Bull. Aust. Math. Soc. **106** (2022), 269–272
doi:10.1017/S0004972722000041

AN APPLICATION OF BINARY QUADRATIC FORMS OF DISCRIMINANT −31 TO MODULAR FORMS

ZAFER SELCUK AYGIN and KENNETH S. WILLIAMS

(Received 26 November 2021; accepted 6 December 2021; first published online 8 February 2022)

Abstract

In this note, we use Dedekind's eta function to prove a congruence relation between the number of representations by binary quadratic forms of discriminant −31 and Fourier coefficients of a weight 16 cusp form. Our result is analogous to the classical result concerning Ramanujan's tau function and binary quadratic forms of discriminant −23.

2020 *Mathematics subject classification*: primary 11F11; secondary 11E25, 11F20, 11F27.

Keywords and phrases: binary quadratic forms, cusp forms.

The Ramanujan tau function τ is defined by

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

where $q := e^{2\pi iz}$ ($z \in \mathbb{C}$, $\mathrm{Im}(z) > 0$). In 1930, Wilton [5] determined $\tau(n)$ modulo 23 for all positive integers n. In 2006, Sun and Williams [3, Corollary 2.2, page 357] obtained Wilton's congruence for $\tau(n)$ modulo 23 as a consequence of their work on binary quadratic forms. Recently Dr. Pieter Moree of the Max Planck Institute for Mathematics in Bonn, Germany, in relation to his recent work [1] with Ciolan and Languasco, asked the second author if the analogous congruence modulo 31 could be obtained using the ideas of [3] for the function $\tau_{16}(n)$, where

$$\Delta(z)E_4(z) := \sum_{n=1}^{\infty}\tau_{16}(n)q^n$$

and $E_4(q)$ is the Eisenstein series

$$E_4(z) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n.$$

Swinnerton-Dyer [4, page 34], before giving the arguments that prove the congruence relation (3), notes that '*there seems little prospect*' of proving this congruence using Dedekind's eta function. In this note, we show that it can be done by giving an explicit proof of the congruence relation (3) using Dedekind's eta function. Then we combine our results with [2, Theorem 10.2, page 166] to obtain the following congruence for $\tau_{16}(n)$ modulo 31.

THEOREM 1. *For any positive integer n,*

$$\tau_{16}(n) \equiv \begin{cases} 0 \pmod{31} & \text{if there is a prime } p \mid n \text{ with } (\frac{p}{31}) = -1 \\ & \text{and } \nu_p(n) \equiv 1 \pmod 2, \text{ or } (\frac{p}{31}) = 1, \\ & p = 2x^2 + xy + 4y^2 \text{ and } \nu_p(n) \equiv 2 \pmod 3, \\ (-1)^{\mu} \prod\limits_{\substack{p \mid n, \\ (\frac{p}{31})=1, \\ p=x^2+xy+8y^2}} (1 + \nu_p(n)) \pmod{31} & \text{otherwise,} \end{cases}$$

where

$$\mu = \sum_{\substack{p \mid n, \\ (\frac{p}{31})=1, \\ p=2x^2+xy+4y^2, \\ \nu_p(n) \equiv 1 \pmod 3}} 1.$$

PROOF. We use the Dedekind eta function which is defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

We have

$$\begin{aligned} \Delta(z)E_4(z) &= \frac{\eta^{32}(z)\eta^{32}(4z)}{\eta^{32}(2z)} + \frac{1}{2}\Big(\frac{\eta^{64}(2z)}{\eta^{32}(4z)} - \frac{\eta^{64}(z)}{\eta^{32}(2z)}\Big) \\ &\quad + 31\Big(69271552\frac{\eta^{64}(4z)}{\eta^{32}(2z)} - 34095104\frac{\eta^{48}(4z)}{\eta^{8}(z)\eta^{8}(2z)} + 7915008\frac{\eta^{16}(2z)\eta^{32}(4z)}{\eta^{16}(z)} \\ &\quad - 1050688\frac{\eta^{40}(2z)\eta^{16}(4z)}{\eta^{24}(z)} + 82977\frac{\eta^{64}(2z)}{\eta^{32}(z)} - 3840\frac{\eta^{88}(2z)}{\eta^{40}(z)\eta^{16}(4z)} \\ &\quad + 96\frac{\eta^{112}(2z)}{\eta^{48}(z)\eta^{32}(4z)} - \frac{\eta^{136}(2z)}{\eta^{56}(z)\eta^{48}(4z)}\Big). \end{aligned}$$

https://doi.org/10.1017/S0004972722000041 Published online by Cambridge University Press

All of the functions in this modular equation are in $M_{16}(\Gamma_0(4))$ and the identity can be proved using Sturm's theorem. Thus, we have

$$\Delta(z)E_4(z) \equiv \frac{\eta^{32}(z)\eta^{32}(4z)}{\eta^{32}(2z)} + \frac{1}{2}\Big(\frac{\eta^{64}(2z)}{\eta^{32}(4z)} - \frac{\eta^{64}(z)}{\eta^{32}(2z)}\Big) \pmod{31}. \tag{1}$$

If $f := ax^2 + bxy + cy^2$ is a positive-definite integral binary quadratic form, we denote by $r(f; n)$ the number of representations of a nonnegative integer n by f. We set

$$\begin{aligned}\phi_2(z) &:= \frac{1}{2}\sum_{n\geq 0}(r(x^2+xy+8y^2;n) - r(2x^2+xy+4y^2;n))q^n \\ &= \frac{1}{2}\sum_{x,y=-\infty}^{\infty}(q^{x^2+xy+8y^2} - q^{2x^2+xy+4y^2}).\end{aligned}$$

The theta functions $\sum_{x,y=-\infty}^{\infty} q^{x^2+xy+8y^2}$ and $\sum_{x,y=-\infty}^{\infty} q^{2x^2+xy+4y^2}$ belong to the space $M_1(\Gamma_0(124), (\frac{-31}{*}))$ as do the eta quotients

$$\frac{\eta(z)\eta(4z)\eta(31z)\eta(124z)}{\eta(2z)\eta(62z)}, \quad \frac{\eta^2(2z)\eta^2(62z)}{\eta(4z)\eta(124z)} \quad \text{and} \quad \frac{\eta^2(z)\eta^2(31z)}{\eta(2z)\eta(62z)}.$$

Then it is straightforward to prove the modular identity

$$\phi_2(z) = \frac{\eta(z)\eta(4z)\eta(31z)\eta(124z)}{\eta(2z)\eta(62z)} + \frac{1}{2}\Big(\frac{\eta^2(2z)\eta^2(62z)}{\eta(4z)\eta(124z)} - \frac{\eta^2(z)\eta^2(31z)}{\eta(2z)\eta(62z)}\Big)$$

using Sturm's theorem. We have $1 - A^{31} \equiv (1 - A)^{31} \pmod{31}$ by the binomial theorem, so that

$$\phi_2(z) \equiv \frac{\eta^{32}(z)\eta^{32}(4z)}{\eta^{32}(2z)} + \frac{1}{2}\Big(\frac{\eta^{64}(2z)}{\eta^{32}(4z)} - \frac{\eta^{64}(z)}{\eta^{32}(2z)}\Big) \pmod{31}. \tag{2}$$

From (1) and (2), we deduce that

$$\Delta(z)E_4(z) \equiv \phi_2(z) \pmod{31}. \tag{3}$$

Appealing to the formula for $\frac{1}{2}(r(x^2 + xy + 8y^2; n) - r(2x^2 + xy + 4y^2; n))$ given in [2, Theorem 10.2, page 166], we obtain from (3) the congruence for $\tau_{16}(n)$ stated in the theorem. □

References

[1] A. Ciolan, A. Languasco and P. Moree, 'Landau and Ramanujan approximations for divisor sums and coefficients of cusp forms', Preprint, 2021, arXiv:2109.03288, 43 pages.

[2] Z.-H. Sun and K. S. Williams, 'On the number of representations of n by $ax^2 + bxy + cy^2$', *Acta Arith.* **122**(2) (2006), 101–171.

[3] Z.-H. Sun and K. S. Williams, 'Ramanujan identities and Euler products for a type of Dirichlet series', *Acta Arith.* **122**(4) (2006), 349–393.

[4] H. P. F. Swinnerton-Dyer, 'On ℓ-adic representations and congruences for coefficients of modular forms', in: *Modular Functions of One Variable III (Proceedings International Summer School*

https://doi.org/10.1017/S0004972722000041 Published online by Cambridge University Press

University of Antwerp, RUCA July 17–August 3, 1972), Lecture Notes in Mathematics, 350 (eds. W. Kuijk and J.-P. Serre) (Springer, Berlin, 1973), 1–55.

[5] J. R. Wilton, 'Congruence properties of Ramanujan's function $\tau(n)$', *Proc. Lond. Math. Soc. (3)* **31** (1930), 1–10.

ZAFER SELCUK AYGIN, Centre for Research in Algebra and Number Theory,
School of Mathematics and Statistics, Carleton University,
Ottawa, Ontario K1S 5B6, Canada
and
University Studies Department, Northern Lakes College,
Slave Lake, Alberta T0G2A3, Canada
e-mail: selcukaygin@gmail.com

KENNETH S. WILLIAMS, Centre for Research in Algebra and Number Theory,
School of Mathematics and Statistics, Carleton University,
Ottawa, Ontario K1S 5B6, Canada
e-mail: kennethwilliams@cunet.carleton.ca

https://doi.org/10.1017/S0004972722000041 Published online by Cambridge University Press