ON A CLASS OF SEMI-MARKOV RISK MODELS OBTAINED AS CLASSICAL RISK MODELS IN A MARKOVIAN ENVIRONMENT

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ABSTRACT

We consider a risk model in which the claim inter-arrivals and amounts depend on a markovian environment process. Semi-Markov risk models are so introduced in a quite natural way. We derive some quantities of interest for the risk process and obtain a necessary and sufficient condition for the fairness of the risk (positive asymptotic non-ruin probabilities). These probabilities are explicitly calculated in a particular case (two possible states for the environment, exponential claim amounts distributions).

KEYWORDS
Semi-Markov processes, ruin theory.

1. INTRODUCTION

Several authors have used the semi-Markov processes in Queuing Theory and in Risk Theory [e.g., CINLAR (1967), NEUTS (1966), NEUTS and SHUN-ZER CHEN (1972), PURDUE (1974), JANSSEN (1980), REINHARD (1981)]. Besides, some duality results lead to nice connections between the two theories [FELLER (1971), JANSSEN and REINHARD (1982)].

Semi-Markov risk models may be defined as follows. Consider a risk model in continuous time; let B_n $(n \in N_0)$* and U_n $(n \in N_0)$ denote respectively the amount and the arrival time of the nth claim. Put $A_0 = B_0 = U_0 = 0$ and define $A_n = U_n - U_{n-1}$ $(n \in N_0)$. We suppose that the A_n and B_n are random variables defined on a complete probability space $(\Omega, \mathcal{A}, P)$; the variables A_n $(n \in N_0)$ are a.s. positive. Let now J_n $(n \in N)$ be random variables defined on $(\Omega, \mathcal{A}, P)$ and taking their values in $J = \{1, \dots, m\}$ $(m \in N_0)$. Suppose finally that $\{(J_n, A_n, B_n); n \in N\}$ is a Markov chain with transition probabilities defined by a bivariate semi-Markov kernel:

$$P[J_{n+1} = j, A_{n+1} \leq t, B_{n+1} \leq x \mid J_k, A_k, B_k; k = 0, \dots, n] = Q_{J_n j}(x, t) \quad \text{a.s.} \tag{1.1}$$
$$(j \in J, \quad t \geq 0, \quad x \in R, \quad n \in N)$$

where $Q_{ij}(x, \cdot)$ and $Q_{ij}(\cdot, t)$ are right continuous nondecreasing functions satisfying:

$$Q_{ij}(x, t) \geq 0, \quad Q_{ij}(\infty, 0) = 0 \qquad (i, j \in J; t \geq 0)$$
$$\sum_{j=1}^{m} Q_{ij}(\infty, \infty) = 1 \qquad (i \in J)$$
$$Q_{ij}(-\infty, \infty) = 0 \qquad (i, j \in J).$$

* $N_0 = \{1, 2, 3, \dots\}$; $N = \{0, 1, 2, 3, \dots\}$.

Such processes, called $(J\text{-}Y\text{-}X)$ processes, were studied by JANSSEN and REINHARD (1982) and REINHARD (1982). In the particular case where

(1.2) $$Q_{ij}(x,t)=(1-e^{-\lambda t})Q_{ij}(x), \qquad \lambda>0,$$

the processes $\{A_n\}$ and $\{(J_n, B_n)\}$ being independent, JANSSEN (1980) interpreted the variables J_n as the types of the successive claims. The next section will show that another subclass of semi-Markov kernels appears if we assume that the risk depends on an environment process.

2. RISK PROCESSES IN A MARKOVIAN ENVIRONMENT

Suppose that the claim frequency and amounts depend on the external environment (economic situation . . .) and that the external environment may be characterized at any time by one of the m states $1, \dots, m$ $(m \in N_0)$. Let I_0 denote the state of the environment at time $t=0$ and let I_n, $n=1, \dots,$ be the state of the environment after its nth transition. Put $T_0=0$ and let T_n $(n \in N_0)$ be the time at which occurs the nth transition of the environment process. We suppose that I_n and T_n $(n \in N)$ are random variables defined on $(\Omega, \mathcal{A}, P)$ and taking their values in J and R^+ respectively. Define now $Y_n=T_n-T_{n-1}$ $(n \in N_0)$, $Y_0=0$ and assume that

(2.1) $$P[I_{n+1}=j,\ Y_{n+1}\le t\,|\,(I_k, Y_k), k=0,\dots,n, I_n=i]=h_{ij}(1-e^{-\lambda_i t})$$
$$(i,j\in J;\quad t\ge 0;\quad n\in N)$$

where the λ_i are strictly positive real numbers and $H=(h_{ij})$ is a transition matrix:

$$h_{ij}\ge 0, \qquad \sum_{k=1}^{m} h_{ik}=1 \qquad (i,j\in J).$$

$\{I_n, n\in N\}$ is then a Markov chain with a matrix of transition probabilities $H=(h_{ij})$:

(2.2) $$h_{ij}=P[I_{n+1}=j\,|\,I_n=i].$$

Define $N_e(t)=\sup\{n: T_n\le t\}$ and $I(t)=I_{N_e(t)}$ $(t\ge 0)$. The process $\{I(t), t\ge 0\}$ is a finite-state Markov process; it is known that the number of transitions of the environment process $\{I(t)\}$ in any finite interval $(s,t]$, i.e., $N_e(t)-N_e(s)$, is a.s. finite.

Denote now by J_n the state of the environment process at the arrival of the nth claim:

(2.3) $$J_n=I(U_n) \qquad (n\in N).$$

We will suppose that the following assumptions are satisfied:

(H1) The sequences of random variables (A_n) and (B_n) are conditionally independent given the variables J_n.

(H2) The distribution of a claim depends uniquely on the state of the environment at the time of arrival of that claim. Let

(2.4) $$F_i(x)=P[B_n\le x\,|\,J_n=i] \qquad (i\in J,\quad n\in N,\quad x\in R)$$

https://doi.org/10.1017/S0515036100004785 Published online by Cambridge University Press

(H3) Let $N(t)$ be the number of claims occurring in $(0, t]$. If $I(u) = i$ for all u in some interval $(t, t+h]$, then the number of claims occurring in that interval, i.e., $N(t+h) - N(t)$, has a Poisson distribution with parameter α_i ($\alpha_i > 0$); we assume further that given the process $\{I(t)\}$ the process $\{N(t)\}$ has independent increments. So

(2.5) $$P[N(t+h) = n+1 \mid N(t) = n, I(u) = i \text{ for } t < u \leq t+h] = \alpha_i h + o(h).$$

The process $\{N(t); t \geq 0\}$ appears thus as a Poisson process with parameter modified by the transitions of the environment process.

Under the above assumptions it may be shown that $\{(J_n, A_n, B_n), n \in N\}$ is a $(J\text{–}Y\text{–}X)$ process with semi-Markov kernel $\mathcal{Q}$ defined by (1.1). $\{(J_n, A_n), n \in N\}$ is a Markov renewal process [see PYKE (1961)]; we denote its kernel by $\mathcal{V} = (V_{ij}(\cdot))$:

(2.6) $$V_{ij}(t) = P[J_{n+1} = j, A_n \leq t \mid (J_k, A_k), k = 0, \ldots, n; J_n = i]$$
$$(i, j \in J, \quad n \in N, \quad t \geq 0).$$

Moreover it follows from the assumptions that

(2.7) $$Q_{ij}(x, t) = V_{ij}(t) F_j(x) \qquad (i, j \in J, \quad t \geq 0, \quad x \in R).$$

$\{J_n, n \in N\}$ is a Markov chain with matrix P of transition probabilities defined by

(2.8) $$P_{ij} = P[J_{n+1} = j \mid J_n = i] = Q_{ij}(\infty, \infty) = V_{ij}(\infty) \qquad (i, j \in J).$$

In the next section it will be shown how the semi-Markov kernel $\mathcal{Q}$ (or equivalently $\mathcal{V}$) can be deduced from the instantaneous rates α_i, the transition matrix H, the constants λ_i and the distributions $F_i(\cdot)$.

3. COMPUTATION OF THE KERNEL

Let us first introduce some notations: for any mass function (i.e., right continuous and non-decreasing) $G(t)$ defined on R^+ let

$$\tilde{G}(s) = \int_0^\infty e^{-st} G(t)\, dt, \qquad g(s) = \int_{0-}^\infty e^{-st}\, dG(t)$$

provided the above integrals converge.

The following system of integral equations may be easily deduced from the hypothesis

(3.1) $$V_{ij}(t) = \delta_{ij} \frac{\alpha_i}{\alpha_i + \lambda_i}(1 - e^{-(\alpha_i + \lambda_i)t}) + \lambda_i \sum_{k=1}^{m} h_{ik} \int_0^t e^{-(\alpha_i + \lambda_i)u} V_{kj}(t-u)\, du$$
$$(i, j \in J; \quad t \geq 0).$$

The first term in the right side of (3.1) corresponds to the case where a claim occurs before the environment changes, the second term to the case where the environment changes before a claim occurs.

For $s \geq 0$, define now the following matrices:

$$L(s) = (h_{ij}\lambda_i/(\alpha_i + s + \lambda_i)), \qquad E(s) = (\delta_{ij}\alpha_i/(\alpha_i + s + \lambda_i)).$$

By taking the Laplace transforms of both sides in (3.1) we obtain

$$\tilde{V}_{ij}(s) = \delta_{ij}\frac{\alpha_i}{s(\alpha_i+\lambda_i+s)} + \frac{\lambda_i}{\alpha_i+\lambda_i+s}\sum_{k=1}^{m} h_{ik}\tilde{V}_{kj}(s) \tag{3.2}$$

$$(i, j \in J; \quad s > 0),$$

or, in matrix notation,

$$[I - L(s)]\tilde{V}(s) = (1/s)E(s) \qquad (s > 0) \tag{3.3}$$

(we will always use the same symbol for a matrix and its elements whenever this causes no ambiguity). As for any $s \geq 0$

$$L_i(s) = \sum_{j=1}^{m} L_{ij}(s) = \frac{\lambda_i}{\alpha_i+\lambda_i+s} < 1,$$

$I - L(s)$ is regular for $s \geq 0$ and consequently (3.3) has as unique solution

$$\tilde{V}(s) = (1/s)[I - L(s)]^{-1}E(s) \qquad (s > 0), \tag{3.4}$$

or equivalently

$$v(s) = [I - L(s)]^{-1}E(s) \qquad (s > 0). \tag{3.5}$$

As $p_{ij} = V_{ij}(\infty) = \lim_{s\searrow 0} v_{ij}(s)$, the matrix P of the transition probabilities of the chain $\{J_n\}$ can be directly deduced from (3.5):

$$P = [I - L(0)]^{-1}E(0). \tag{3.6}$$

Notice that the semi-Markov kernel $\mathscr{V}$ is solution of a first order linear differential system: by deriving (3.1) with respect to t we obtain

$$V'_{ij}(t) = \alpha_i\delta_{ij} + \sum_{k=1}^{m} [\lambda_i h_{ik} - (\alpha_i+\lambda_i)\delta_{ik}]V_{kj}(t) \qquad (i, j \in J; \quad t \geq 0). \tag{3.7}$$

4. SOME RESULTS ABOUT QUANTITIES RELATED TO THE RISK PROCESS

In this section we derive some explicit expressions or equations related to the semi-Markov risk-process defined in the preceding sections.

4.1. *Stationary Probabilities of the Chain* $\{J_n\}$

From now on we suppose that the chain $\{J_n\}$ is irreducible. As m is finite there exists a unique probability distribution $\bar{\eta} = (\eta_1, \ldots, \eta_m)$ such that

$$\eta_i > 0 \qquad (i \in J), \tag{4.1}$$

$$\sum_{i=1}^{m} \eta_i h_{ij} = \eta_j \qquad (j \in J).$$

We have then:

THEOREM 1

The Markov chain $\{J_n; n \in N\}$ is irreducible and aperiodic (thus ergodic as $m < \infty$). Its stationary probabilities are given by

$$\pi_i = \frac{\alpha_i \eta_i}{\lambda_i} \left\{ \sum_{j=1}^{m} \frac{\alpha_j \eta_j}{\lambda_j} \right\}^{-1} \qquad (i \in J). \tag{4.2}$$

Proof

Let $i, j \in J$. As the chain $\{I_n\}$ is irreducible, there exists $n \in N$ such that $h_{ij}^{(n)} > 0$. It may be easily seen that this implies $(L^n(0))_{ij} > 0$. Now we obtain from (3.6):

$$p_{ij} = \sum_{n=0}^{\infty} (L^n(0))_{ij} \frac{\alpha_j}{\alpha_j + \lambda_j}. \tag{4.3}$$

The probabilities p_{ij} are thus strictly positive for all $i, j \in J$.

It remains to show that $\bar{\pi} P = \bar{\pi}$. Define the diagonal matrices

$$D = \left\langle \delta_{ij} \frac{\lambda_i}{\alpha_i + \lambda_i} \right\rangle, \qquad A = \left\langle \delta_{ij} \frac{\alpha_i}{\lambda_i} \right\rangle. \tag{4.4}$$

We have then $L(0) = DH$, $E(0) = I - D$, $\bar{\pi} = K\bar{\eta}A$ (where K is the norming factor in the right side of (4.2)), $AD = I - D$; (3.6) may be written as follows:

$$P = I - D + DHP. \tag{4.5}$$

Now

$$\bar{\pi}P = \bar{\pi} - \bar{\pi}D + \bar{\pi}DHP = \bar{\pi} - K[\bar{\eta}(I - D) - \bar{\eta}(I - D)HP].$$

As $\bar{\eta}H = \bar{\eta}$, we obtain

$$\bar{\pi}P = \bar{\pi} - K\bar{\eta}[(I - D) - (I - DH)P] = \bar{\pi}, \tag{4.6}$$

the last equality resulting from (4.5).

Note that (4.2) has an immediate intuitive interpretation: η_i is the asymptotic probability of finding the chain $\{I_n; n \in N\}$ in state i; $(\lambda_i)^{-1}$ is the mean time spent by the process $\{I(t); t \geq 0\}$ in state i before its next transition; α_i is the mean number of claims occurring per time unit when the process $\{I(t); t \geq 0\}$ sojourns in state i; π_i appears thus well as the asymptotic average number of claims occurring in environment i.

4.2. *Number of Claims Occurring in* $(0, t)$

The equations obtained here could be derived from the general theory of semi-Markov processes. It is, however, interesting to restate them directly as the semi-Markov kernel $\mathscr{V}$ is itself expressed as the solution of the differential system (3.7)

Define

$$N_j(t)=\begin{cases}\sum_{k=1}^{N(t)} 1_{[J_k=j]} & \text{if } N(t)>0,\\ 0 & \text{if } N(t)=0,\end{cases} \tag{4.7}$$

where as previously $N(t)$ is the number of claims occurring in $(0, t)$. $N_j(t)$ is clearly the number of claims occurring in environment j before t. Let

$$M_{ij}(t)=E[N_j(t)\mid J_0=i]$$

and

$$M_i(t)=E[N(t)\mid J_0=i]=\sum_{j=1}^{m} M_{ij}(t) \qquad (t\geqslant 0).$$

The following system of integral equations is easily obtained:

$$M_{ij}(t)=\delta_{ij}\,e^{-\lambda_i t}\alpha_i t+\int_0^t \lambda_i\,e^{-\lambda_i u}\Big[\delta_{ij}\alpha_i u+\sum_k h_{ik}M_{kj}(t-u)\Big]\,du$$

or

$$M_{ij}(t)=\delta_{ij}\alpha_i\frac{1-e^{-\lambda_i t}}{\lambda_i}+\sum_{k=1}^{m}\lambda_i h_{ik}\int_0^t e^{-\lambda_i u}M_{kj}(t-u)\,du \qquad (t\geqslant 0). \tag{4.8}$$

Taking the derivatives of both sides with respect to t we obtain

$$M'_{ij}(t)=\alpha_i\delta_{ij}-\lambda_i M_{ij}(t)+\lambda_i\sum_{k=1}^{m} h_{ik}M_{kj}(t) \qquad (t\geqslant 0), \tag{4.9}$$

and after summation over j

$$M'_i(t)=\alpha_i-\lambda_i M_i(t)+\lambda_i\sum_{k=1}^{m} h_{ik}M_k(t) \qquad (t\geqslant 0). \tag{4.10}$$

(4.9) with the boundary condition $M_{ij}(0)=0$ $(i, j\in J)$ has a unique solution.

4.3. *Further Properties of the Claim Arrival Process*

We extend first to the $(J\text{–}Y\text{–}X)$ processes a well known property of Markov chains and $(J\text{–}X)$ processes.

THEOREM 2

Let $\{(J_n, A_n, B_n);\ n\in N\}$ be a $(J-Y-X)$ process with state space $J\times R^+\times R$ and kernel $\mathscr{Q}$ defined by (1.1). Suppose that the Markov chain $\{J_n\}$ is irreducible (and thus positive recurrent as m is finite). Let $Z_{ij}(x, t)$, $i, j\in J$, be real measurable functions defined on $R \times R^+$ such that the integrals

$$\int_{-\infty}^{\infty} \int_0^{\infty} |Z_{ij}(x, t)| Q_{ij}(dx, dt) \qquad (i, j \in J)$$

are finite. Let

$$z_i = \sum_{j=1}^{m} \int_{-\infty}^{\infty} \int_0^{\infty} Z_{ij}(x, t) Q_{ij}(dx, dt) = E(Z_{J_{n-1}J_n}(B_n, A_n) | J_{n-1} = i).$$

Define then $n_{i,0} = 0$, $n_{i,k} = \inf \{n > n_{i,k-1}: J_n = i\}$ for $k \in N_0$ (recurrence indices of state i) and let

$$\zeta_{i,r} = E\left(\sum_{k=n_{i,r}+1}^{n_{i,r+1}} Z_{J_{k-1}J_k}(B_k, A_k)\right) \qquad (i \in J, \quad r \in N).$$

The random variables $\zeta_{i,r}$, $r = 1, 2, \ldots$, are i.i.d. and we have

$$E(\zeta_{i,r}) = \frac{1}{\pi_i} \sum_{j=1}^{m} \pi_j z_j \qquad (i \in J, \quad r \in N_0) \tag{4.11}$$

where the π_i are the stationary probabilities of the chain $\{J_n\}$.

Proof

Define

$${}_ip_{ij}^{(n)} = P[J_n = j, J_k \neq i \text{ for } k = 1, \ldots, n-1 | J_0 = i] \qquad (i, j \in J; \quad n \in N_0).$$

We have then

$$E(\zeta_{i,r}) = \sum_{k \neq i} \sum_{n=1}^{\infty} {}_ip_{ik}^{(n)} z_k + z_i \qquad (i \in J, \quad r \in N_0).$$

(4.11) follows since we know from Markov chain theory that $\sum_{n=1}^{\infty} {}_ip_{ik}^{(n)} = \pi_k / \pi_i$.

Mean Recurrence Time of Claims Occurring in a Given Environment

We return now to the risk model. Define

$$G_{ij}(t) = P[N_j(t) > 0 | J_0 = i] \qquad (i, j \in J; \quad t \geq 0). \tag{4.12}$$

$G_{ij}(\cdot)$ is the distribution function of the first time at which a claim occurs in environment j given that the initial environment is i. Let

$$\gamma_{ij} = \int_{0-}^{\infty} t \, dG_{ij}(t) \qquad (i, j \in J). \tag{4.13}$$

We could obtain a system of integral equations for the distributions $G_{ij}(\cdot)$ and derive from it after passage to the Laplace–Stieltjes transforms a linear system for the γ_{ij}. We may, however, proceed more directly as follows:

$$\text{(4.14)} \qquad \gamma_{ij} = \sigma_{ij} \int_0^\infty e^{-(\alpha_i+\lambda_i)t} \left[\alpha_i t + \lambda_i \sum_{k=1}^m h_{ik}(t+\gamma_{kj}) \right] dt$$

$$+ (1-\delta_{ij}) \int_0^\infty e^{-(\alpha_i+\lambda_i)t} \left[\alpha_i(t+\gamma_{ij}) + \lambda_i \sum_{k=1}^m h_{ik}(t+\gamma_{kj}) \right] dt;$$

we thus get a linear system:

$$\text{(4.15)} \qquad \frac{\lambda_i + \delta_{ij}\alpha_i}{\alpha_i+\lambda_i} \gamma_{ij} = \frac{1}{\alpha_i+\lambda_i} + \frac{\lambda_i}{\alpha_i+\lambda_i} \sum_{k=1}^m h_{ij}\gamma_{kj} \qquad (i, j \in J).$$

The diagonal elements γ_{ii} (mean recurrence time of claims occurring in state i) may be explicitly expressed by using Theorem 2. Define $Z_{ij}(x, t) = t$; then $z_i = E(A_1 | J_0 = i)$. We have

$$z_i = \int_0^\infty e^{-(\alpha_i+\lambda_i)t} \left[\alpha_i t + \lambda_i \sum_{j=1}^m h_{ij}(t+z_j) \right] dt \qquad (i \in J).$$

Hence

$$z_i = \frac{1}{\alpha_i+\lambda_i} + \frac{\lambda_i}{\alpha_i+\lambda_i} \sum_{j=1}^m h_{ij} z_j \qquad (i \in J),$$

or, if $\bar{z} = (z_1, \ldots, z_m)^t$ and $\bar{y} = (\alpha_1^{-1}, \ldots, \alpha_m^{-1})^t$,

$$\bar{z} = (I - L(0))^{-1} E(0) \bar{y} = P\bar{y};$$

we have thus

$$\text{(4.16)} \qquad z_i = E(A_1 | J_0 = i) = \sum_{j=1}^m p_{ij} \frac{1}{\alpha_j} \qquad (i \in J)$$

and consequently

$$\text{(4.17)} \qquad \sum_{i=1}^m \pi_i z_i = E_\pi(A_1) = \sum_{j=1}^m \pi_j \frac{1}{\alpha_j}.$$

Using finally theorem 2 we have:

THEOREM 3

For any $i \in J$:

$$\text{(4.18)} \qquad \gamma_{ii} = \frac{1}{\pi_i} \sum_{j=1}^m \pi_j \frac{1}{\alpha_j}.$$

Renewal Theorem—Stationary Probabilities

Given that $J_0 = i$, the times at which claims occur in environment j form a pure renewal process if $i = j$ and a delayed renewal process if $i \neq j$. We have the

https://doi.org/10.1017/S0515036100004785 Published online by Cambridge University Press

classical renewal equations:

$$M_{ij}(t)=\int_0^t [1+M_{jj}(t-u)]\,d\dot{G}_{ij}(u) \qquad (i,j\in J;\quad t\geq 0). \tag{4.19}$$

As the distribution functions $G_{jj}(\cdot)$ are clearly not arithmetic, the expected number of claims occurring in environment j within $(t,t+h)$ tends to $h(\gamma_{jj})^{-1}$ when $t\to\infty$ whatever the initial environment i, i.e.,

$$\lim_{t\to\infty}[M_{ij}(t+h)-M_{ij}(t)]=\frac{h}{\gamma_{jj}} \qquad (i,j\in J;\quad h\geq 0). \tag{4.20}$$

[see FELLER (1971), Chapt. XI]. From (4.20) it follows that

$$\lim_{t\to\infty}\frac{M_{ij}(t)}{t}=\frac{1}{\gamma_{jj}} \qquad (i,j\in J). \tag{4.21}$$

Define now

$$F_{ij}(t)=(p_{ij})^{-1}V_{ij}(t) \tag{4.22}$$

$$R_{jk}^{(i)}(u,t)=P[J_{N(t)}=j,J_{N(t)+1}=k,\,U_{N(t)+1}\leq t+u\,|\,J_0=i];$$

the last quantity is thus the probability, given that $J_0=i$, that the last claim before t occurred in environment j and that the next claim will occur in environment k before time $t+u$. We deduce immediately from Theorem 7.1 of PYKE (1961b) that

$$\lim_{t\to\infty}R_{jk}^{(i)}(u,t)=p_{jk}\frac{1}{\gamma_{jj}}\int_0^u [1-F_{jk}(y)]\,dy, \tag{4.23}$$

which limit is independent of i; we denote it by $R_{jk}^{0}(u)$. Let now

$$V_{ij}^{*}(u)=\gamma_{ii}z_i^{-1}R_{ij}^{0}(u)$$

and define a chain $\{(\bar{J}_n,\bar{A}_n,\bar{B}_n);\,n\in N\}$ as follows:

$$\begin{cases}\bar{A}_0=\bar{B}_0=0 \quad \text{a.s.}\\ P[\bar{J}_1=j,\bar{A}_1\leq u,\bar{B}_1\leq x\,|\,\bar{A}_0,\bar{B}_0;\bar{J}_0=i]=V_{ij}^{*}(u)F_j(x)\\ P[\bar{J}_n=j,\bar{A}_n\leq u,\bar{B}_n\leq x\,|\,\bar{A}_k,\bar{B}_k,\bar{J}_k(k=0,\dots,n-1);\bar{J}_{n-1}\\ \qquad =i]=V_{ij}(u)F_j(x)\end{cases} \tag{4.24}$$

$$(i,j\in J;\quad u\in R^{+},\quad x\in R,\quad n\geq 2).$$

where z_i is defined by (4.16).

We define for that chain the same quantities and adopt the same notations as for the chain $\{(J_n,A_n,B_n);\,n\in N\}$. The risk processes associated with the two chains are identical except that for the second one the time of occurrence of the first claim is distributed according to the semi-Markov kernel $(V_{ij}^{*}(\cdot))$ instead of $(V_{ij}(\cdot))$. Suppose now that

$$a_i=P[\bar{J}_0=i]=\frac{z_i}{\gamma_{ii}} \qquad (i\in J). \tag{4.25}$$

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Then [see PYKE (1961b)]:

(4.26) $$P[\bar{J}_{\tilde{N}(t)}=j, \bar{J}_{\tilde{N}(t)+1}=k, \bar{U}_{\tilde{N}(t)+1}\leq t+u]=R^0_{jk}(u).$$

5. PREMIUM INCOME—RUIN PROBABILITIES

We assume that the company managing the risk receives premiums at a constant rate $c_i>0$ during any time interval the environment process remains in state i. The premium income process is thus characterized by a vector $(c_1, \ldots, c_m)$ with positive entries. Denote by $A^c(t)$ the aggregate premium received during $(0, t)$:

(5.1) $$A^c(t)=\sum_{k=1}^{N_e(t)} c_{I_{k-1}}(T_k-T_{k-1})+c_{I_{N_e(t)}}(t-T_{N_e(t)})$$

and by $B(t)$ the aggregate amount of the claims occurring in $(0, t)$:

(5.2) $$B(t)=\sum_{k=0}^{N(t)} B_k \qquad (t\geq 0).$$

Assume now that the initial amount of free assets of the company is $u\geq 0$. The amount of free assets at time t is then

(5.3) $$Z_u(t)=u+S(t)$$

where

(5.4) $$S(t)=A^c(t)-B(t).$$

Define then

(5.5) $$R_i(u, t)=P[Z_u(v)\geq 0 \text{ for } 0\leq v\leq t \mid J_0=i] \qquad (i\in J; \quad u, t\geq 0),$$

(5.6) $$R_i(u)=R_i(u, \infty)=P[Z_u(v)\geq 0 \text{ for all } v\geq 0 \mid J_0=i] \qquad (i\in J, \quad u\geq 0).$$

We will refer to the probabilities (5.5) as to the finite time non-ruin probabilities and to the probabilities (5.6) as to the asymptotic non-ruin probabilities.

5.1. *Random Walk of the Free Assets*

Denote by A^c_n the premium received between the occurrences of the $(n-1)$th and nth claims $(n\geq 1)$. Define then

(5.7) $$X_k=A^c_k-B_k \qquad (k=1, 2, \ldots); \quad X_0=0 \quad \text{a.s.},$$

(5.8) $$S_n=\sum_{k=0}^{n} X_k \qquad (n\in N).$$

Clearly the chain $\{(J_k, X_k); k\in N\}$ is a $(J\text{-}X)$ process, $\{S_n\}$ is a random walk defined on the finite Markov chain $\{J_n\}$ [see JANSSEN (1970); MILLER (1962); NEWBOULD (1973)]. The amount of free assets just after the occurrence of the

nth claim is given by

$$Z_u(A_0+\cdots+A_n)=u+S_n$$

and clearly

(5.9) $$R_i(u)=P\left[\inf_k S_k \geqslant -u \mid J_0=i\right].$$

From now on we assume that the d.f. $F_i(\cdot)$ has a finite expectation μ_i $(i\in J)$. We get then

(5.10) $$b_i=E[B_k \mid J_{k-1}=i]=\sum_{j=1}^m p_{ij}\mu_j$$

and

$$z_i^c=E[A_k^c \mid J_{k-1}=i]=\int_0^\infty e^{-(\alpha_i+\lambda_i)t}\left[\alpha_i c_i t+\lambda_i \sum_{j=1}^m h_{ij}(c_i t+z_j^c)\right]dt$$

so that, concluding as to obtain (4.16),

(5.11) $$z_i^c=\sum_{j=1}^m p_{ij}\frac{c_j}{\alpha_j} \qquad (i\in J).$$

If the premium rates are constant whatever the state of the environment, i.e., if $\bar{c}=(c,\ldots,c)$, we obtain naturally $z_i^c=cz_i$. We conclude from (5.10) and (5.11) that

(5.12) $$\zeta_i=E[X_k \mid J_{k-1}=i]=\sum_{j=1}^m p_{ij}\left(\frac{c_j}{\alpha_j}-\mu_j\right).$$

Notice that we would obtain the same result for a semi-Markov risk model with kernel $\mathcal{Q}^*$ defined by

(5.13) $$Q_{ij}^*(x,t)=p_{ij}(1-e^{-\alpha_j t})F_j(x).$$

Define now

$$D_{i,r}=\sum_{k=n_{i,r}+1}^{n_{i,k+1}} X_k \qquad (i\in J, \quad r\in N_0)$$

where the $n_{i,r}$ are the recurrence indices of claims occurring in environment i as defined in section 4.3; for i fixed the variables $D_{i,r}$ $(r=1,2,\ldots)$ are i.i.d.; $D_{i,r}$ is clearly the variation of the free assets between the rth and $(r+1)$th claims occurring in environment i. We obtain from theorem 2

(5.14) $$E(D_{i,r})=\frac{1}{\pi_i}\sum_{j=1}^m \pi_j\left(\frac{c_j}{\alpha_j}-\mu_j\right) \qquad (i\in J, \quad r\in N_0).$$

As the variables A_k^c are absolutely continuous and conditionally (given the J_k) independent of the variables B_k, the process $\{(J_n,S_n); n\in N\}$ is not degenerate [see NEWBOULD (1973)], i.e., there exist no constants $w_1, \ldots, w_m$ such that $P[X_n = w_j - w_i | J_{n-1} = i, J_n = j] = 1$, or equivalently there exists no i such that $D_{i,r} = 0$ a.s. (NEWBOULD (1973), lemma 2). Using Proposition 3A of JANSSEN (1970) we obtain then

THEOREM 4

Let

$$d = \sum_{j=1}^{m} \pi_j \left(\frac{c_j}{\alpha_j} - \mu_j \right). \tag{5.15}$$

Then (i) If $d > 0$, the random walk $\{S_n\}$ drifts to $+\infty$, i.e. $\lim_{n\to\infty} S_n = \infty$ a.s.; $R_i(u) > 0$, $\forall u \geq 0$, $i \in J$. (ii) If $d < 0$, the random walk $\{S_n\}$ drifts to $-\infty$, i.e. $\lim_{n\to\infty} S_n = -\infty$ a.s.: $R_i(u) = 0$, $\forall u \geq 0$, $i \in J$. (iii) If $d = 0$, the random walk $\{S_n\}$ is oscillating, i.e. $\limsup S_n = +\infty$ a.s. and $\liminf S_n = -\infty$ a.s.; $R_i(u) = 0$, $\forall u \geq 0$, $i \in J$.

Notice that when $m = 1$ theorem 4 reduces evidently to the classical result for the Poisson model.

5.2. *Distribution of the Aggregate Net Pay-out in* $(0, t)$

From now on we suppose that the claim amounts are a.s. positive:

$$F_i(0-) = 0, \qquad F_i(0) < 1 \qquad \forall i \in J. \tag{5.16}$$

Recall that $A^c(t)$ and $B(t)$ denote respectively the aggregate premium received and the aggregate amount of claims occurred during $(0, t)$. Then denote by $C(t)$ the net pay-out of the company in $(0, t)$:

$$C(t) = B(t) - A^c(t) = -S(t) \qquad (t \geq 0)$$

Let then

$$W_{ij}(x, t) = P[C(t) \leq x, I(t) = j | I(0) = i] \qquad (i, j \in J; \quad t \geq 0). \tag{5.17}$$

Define now

$$c_0 = \max\{c_i; i \in J\}, \qquad J_0 = \{i \in J : c_i = c_0\}.$$

It is easy to prove the following

LEMMA

(i) $W_{ij}(x, t) = 0$ for $i, j \in J$ and $x < -c_0 t$;
(ii) $W_{ij}(x, t) > 0$ for $i, j \in J$ and $x > -c_0 t$;
(iii) $W_{ij}(-c_0 t, t) > 0$ if $i, j \in J_0$ and either $i = j$ or there exist $r \in N_0$ and $i_1, \ldots, i_r \in J_0$ such that $h_{ii_1} h_{i_1 i_2} \ldots h_{i_r j} > 0$; $W_{ij}(-c_0 t, t) = 0$ otherwise.

Let now

$$\tilde{W}_{ij}(s,t)=\int_{-c_0t}^{\infty} e^{-sx}W_{ij}(x,t)\,dx;\qquad \tilde{W}(s,t)=(\tilde{W}_{ij}(s,t))\qquad (s>0),$$

$$w_{ij}(s,t)=\int_{-c_0t-}^{\infty} e^{-sx}\,d_xW_{ij}(x,t)=s\tilde{W}_{ij}(s,t);\qquad w(s,t)=(w_{ij}(s,t))\qquad (s>0),$$

$$\varphi_i(s)=\int_{0-}^{\infty} e^{-sx}\,dF_i(x)\qquad (s\geq 0).$$

The following theorem gives an explicit expression for the transform matrix $\tilde{W}(s,t)$.

THEOREM 5

For $s>0$ and $t\geq 0$,

$$\tilde{W}(s,t)=1/s\exp\{-T(s)t\} \tag{5.18}$$

where

$$T_{ij}(s)=\delta_{ij}(\alpha_i+\lambda_i-\alpha_i\varphi_i(s)-c_is)-\lambda_ih_{ij}. \tag{5.19}$$

Proof

For $x\geq -c_0t$, $t\geq 0$ and $h>0$ we obtain easily

$$\begin{aligned} W_{ij}(x,t+h)&=(1-(\alpha_i+\lambda_i)h)W_{ij}(x+c_ih,t)\\ &\quad+\alpha_ih\int_{0-}^{x+c_ih+c_0t} W_{ij}(x+c_ih-y,t)\,dF_i(y)\\ &\quad+\lambda_ih\sum_{k=1}^{m} h_{ik}W_{kj}(x+c_ih,t)+o(h). \end{aligned} \tag{5.20}$$

Dividing (5.20) by h and letting h tend to 0, we get

$$\begin{aligned} \frac{\partial}{\partial t}W_{ij}(x,t)-c_i\frac{\partial}{\partial x}W_{ij}(x,t)&=-(\alpha_i+\lambda_i)W_{ij}(x,t)\\ &\quad+\alpha_i\int_{0-}^{x+c_0t} W_{ij}(x-y,t)\,dF_i(y)\\ &\quad+\lambda_i\sum_{k=1}^{m} h_{ik}W_{kj}(x,t)\\ &\qquad (x\geq -c_0t,\ t\geq 0). \end{aligned} \tag{5.21}$$

We multiply now each term in (5.21) by e^{-sx} and integrate from $-c_0 t$ to ∞. We obtain so

$$\frac{\partial}{\partial t}\tilde{W}_{ij}(s,t)+\sum_{k=1}^{m}[\delta_{ik}(\alpha_i+\lambda_i-\alpha_i\varphi_i(s)-c_is)-\lambda_ih_{ik}]\tilde{W}_{kj}(s,t) = (c_0-c_i)\,e^{sc_0t}W_{ij}(-c_0t,t) \qquad (s>0,\, t\geqslant 0). \tag{5.22}$$

According to the above lemma the right side of (5.22) is always zero. In matrix notation, the solution of (5.22) is then easily seen to be

$$\tilde{W}(s,t)=\exp\{-T(s)t\}K \tag{5.23}$$

where

$$K=\tilde{W}(s,0)=(1/s)w(s,0)=(1/s)I \qquad (s>0).$$

The proof is complete.

Notice that when $m=1$ (5.18) reduces to the known result for the classical Poisson model.

5.3. *Seal's Integral Equation for the Finite Time non-ruin Probabilities*

We show in this subsection that the SEAL's integral equation (1974) may be extended to the here considered semi-Markov model. We still assume that the claim amounts are a.s. positive.

Define for $u, t\geqslant 0$ and $i, j\in J$

$$R_{ij}(u,t)=P[Z_u(v)\geqslant 0 \text{ for } 0\leqslant v\leqslant t,\, I(t)=j\,|\,I(0)=i]; \tag{5.24}$$

we have clearly

$$R_i(u,t)=\sum_{j=1}^{m}R_{ij}(u,t) \qquad (i\in J;\quad u,t\geqslant 0).$$

Define further for $s>0$ and $t\geqslant 0$

$$\tilde{R}_{ij}(s,t)=\int_0^\infty e^{-su}R_{ij}(u,t)\,du; \qquad \tilde{R}(s,t)=(\tilde{R}_{ij}(s,t)),$$

$$r_{ij}(s,u)=\int_{0-}^\infty e^{-su}\,d_uR_{ij}(u,t)=s\tilde{R}_{ij}(s,t); \qquad r(s,t)=(r_{ij}(s,t)).$$

We obtain easily for $u, t\geqslant 0$ and $h>0$

$$\begin{aligned} R_{ij}(u,t+h) &= [1-(\alpha_i+\lambda_i)h]R_{ij}(u+c_ih,t) \\ &\quad +\alpha_ih\int_{0-}^{u+c_ih}R_{ij}(u+c_ih-y,t)\,dF_i(y) \\ &\quad +\lambda_ih\sum_{k=1}^{m}h_{ik}R_{kj}(u+c_ih,t)+o(h). \end{aligned} \tag{5.25}$$

Dividing (5.25) by h and letting h tend to 0, we find

$$(5.26)\qquad \frac{\partial}{\partial t}R_{ij}(u,t)-c_i\frac{\partial}{\partial u}R_{ij}(u,t)=-(\alpha_i+\lambda_i)R_{ij}(u,t)+\alpha_i\int_0^u R_{ij}(u-y,t)\,dF_i(y)+\lambda_i\sum_{k=1}^m h_{ik}R_{kj}(u,t)\qquad (u,t\geqslant 0).$$

Taking the Laplace transform of each term ir (5.26), we obtain

$$(5.27)\qquad \frac{\partial}{\partial t}\tilde{R}_{ij}(s,t)+\sum_{k=1}^m[\delta_{ik}(\alpha_i+\lambda_i-c_is-\alpha_i\varphi_i(s))-\lambda_ih_{ik}]\tilde{R}_{kj}(s,t)+c_iR_{ij}(0,t)=0\qquad (s>0,\quad t\geqslant 0).$$

The solution of the differential system (5.27) is easily seen to be

$$(5.28)\qquad \tilde{R}(s,t)=\exp\{-T(s)t\}K-\int_0^t\exp\{-T(s)(t-u)\}CR(0,u)\,du\qquad (s>0,\quad t\geqslant 0)$$

where $C=(\delta_{ij}c_i)$; the constant matrix K is determined by the boundary condition $r(s,0)=s\tilde{R}(s,0)=sI$. Thus $K=s^{-1}I$. Using finally (5.18), (5.28) may be written as follows

$$(5.29)\ \tilde{R}_{ij}(s,t)=\tilde{W}_{ij}(s,t)-s\sum_{k=1}^m\int_0^t\tilde{W}_{ik}(s,t-u)c_kR_{kj}(0,u)\,du\qquad (s>0,\quad t\geqslant 0).$$

Suppose now that the distributions $F_i(\cdot)$ are absolutely continuous and denote their densities by $f_i(\cdot)$. The mass functions $W_{ij}(\cdot,t)$ are then absolutely continuous too; we denote their densities by $W'_{ij}(\cdot,t)$ $(t\geqslant 0)$. Taking the inverse Laplace transforms in (5.29) we obtain then

$$(5.30)\quad R_{ij}(x,t)=W_{ij}(x,t)-\sum_{k=1}^m c_k\int_0^t W'_{ik}(x,u)R_{kj}(0,t-u)\,du\qquad (x,t\geqslant 0).$$

The unknown constants (with respect to x) $R_{kj}(0,u)$ are solutions of the Volterra type integral system obtained by putting $x=0$ in (5.30):

$$(5.31)\ R_{ij}(0,t)=W_{ij}(0,t)-\sum_{k=1}^m c_k\int_0^t W'_{ik}(0,u)R_{kj}(0,t-u)\,du\qquad (t\geqslant 0).$$

Define now

$$S_{ij}(x,t)=P[B(t)\leqslant x,I(t)=j\,|\,I(0)=i]\qquad (x,t\geqslant 0)$$

and denote the corresponding densities by $S'_{ij}(x,t)$. In the particular case where

$c_i = c$ $(i \in J)$ we have clearly $W_{ij}(x, t) = S_{ij}(x + ct, t)$; (5.30) and (5.31) become then

$$\text{(5.32)} \quad R_{ij}(x, t) = S_{ij}(x + ct, t) - c \sum_{k=1}^{m} \int_0^t S'_{ik}(x + cu, u) R_{kj}(0, t - u)\, du \qquad (x, t \geq 0),$$

$$\text{(5.33)} \quad R_{ij}(0, t) = S_{ij}(ct, t) - c \sum_{k=1}^{m} \int_0^t S'_{ik}(cu, u) R_{kj}(0, t - u)\, du \qquad (t \geq 0).$$

When $m = 1$ (5.32) and (5.33) reduce exactly to Seal's system.

5.4. *Asymptotic Non-ruin Probabilities*

We suppose here that the number d defined by (5.15) is strictly positive; then for all $i \in J$ and $u \geq 0$, $R_i(u) > 0$ and $R_i(\cdot)$ is a probability distribution. After summation over j (5.26) gives for $t = \infty$:

$$\text{(5.34)} \quad c_i R'_i(u) = (\alpha_i + \lambda_i) R_i(u) - \alpha_i \int_{0-}^{u} R_i(u - y)\, dF_i(y) - \lambda_i \sum_{k=1}^{m} h_{ik} R_k(u)$$
$$(i \in J; \quad u \geq 0).$$

It can be shown that (5.34) has a unique solution such that $R_i(\infty) = 1$, $\forall i \in J$. Integrating (5.34) from 0 to t we get

$$\text{(5.35)} \qquad c_i R_i(t) = c_i R_i(0) + \alpha_i \int_0^t R_i(t - y)[1 - F_i(y)]\, dy$$
$$+ \lambda_i \int_0^t \left[R_i(u) - \sum_{k=1}^{m} h_{ik} R_k(u) \right] du \qquad (i \in J, \quad t \geq 0).$$

For $m = 1$ (5.35) is the well known defective renewal equation from which the famous Cramer estimate may be derived (see FELLER, Chapter XI). For $m > 1$, (5.35) is unfortunately not more a renewal type equation. Letting t tend to ∞ in (5.35) does not give an explicit value for the probabilities $R_i(0)$ as is the case when $m = 1$:

$$\text{(5.36)} \qquad R_i(0) = 1 - \frac{\alpha_i \mu_i}{c_i} - \frac{\lambda_i}{c_i} \int_0^\infty \left[R_i(u) - \sum_{k=1}^{m} h_{ik} R_k(u) \right] du.$$

However, when the claim amounts distributions are exponential,

$$F_i(x) = 1 - e^{-x/\mu_i} \qquad (x \geq 0),$$

a further differentiation of both sides of (5.34) shows that the asymptotic non-ruin probabilities are solution of the differential system

$$\text{(5.37)} \qquad R''_i(u) = \left(\frac{\alpha_i + \lambda_i}{c_i} - \frac{1}{\mu_i} \right) R'_i(u) - \frac{\lambda_i}{c_i} \sum_{j=1}^{m} h_{ij} R'_j(u) + \frac{\lambda_i}{c_i \mu_i} R_i(u)$$
$$- \frac{\lambda_i}{c_i \mu_i} \sum_{j=1}^{m} h_{ij} R_j(u) \qquad (i \in J, \quad u \geq 0)$$

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with the boundary conditions

(5.38) $$R_i(\infty)=1; \qquad R_i'(0)=\frac{\alpha_i+\lambda_i}{c_i}R_i(0)-\frac{\lambda_i}{c_i}\sum_{j=1}^{m} h_{ij}R_j(0) \qquad (i\in J).$$

6. EXAMPLE

Assume that

(6.1) $$m=2, \qquad h_{12}=h_{21}=1, \qquad h_{11}=h_{22}=0;$$

there are thus two possible states for the environment, the sojourn times in each state being exponentially distributed.

The solution of system (3.7) is then

(6.2) $$\begin{cases} V_{11}(t)=-\dfrac{\alpha_1(\alpha_1+\lambda_2+r_1)}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\alpha_1(\alpha_2+\lambda_2+r_2)}{r_2(r_1-r_2)}(1-e^{r_2t}), \\ V_{12}(t)=-\dfrac{\lambda_1\alpha_2}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\lambda_1\alpha_2}{r_2(r_1-r_2)}(1-e^{r_2t}), \\ V_{22}(t)=-\dfrac{\alpha_2(\alpha_1+\lambda_1+r_1)}{r_1(r_1-r_2)}\cdot(1-e^{r_1t})+\dfrac{\alpha_2(\alpha_1+\lambda_1+r_2)}{r_2(r_1-r_2)}(1-e^{r_2t}), \\ V_{21}(t)=-\dfrac{\lambda_2\alpha_1}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\lambda_2\alpha_1}{r_2(r_1-r_2)}(1-e^{r_2t}) \qquad (t\geqslant 0), \end{cases}$$

where r_1 and r_2 are the solutions (always distinct and negative as α_i, $\lambda_i>0$) of

(6.3) $$(\alpha_1+\lambda_1+r)(\alpha_2+\lambda_2+r)=\lambda_1\lambda_2.$$

The stationary probabilities for the chain $\{J_n\}$ are given by (4.2) which becomes here

(6.4) $$\pi_1=\frac{\alpha_1\lambda_2}{\alpha_1\lambda_2+\alpha_2\lambda_1}, \qquad \pi_2=\frac{\alpha_2\lambda_1}{\alpha_1\lambda_2+\alpha_2\lambda_1}.$$

Expectations of the number of claims occurring in environment i ($i=1,2$) before t are obtained by solving system (4.9) with the boundary conditions $M_{ij}(0)=0$:

(6.5) $$M_{11}(t)=\frac{\alpha_1\lambda_2}{\lambda_1+\lambda_2}t+\frac{\alpha_1\lambda_1}{(\lambda_1+\lambda_2)^2}(1-e^{-(\lambda_1+\lambda_2)t}),$$

$$M_{12}(t)=\frac{\alpha_2\lambda_1}{\lambda_1+\lambda_2}t-\frac{\alpha_2\lambda_1}{(\lambda_1+\lambda_2)^2}(1-e^{-(\lambda_1+\lambda_2)t}).$$

$M_{22}(t)$ and $M_{21}(t)$ are obtained by replacing in the expressions of $M_{11}(t)$ and $M_{12}(t)$ respectively $\alpha_{1(2)}$ by $\alpha_{2(1)}$ and $\lambda_{1(2)}$ by $\lambda_{2(1)}$.

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The mean recurrence time of claims occurring in environment i $(i=1,2)$ is given by (4.18):

$$\gamma_{11}=\frac{\lambda_1+\lambda_2}{\alpha_1\lambda_2},\qquad \gamma_{22}=\frac{\lambda_1+\lambda_2}{\alpha_2\lambda_1}; \tag{6.6}$$

We obtain then from (4.15)

$$\gamma_{12}=\frac{\alpha_2+\lambda_1+\lambda_2}{\alpha_2\lambda_1},\qquad \gamma_{21}=\frac{\alpha_1+\lambda_1+\lambda_2}{\alpha_1\lambda_2}. \tag{6.7}$$

The characteristic number d defined by (5.15) takes the following form:

$$d=\frac{\lambda_2(c_1-\alpha_1\mu_1)+\lambda_1(c_2-\alpha_2\mu_2)}{\alpha_1\lambda_2+\alpha_2\lambda_1}. \tag{6.8}$$

From now on we assume that $d>0$ and that the claim amount distributions $F_i(\cdot)$ are exponential, i.e.,

$$F_i(x)=1-e^{-x/\mu_i}\qquad (x\geq 0;\quad i=1,2). \tag{6.9}$$

From (5.37) and (5.38) we obtain that the asymptotic non-ruin probabilities are solution of the following differential system

$$\begin{cases} c_1R_1''(u)=\left(\alpha_1+\lambda_1-\dfrac{c_1}{\mu_1}\right)R_1'(u)+\dfrac{\lambda_1}{\mu_1}R_1(u)-\dfrac{\lambda_1}{\mu_1}R_2(u)-\lambda_1R_2'(u)\\ c_2R_2''(u)=\left(\alpha_2+\lambda_2-\dfrac{c_2}{\mu_2}\right)R_2'(u)+\dfrac{\lambda_2}{\mu_2}R_2(u)-\dfrac{\lambda_2}{\mu_2}R_1(u)-\lambda_2R_1'(u)\end{cases} \tag{6.10}$$

$$(u\geq 0)$$

with the boundary conditions

$$\begin{cases} R_1(\infty)=R_2(\infty)=1\\ c_1R_1'(0)-(\alpha_1+\lambda_1)R_1(0)+\lambda_1R_2(0)=c_2R_2'(0)\\ \qquad -(\alpha_2+\lambda_2)R_2(0)+\lambda_2R_1(0)=0.\end{cases} \tag{6.11}$$

Define

$$\rho_i=\frac{1}{\mu_i}-\frac{\alpha_i}{c_i}\qquad (i=1,2) \tag{6.12}$$

and assume without restriction that $\rho_1\geq\rho_2$.
The condition $d>0$ is then equivalent to the following

$$\frac{\lambda_2}{c_2\mu_2}\rho_1+\frac{\lambda_1}{c_1\mu_1}\rho_2>0. \tag{6.13}$$

As $\rho_1 \geqslant \rho_2$, then ρ_1 is clearly strictly positive. We obtain then that the general solution of (6.10) takes the form

$$(6.14)\qquad \begin{cases} R_1(u) = A_0 + A_1 e^{k_1 u} + A_2 e^{k_2 u} + A_3 e^{k_3 u}, \\ R_2(u) = A_0 - D(k_1) A_1 e^{k_1 u} - D(k_2) A_2 e^{k_2 u} \\ \qquad - D(k_3) A_3 e^{k_3 u}, \end{cases}$$

where

$$(6.15)\qquad D(k_i) = \frac{c_1 \mu_1 k_i^2 + (c_1 - \alpha_1 \mu_1 - \lambda_1 \mu_1) k_i - \lambda_1}{\lambda_1 \mu_1 k_i + \lambda_1} = \frac{\lambda_2 \mu_2 k_i + \lambda_2}{c_2 \mu_2 k_i^2 + (c_2 - \alpha_2 \mu_2 - \lambda_2 \mu_2) k_i - \lambda_2},$$

and where k_1, k_2, k_3 are the roots of the characteristic equation

$$(6.16)\qquad P(k) = k^3 + \left(\rho_1 + \rho_2 - \frac{\lambda_1}{c_1} - \frac{\lambda_2}{c_2}\right) k^2 + \left[\left(\rho_1 - \frac{\lambda_1}{c_1}\right)\left(\rho_2 - \frac{\lambda_2}{c_2}\right) - \frac{\lambda_2}{c_2 \mu_2} - \frac{\lambda_1}{c_1 \mu_1} - \frac{\lambda_1 \lambda_2}{c_1 c_2}\right] k - \left(\frac{\lambda_2}{c_2 \mu_2} \rho_1 + \frac{\lambda_1}{c_1 \mu_1} \rho_2\right) = 0.$$

From (6.13) we see that $k_1 k_2 k_3 > 0$. It is easily verified that

$$P(-\rho_1) = \frac{\alpha_1 \lambda_1}{c_1^2} (\rho_1 - \rho_2) \geqslant 0; \qquad P(-\rho_2) = \frac{\alpha_2 \lambda_2}{c_2^2} (\rho_2 - \rho_1) \leqslant 0;$$

$$P(0) < 0.$$

From this we may deduce that $P(k)$ has a negative root, say k_2, between $-\rho_1$ and $-\rho_2$. As the product of the three roots is positive we deduce further that the two other roots, k_1 and k_3, are real (if k_1 and k_3 were complex conjugate roots, their product would be positive; we would then have $k_1 k_2 k_3 < 0$). As $P(+\infty) = +\infty$ and $P(-\infty) = -\infty$, we conclude finally that when $\rho_1 > \rho_2$ one of the roots, say k_1, is strictly less than $-\rho_1$ and that the other, k_3, is positive. When $\rho_1 = \rho_2 = \rho$ (we have then $k_2 = -\rho$), we obtain the same conclusions by verifying that $P'(-\rho) < 0$. We summarize this as follows:

$$(6.17)\qquad \begin{array}{lll} k_1 < -\rho_1 < k_2 < \min\{0, -\rho_2\}, & k_3 > 0 & \text{if } \rho_1 > \rho_2, \\ k_1 < k_2 = -\rho < 0 < k_3 & & \text{if } \rho_1 = \rho_2 = \rho. \end{array}$$

From the boundary conditions (6.11) we obtain that

$$(6.18)\qquad A_0 = 1, \qquad A_3 = 0$$

and that A_1 and A_2 are the solutions of

$$[c_1k_1-\alpha_1-\lambda_1-\lambda_1D(k_1)]A_1+[c_1k_2-\alpha_1-\lambda_1-\lambda_1D(k_2)]A_2=\alpha_1$$
$$[(-c_2k_1+\alpha_2+\lambda_2)D(k_1)+\lambda_2]A_1+[(-c_2k_2+\alpha_2+\lambda_2)D(k_2)+\lambda_2]A_2=\alpha_2$$

or, which is equivalent in view of (6.15),

$$\text{(6.19)}\qquad \begin{cases} \dfrac{A_1}{\mu_1k_1+1}+\dfrac{A_2}{\mu_1k_2+1}=-1 \\ \dfrac{D(k_1)}{\mu_2k_1+1}A_1+\dfrac{D(k_2)}{\mu_2k_2+1}A_2=1. \end{cases}$$

We can obtain a lower bound for k_1. Verify first that $P(\mu_1^{-1})<0$ if $\mu_1\leq\mu_2$ and that $P(\mu_2^{-1})<0$ if $\mu_2\leq\mu_1$. We can then easily conclude that

$$\text{(6.20)}\qquad -\min\{\mu_1,\mu_2\}^{-1}<k_1.$$

We summarize the above results in

THEOREM 6

If $m=2$, $h_{12}=h_{21}=1$, $d>0$ and if the claim amount distributions are exponential, the asymptotic non-ruin probabilities are given by

$$R_1(u)=1+A_1e^{k_1u}+A_2e^{k_2u},$$
$$R_2(u)=1-D(k_1)A_1e^{k_1u}-D(k_2)A_2e^{k_2u}\qquad(u\geq0),$$

where k_1 and k_2 are the two negative roots of (6.16), where the constants $D(k_i)$ are given by (6.15) and where A_1 and A_2 are solutions of (6.19).

When $\alpha_1=\alpha_2=\alpha$, $\mu_1=\mu_2=\mu$, $c_1=c_2=c$ and if λ_1 and λ_2 are arbitrary positive numbers, then $k_2=-\rho$ and k_1 is the negative root of

$$\text{(6.21)}\qquad k^2+\left(\rho-\frac{\lambda_1+\lambda_2}{c}\right)k-\frac{\lambda_1+\lambda_2}{c\mu}=0.$$

When obtain then $D(k_2)=-1$, $D(k_1)=\lambda_2/\lambda_1$ and the solution of (6.19) is $A_1=0$, $A_2=-\alpha\mu/c$. As expected the ruin probabilities $R_1(u)$ and $R_2(u)$ are in this case identical and equal to the ruin probabilities obtained for the classical Poisson model with exponentially distributed claim amounts:

$$\text{(6.22)}\qquad R_1(u)=R_2(u)=1-\frac{\alpha\mu}{c}e^{-\rho u}.$$

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