left\langle G_\alpha^{(k,m)}(f), g\right\rangle_{L^2_{\mathbb{B}}(A^c, d\gamma_\alpha)} &= \int_{A^c} \left\langle G_\alpha^{(k,m)}(f), g(x,\cdot)\right\rangle_{L^2((0,\infty), t^{2(m+k)-1}dt)} d\gamma_\alpha(x)\\ &= \int_{A^c}\int_0^\infty \frac{\partial^{m+k}}{\partial x^m \partial t^k} P_{\alpha,t}(f)(x)\overline{g(x,t)}t^{2(m+k)-1}dtd\gamma_\alpha(x)\\ &= \int_{A^c}\int_0^\infty\int_0^\infty \frac{\partial^{m+k}}{\partial x^m \partial t^k} P_\alpha(t,x,y)f(y)d\gamma_\alpha(y)\overline{g(x,t)}\\ &\quad \times t^{2(m+k)-1}dtd\gamma_\alpha(x)\\ &= \left\langle \int_0^\infty K_\alpha^{(k,m)}(x,y)f(y)d\gamma_\alpha(y), g(x)\right\rangle_{L^2_{\mathbb{B}}(A^c, d\gamma_\alpha)},\end{aligned}$$

provided that the differentiation under integral sign is justified. This differentiation can be made because, according to (2.4), for every $x \notin A$ and $t \in (0,\infty)$, there exists $C > 0$ depending only on the distance $d(x,A)$ of x to A and on $t \in (0,\infty)$ for which

$$\begin{aligned}&\int_0^\infty \left|\frac{\partial^{m+k}}{\partial x^m \partial t^k} P_\alpha(t,x,y)\right| |f(y)|d\gamma_\alpha(y)\\ &\quad \leq \left\{\int_A \left|\frac{\partial^{m+k}}{\partial x^m \partial t^k} P_\alpha(t,x,y)\right|^2 d\gamma_\alpha(y)\right\}^{\frac{1}{2}} \left\{\int_0^\infty |f(y)|^2 d\gamma_\alpha(y)\right\}^{\frac{1}{2}}\\ &\quad \leq C\|f\|_{L^2((0,\infty), d\gamma_\alpha)}.\end{aligned}$$

□

LEMMA 2.4. *If $K_\alpha^{(k,m)}$ is given by* (1.6)*, then the properties* (i) *and* (ii) *in Theorem* 1.3 *are satisfied.*

Proof. Firstly, note that (2.7) implies (i) provided that $0 < y < \frac{x}{b}$ or $\frac{x}{b} < y < \infty$.

https://doi.org/10.1017/S0017089508004539 Published online by Cambridge University Press

Also, by repeating the analysis developed in the proof of Lemma 2.2 without introducing logarithmic, terms, we get, for $0 < \frac{x}{b} < y < bx$,

$$\left\| \frac{\partial^{m+k}}{\partial t^k \partial x^m} P_\alpha(t, x, y) - \frac{1}{\pi} \chi_{\{\frac{x}{b} < y < bx\}}(y)(xy)^{-\alpha-\frac{1}{2}} \frac{\partial^{m+k}}{\partial t^k \partial x^m} \left(\frac{t}{(x-y)^2 + t^2} \right) \right\|_{\mathbb{B}} \\ \leq \frac{C}{|x-y| \max\{x^{2\alpha+1}, y^{2\alpha+1}\}}.$$

Moreover, proceeding as in the estimate of H_2 in the proof of Lemma 2.2, we can obtain

$$\left\| \frac{\partial^{m+k}}{\partial t^k \partial x^m} \left(\frac{t}{(x-y)^2 + t^2} \right) \right\|_{\mathbb{B}} \leq \frac{C}{|x-y|}, \qquad x, y \in (0, \infty).$$

Then, for every $0 < \frac{x}{b} < y < bx$, (i) holds. Thus, it is established that $K_\alpha^{(k,m)}$ satisfies condition (i) for every $x, y \in (0, \infty)$.

To see that $K_\alpha^{(k,m)}$ verifies condition (ii) we can proceed by differentiating as in (2.4) and by using the arguments presented in the proof of Lemma 2.2. $\square$

Theorem 1.1 implies now that the g-function $g_\alpha^{(k,m)}$ satisfies the boundedness properties listed in Theorem 1.4.

Acknowledgements. The authors thanks J. L. Torrea, M. T. Martínez and L. Rodríguez-Mesa for very fruitful comments about harmonic analysis in the Bessel setting that allowed us to improve the results of this paper. Also, we thank the referee for helpful suggestions that greatly improved the content of this paper. Partially supported by MTM2004/05878 and MTM2007/65609. Third author (A. S.) was also supported by Consejería de Educación Gobierno de Canarias PI 04 2004/067.

REFERENCES

1. K. F. Andersen and R. A. Kerman, Weighted norm inequalities for generalized Hankel conjugate transformations, *Stud. Math.* **71**(1) (1981/82), 15–26.

2. J. J. Betancor, D. Buraczewski, J. C. Fariña, M. Martínez and J. L. Torrea, Riesz transforms related to Bessel operators, *Proc. Roy. Soc. Edinb.* **137A** (2007), 701–725.

3. D. Buraczewski, M. Martínez, J. L. Torrea and R. Urban, On the Riesz transform associated to the ultraspherical polynomials, *J. d'Analyse Math.* **98** (2006), 113–143.

4. D. Buraczewski, M. Martínez and J. L. Torrea, Calderón-Zygmund operators associated to ultraspherical expansions, *Can. J. Math.* **59**(6) (2007), 1223–1244.

5. R. Coifman and Y. Meyer, *Au dela des operateurs pseudo-differentielles*. (Astérisque 57 (Société Mathématique de France, Paris, 1978).

6. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*. Lecture Notes in Mathematics, Vol. 242 (Springer-Verlag, Berlin-New York, 1971).

7. J. Diestel and J. Ull, *Vector measures*, Mathematical Surveys, No. 15. (American Mathematical Society, Providence, RI, 1977).

8. J. Duoandikoetxea, Fourier analysis. *Graduate Studies in Mathematics, Vol.* 29 (American Mathematical Society, Providence, RI, 2001).

9. J. García-Cuerva, G. Mauceri, P. Sjögren and J. L. Torrea, Spectral multipliers for the Ornstein-Uhlenbeck semigroup, *J. Anal. Math.* **78** (1999), 281–305.

10. R. Macías and C. Segovia, A decomposition into atoms of distributions on spaces of homogeneous type, *Adv. Math.* **33**(3) (1979), 271–309.

11. R. Macías and C. Segovia, Lipschitz functions on spaces of homogeneous type, *Adv. Math.* **33**(3) (1979), 257–270.

12. R. Macías, C. Segovia and J. L. Torrea, Heat-diffusion maximal operators for Laguerre semigroups with negative parameters, *J. Funct. Anal.* **229**(2) (2005), 300–316.

13. B. Muckenhoupt and E. Stein, Classical expansions and their relations to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17–92.

14. A. Nowak and K. Stempak, Riesz transforms for multi-dimensional Laguerre function expansions, *Adv. Math.* **215** (2007), 642–678.

15. J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, Les opérateurs de Calderón-Zygmund vectoriels, *C. R. Acad. Sci. Paris* Serie I **297** (1983), 477–480.

16. F. J. Ruiz and J. L. Torrea, Vector-valued Calderón-Zygmund theory and Carleson measures on spaces of homogeneous nature, *Stud. Math.* **88** (3) (1988), 221–243.

17. E. M. Stein, Singular integrals and differentiability properties of functions. *Princeton Mathematical Series, No*. 30 (Princeton University Press, Princeton, NJ 1970).

18. E. M. Stein, Topics in harmonic analysis related to the Littlewood Paley theory, (Princeton University. Press, Princeton, NJ 1970).

19. K. Stempak, *The Littlewood-Paley theory for the Fourier-Bessel transform.* Preprint no. 45 (Mathematical Institute of Wroclaw, Poland, 1985).

20. K. Stempak and J. L. Torrea, Poisson integrals and Riesz transforms for Hermite function expansions with weights, *J. Funct. Anal.* **202**(2) (2003), 443–472.

21. K. Stempak and J. L. Torrea, On g-functions for Hermite function expansions, *Acta Math. Hungar.* **109**(1–2) (2005), 99–125.

22. K. Stempak and J. L. Torrea, BMO results for operators associated to Hermite expansions, *Illinois J. Math.* **49**(4) (2005), 1111–1131.

23. K. Stempak and J. L. Torrea, Higher Riesz transforms and imaginary powers associated to the harmonic oscillator, *Acta Math. Hungar.* **111** (1–2) (2006), 43–64.

24. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, (Cambridge, England; The Macmillan Company, New York, 1944).

25. A. Weinstein, Discontinuous integrals and generalized potential theory, *Trans. Amer. Math. Soc.* **63** (1948), 342–354.