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QUANTUM DEFORMATIONS OF SIMPLE LIE ALGEBRAS

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ABSTRACT. It is shown that every simple complex Lie algebra $\mathfrak{g}$ admits a 1-parameter family $\mathfrak{g}_q$ of deformations outside the category of Lie algebras. These deformations are derived from a tensor product decomposition for $U_q(\mathfrak{g})$-modules; here $U_q(\mathfrak{g})$ is the quantized enveloping algebra of $\mathfrak{g}$. From this it follows that the multiplication on $\mathfrak{g}_q$ is $U_q(\mathfrak{g})$-invariant. In the special case $\mathfrak{g} = \mathfrak{sl}(2)$, the structure constants for the deformation $\mathfrak{sl}(2)_q$ are obtained from the quantum Clebsch-Gordan formula applied to $V(2)_q \otimes V(2)_q$; here $V(2)_q$ is the simple 3-dimensional $U_q(\mathfrak{sl}(2))$-module of highest weight q^2.

1. **Introduction.** Lyubashenko and Sudbery [LS] have suggested that the quantized enveloping algebra $U_q(\mathfrak{g})$ of a simple complex (finite dimensional) Lie algebra $\mathfrak{g}$ ought to be regarded as the universal associative enveloping algebra of some (as yet undetermined) non-associative algebra $\mathfrak{g}_q$. The relation between $\mathfrak{g}_q$ and $\mathfrak{g}$ should be analogous to that between $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$, and there should be a PBW-type theorem relating $U_q(\mathfrak{g})$ and $\mathfrak{g}_q$. For other work along these lines, see [DH] and [DHGZ].

The purpose of this note is to show that a natural candidate for the "quantum Lie algebra" $\mathfrak{g}_q$ can be obtained from the decomposition of the tensor square of the $U_q(\mathfrak{g})$-module V_q corresponding to the adjoint representation V of $\mathfrak{g}$. Thus in every case $\dim(\mathfrak{g}_q) = \dim(\mathfrak{g})$; deformations of $\mathfrak{g}$ satisfying this condition appear to be new except when $\mathfrak{g} = \mathfrak{sl}(n)$. The structure constants of $\mathfrak{sl}(2)_q$ are worked out in detail using the quantum Clebsch-Gordan formula (§VII.7 of [K]).

The algebras $\mathfrak{g}_q$ defined in this note are not Lie algebras (except for a few special values of q): this is clear since every simple complex Lie algebra $\mathfrak{g}$ has only trivial deformations in the category of Lie algebras (Chapter XVII of [K]). However the algebras $\mathfrak{g}_q$ are structurally very closely related to Lie algebras, and so Lie-theoretic techniques should be applicable to this larger class of non-associative algebras.

General references on quantum groups are [K], [CP], [J] and [Lu]. We assume throughout that q is a complex number with $q \neq 0$ and q not a root of unity.

THEOREM. *Let $\mathfrak{g}$ be a simple complex Lie algebra, and let $U_q(\mathfrak{g})$ be the corresponding quantized enveloping algebra. There exists a deformation $\mathfrak{g}_q$ of $\mathfrak{g}$ such that*

(1) $\mathfrak{g}_q$ *is a* $U_q(\mathfrak{g})$*-module with* $\dim(\mathfrak{g}_q) = \dim(\mathfrak{g})$,

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(2) *the multiplication* $\mathfrak{g}_q \otimes \mathfrak{g}_q \longrightarrow \mathfrak{g}_q$ *is a morphism of* $U_q(\mathfrak{g})$*-modules,*
(3) $\mathfrak{g}_q$ *possesses a* $U_q(\mathfrak{g})$*-invariant bilinear form.*

PROOF. Let V denote the adjoint representation of $\mathfrak{g}$. Then V is also a $U(\mathfrak{g})$-module, so let V_q denote the corresponding $U_q(\mathfrak{g})$-module. We need two properties of $U_q(\mathfrak{g})$-modules:

- Every finite dimensional $U_q(\mathfrak{g})$-module is semisimple. See Theorem 10.1.14 of [CP], Theorem 5.17 of [J], or Theorem 6.2.2 of [Lu].
- The $U_q(\mathfrak{g})$-module V_q has the same formal character as the $U(\mathfrak{g})$-module V (as given by the classical Weyl character formula). See Corollary 10.1.15 of [CP], Theorem 5.15 of [J], or Theorem 33.1.3 of [Lu].

These results imply that the decomposition of any tensor product of $U_q(\mathfrak{g})$-modules is the same as the decomposition of the corresponding tensor product of $U(\mathfrak{g})$-modules. In particular, the Lie bracket and the Killing form on $\mathfrak{g}$ show that $V \otimes V$ contains a copy of V and a copy of $\mathbb{C}$, and so $V_q \otimes V_q$ contains a copy of V_q and a copy of $\mathbb{C}$. The projections $V_q \otimes V_q \longrightarrow V_q$ and $V_q \otimes V_q \longrightarrow \mathbb{C}$ give a multiplication and a bilinear form on $\mathfrak{g}_q = V_q$ satisfying the given conditions. ∎

The case $\mathfrak{g} = \mathfrak{sl}(2)$ Let U_q denote the quantized universal enveloping algebra of $\mathfrak{sl}(2)$ as defined in Chapters VI-VII of [K]. As an algebra U_q has generators E, F, K, K^{-1} and relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [EF] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The coalgebra structure is given by

$$\begin{aligned}
\Delta(K) &= K \otimes K \\
\Delta(K^{-1}) &= K^{-1} \otimes K^{-1} \\
\epsilon(K) &= \epsilon(K^{-1}) = 1 \\
\Delta(E) &= 1 \otimes E + E \otimes K \\
\Delta(F) &= K^{-1} \otimes F + F \otimes 1 \\
\epsilon(E) &= \epsilon(F) = 1.
\end{aligned}$$

The bialgebra U_q becomes a Hopf algebra if we define the antipode by

$$S(K) = K^{-1}, \quad S(K^{-1}) = K, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.$$

Let $V(n)_q$ for $n \geq 0$ denote the unique simple U_q-module with highest weight q^n; then $\dim V(n)_q = n + 1$. If v_0 is a highest weight vector in $V(n)_q$ then the vectors $v_i = \frac{1}{[i]!}F^i v_0$ for $0 \leq i \leq n$ form a basis of $V(n)_q$. Here $[i] = (q^i - q^{-i})/(q - q^{-1})$ and $[i]! = [i][i-1]\cdots[1]$. The quantum Clebsch-Gordan formula (Theorem VII.7.1 of [K]) states that for $n \geq m \geq 0$ there is a U_q-module isomorphism

$$V(n)_q \otimes V(m)_q \cong V(n+m)_q \oplus V(n+m-2)_q \oplus \cdots \oplus V(n-m)_q.$$

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In the special case $n = m = 2$ we obtain

$$(*) \qquad V(2)_q \otimes V(2)_q \cong V(4)_q \oplus V(2)_q \oplus V(0)_q.$$

Let v_0 denote a highest weight vector for the copies of $V(2)_q$ on the left side of $(*)$. Lemma VII.7.2 of [K] gives highest weight vectors x_0, y_0 and z_0 for the summands $V(4)_q$, $V(2)_q$ and $V(0)_q$ on the right side of $(*)$. Let s_i denote the i-th vector in the ordered basis $\{x_0, x_1, x_2, x_3, x_4, y_0, y_1, y_2, z_0\}$ and let t_j be the j-th vector in the ordered basis

$$\{v_0 \otimes v_0, v_0 \otimes v_1, v_0 \otimes v_2, v_1 \otimes v_0, v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_0, v_2 \otimes v_1, v_2 \otimes v_2\}.$$

Then $s_i = \sum_{j=1}^{9} c_{ij} t_j$ where $C = (c_{ij})$ is the matrix of quantum Clebsch-Gordan coefficients:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/q^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/q^4 & 0 & 1/q & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/q^2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1/q^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (q^2+1)/q^3 & 0 & (q^2-1)/q^2 & 0 & -(q^2+1)/q^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1/q^2 & 0 \\
0 & 0 & 1 & 0 & -q/(q^2+1) & 0 & 1/q^2 & 0 & 0
\end{pmatrix}.$$

Rows 1, 6 and 9 give, respectively, the highest weight vectors of $V(4)_q$, $V(2)_q$ and $V(0)_q$:

$$x_0 = v_0 \otimes v_0, \quad y_0 = v_0 \otimes v_1 - \frac{1}{q^2} v_1 \otimes v_0, \quad z_0 = v_0 \otimes v_2 - \frac{q}{q^2+1} v_1 \otimes v_1 + \frac{1}{q^2} v_2 \otimes v_0.$$

Columns 1–5, 6–8 and 9 of C^{-1} are

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & q^2/(q^4+1) & 0 & 0 & 0 \\
0 & 0 & q^4/(q^8+q^6+2q^4+q^2+1) & 0 & 0 \\
0 & q^4/(q^4+1) & 0 & 0 & 0 \\
0 & 0 & (q^7+2q^5+q^3)/(q^8+q^6+2q^4+q^2+1) & 0 & 0 \\
0 & 0 & 0 & q^2/(q^4+1) & 0 \\
0 & 0 & q^8/(q^8+q^6+2q^4+q^2+1) & 0 & 0 \\
0 & 0 & 0 & q^4/(q^4+1) & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 0 & 0 \\
q^4/(q^4+1) & 0 & 0 \\
0 & q^5/(q^6+q^4+q^2+1) & 0 \\
-q^2/(q^4+1) & 0 & 0 \\
0 & (q^4-q^2)/(q^4+1) & 0 \\
0 & 0 & q^4/(q^4+1) \\
0 & -q^5/(q^6+q^4+q^2+1) & 0 \\
0 & 0 & -q^2/(q^4+1) \\
0 & 0 & 0
\end{pmatrix}$$

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$$\begin{pmatrix} 0 \\ 0 \\ q^4/(q^4+q^2+1) \\ 0 \\ -(q^3+q)/(q^4+q^2+1) \\ 0 \\ q^2/(q^4+q^2+1) \\ 0 \\ 0 \end{pmatrix}.$$

Now identify the copies of $V(2)_q$ on the left and right sides of $(*)$. Let X' denote a highest weight vector in $V(2)_q$, and set $H' = FX'$ and $Y' = \frac{1}{[2]!}F^2X'$. Columns 6–8 give the structure constants of $\mathfrak{sl}(2)_q$ (written with brackets although the composition is not anticommutative in general):

$$\begin{aligned}
[X'X'] &= 0 \\
[X'H'] &= \frac{q^4}{q^4+1}X' \\
[X'Y'] &= \frac{q^5}{(q^4+1)(q^2+1)}H' \\
[H'X'] &= \frac{-q^2}{q^4+1}X' \\
[H'H'] &= \frac{q^2(q^2-1)}{q^4+1}H' \\
[H'Y'] &= \frac{q^4}{q^4+1}Y' \\
[Y'X'] &= \frac{-q^5}{(q^4+1)(q^2+1)}H' \\
[Y'H'] &= \frac{-q^2}{q^4+1}Y' \\
[Y'Y'] &= 0.
\end{aligned}$$

Column 9 gives the U_q-invariant bilinear form on $\mathfrak{sl}(2)_q$:

$$\begin{aligned}
(X', Y') &= \frac{q^4}{(q^4+q^2+1)} \\
(H', H') &= \frac{-q(q^2+1)}{(q^4+q^2+1)} \\
(Y', X') &= \frac{q^2}{(q^4+q^2+1)}
\end{aligned}$$

and all other pairings are 0. Now define $H = aH'$, $X = bX'$, $Y = cY'$ where

$$a = b = -q^{-4}(q^6+q^4+q^2+1), \quad c = q^{-5}(q^6+q^4+q^2+1).$$

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Then we obtain

$$
(**) \qquad
\begin{aligned}
[XX] &= 0\\
[XH] &= -(1+q^2)X\\
[XY] &= H\\
[HX] &= (1+q^{-2})X\\
[HH] &= (q^{-2}-q^2)H\\
[HY] &= -(1+q^2)Y\\
[YX] &= -H\\
[YH] &= (1+q^{-2})Y\\
[YY] &= 0.
\end{aligned}
$$

(In the limit case $q = 1$ we obtain the $\mathfrak{sl}(2)$ relations $[HX] = 2X$, $[HY] = -2Y$, $[XY] = H$.) The bilinear form becomes

$$
\begin{aligned}
(X,Y) &= \frac{3(q^2+1)^2(q^4+1)^2}{4q^5(q^4+q^2+1)}\\
(H,H) &= \frac{3(q^2+1)^3(q^4+1)^2}{4q^7(q^4+q^2+1)}\\
(Y,X) &= \frac{3(q^2+1)^2(q^4+1)^2}{4q^3(q^4+q^2+1)}
\end{aligned}
$$

where we now take $-4z_0/3$ as basis for $V(0)_q$. (In the limit case $q = 1$ we obtain the Killing form $(X,Y) = 4$, $(H,H) = 8$, $(Y,X) = 4$.)

FINAL REMARK. The algebra $\mathfrak{sl}(2)_q$ is not a quantum Lie algebra in the sense of [Li] since by definition such an algebra A is anticommutative. The algebras of [Li] also satisfy the quantum Jacobi identity $J_q(x,y,z) = (xy)\sigma(z)+(yz)\sigma(x)+(zx)\sigma(y) = 0$, where $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is $\mathbb{Z}$-graded and $\sigma = (J+J^{-1})/2$, $J(a_n) = q^n a_n$ for any $a_n \in A_n$. The algebra $\mathfrak{sl}(2)_q$ fails to satisfy the quantum Jacobi identity, since any $\mathbb{Z}$-grading of $\mathfrak{sl}(2)_q$ must have $\deg(H) = 0$ and then $J_q(H,H,H) = 3(q^{-2}-q^2)^2H$, which is non-zero for $q \notin \{0, \pm 1, \pm i\}$.

NOTE ADDED IN PROOF. The main theorem of this paper has also been proven by G. W. Delius and M. D. Gould in *Quantum Lie algebras, their existence, uniqueness and q-antisymmetry*, King's College London preprint KCL-TH-96-05, q-alg/9605025, Commun. Math. Phys., to appear.

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