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# SOME DUAL ASPECTS OF THE POISSON KERNEL

*by* F. F. BONSALL

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The Poisson kernel $p(z,\zeta)=(1-|z|^2)|1-\bar{z}\zeta|^{-2}$ is defined for $z$ in the open unit disc $D$ and $\zeta$ in the unit circle $\partial D$. As usually employed, it is integrated with respect to the second variable and a measure on $\partial D$ to yield a harmonic function on $D$. Here, we fix a $\sigma$-finite positive Borel measure $m$ on $D$ and integrate the Poisson kernel with respect to the first variable against a function $\phi$ in $L^1(m)$ to obtain a function $T_m\phi$ on $\partial D$. We ask for what measures $m$ the range of $T_m$ is $L^1(\partial D)$, for what $m$ the kernel of $T_m$ is non-zero, and for what $m$ every positive continuous function on $\partial D$ is of the form $T_m\phi$ with $\phi$ non-negative. When $m$ is the counting measure of a countably infinite subset $\{a_k : k\in\mathbb{N}\}$ of $D$, the function $(T_m\phi)(\zeta)$ is of the form $\sum_{k=1}^{\infty}\lambda_k p(a_k,\zeta)$ with $\sum_{k=1}^{\infty}|\lambda_k|<\infty$. The main results generalize results previously obtained for sums of this form. A related mapping from $L^p(m)$ into $L^p(\partial D)$ with $1<p<\infty$ is briefly considered.

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## 1. Introduction

Let $D$ be the open unit disc in $\mathbb{C}$, $\partial D$ the unit circle, and

$$p(z,\zeta)=p_z(\zeta)=(1-|z|^2)|1-\bar{z}\zeta|^{-2}(z\in D,\zeta\in\partial D),$$

the Poisson kernel for $D$. The Poisson kernel is normally employed by integrating with respect to the second variable against an integrable function $f$ on $\partial D$, or with respect to a Borel measure $\mu$ on $\partial D$, in order to construct harmonic functions on $D$ with boundary values related to $f$ and $\mu$ respectively. In the present article, we fix a measure $m$ on $D$ and integrate the Poisson kernel with respect to the first variable against an integrable function on $D$ to construct a function on $\partial D$. The special case in which the integration is with respect to the counting measure of a countably infinite subset of $D$ gives rise to representations of functions on $\partial D$ as sums of Poisson kernels with the sequence of coefficients in $l^1$, see [**1, 2, 3, 7**]. The results in this special case have guided a large part of this article.

Throughout this article, $m$ is a $\sigma$-finite positive Borel measure on $D$, the $\sigma$-finiteness being required, in particular, to allow the use of Fubini's theorem.

In Section 2, we define $T_m\phi$, for $\phi$ in $L^1(m)$, by

$$(T_m\phi)(\zeta)=\int_D \phi(z)p_z(\zeta)\,dm(z). \tag{1.1}$$

By Fubini's theorem, $T_m$ is a bounded linear mapping of $L^1(m)$ into $L^1(\partial D)$. The adjoint

$T_m^*$ is therefore a bounded linear mapping of $L^\infty(\partial D)$ into $L^\infty(m)$, and is easily identified as the mapping that sends $g$ in $L^\infty(\partial D)$ to its harmonic extension $g^\dagger$, regarded as an element of $L^\infty(m)$.

We ask what measures $m$ give $T_m L^1(m) = L^1(\partial D)$, and prove, Theorem 2.3, that this holds if and only if $m$ has a property of non-tangential density analogous to the corresponding property of a sequence of points. The same theorem also gives other equivalent properties, including the equality $\|g\|_\infty = \|g^\dagger\|_\infty$ for functions $g$ in $L^\infty(\partial D)$ and their harmonic extensions $g^\dagger$.

In Section 3, we ask for what measures $m$ the kernel of $T_m$ is non-zero. It follows easily from Theorem 2.3 that non-tangential density of $m$ implies that $\ker T_m \neq \{0\}$. The remainder of Section 3 is mainly concerned with conditions under which the converse implication holds. If $\phi \in \ker T_m$, we have

$$\int_D \phi(z)(1-wz)^{-1}\,dm(z) = 0 \qquad (w \in D). \tag{1.2}$$

In Theorem 3.3, we show that if $m$ is not non-tangentially dense but a non-zero $\phi$ satisfies (1.2), then there exists a non-void open subset $G$ of $D$ such that $m(G)=0$ and

$$\int_D \phi(z)(z-w)^{-1}\,dm(z) = 0 \qquad (w \in G).$$

In Theorem 3.4, we suppose that the support of $m$ is the union of a sequence $\{E_n\}$ of compact subsets of $D$ with connected complements and void interiors, and that, for each $n$, $E_n$ has void intersection with the closure of the union of the remaining sets $E_k$. For such $m$, $\ker T_m$ is non-zero if and only if $m$ is non-tangentially dense for $\partial D$. The particular case in which $m$ is the counting measure of a countably infinite set without limit points in $D$ is known [3]. In Corollary 3.6, we show that, with $m$ as in Theorem 3.4, $T_m$ has closed range if and only if either $T_m L^1(m) = L^1(\partial D)$ or $T_m^* L^\infty(\partial D) = L^\infty(m)$.

In Section 4, we consider the representation of continuous functions in the form $T_m\phi$. We are unable, even for counting measures, to determine those measures $m$ such that every continuous function on $D$ is of the form $T_m\phi$ with $\phi$ in $L^1(m)$. The following approximation problem is perhaps the most natural question in this context. For what subsets $A$ of $D$ is the linear span of $\{p_a : a \in A\}$ uniformly dense in the space of continuous functions on $D$? It is easily proved (Theorem 4.1) that this question is equivalent to the following question about the space $h^1$ of differences of positive harmonic functions on $D$. For what subsets $A$ of $D$ is the zero function the only member of $h^1$ that vanishes on $A$? It would be very interesting to find a geometric characterization of these sets.

The remainder of Section 4 is concerned with the representation of positive continuous functions on $\partial D$ (that is continuous functions $f$ such that $f(\zeta) > 0$ everywhere on $\partial D$). We say that $m$ is a *positive Poisson representing measure* (PPR measure) if every positive continuous function $f$ on $\partial D$ is of the form

$$f(\zeta)=\int_D \phi(z)p_z(\zeta)\,dm(z) \qquad (\zeta\in\partial D),$$

with $\phi$ a non-negative Borel measurable function on $D$. In [3], a subset $A$ of $D$ is called a *positive Poisson basic set* (PPB set) if every positive continuous function $f$ on $\partial D$ is of the form

$$f(\zeta)=\sum_{k=1}^{\infty}\lambda_k p_{a_k}(\zeta),$$

with $\lambda_k\geqq 0$ and $a_k$ in $A$ for all $k$. We prove in Theorem 4.4 that the following three conditions are equivalent. (i) $m$ is a PPR measure. (ii) For every member $h$ of $h^1$, $\sup_{z\in D}h(z)=\operatorname{ess\,sup}_{z\in D}h(z)$, with the essential supremum relative to $m$. (iii) The support of $m$ is a PPB set. This theorem reduces the problem of representation of positive continuous functions to the problem of characterizing PPB sets, which has been completely solved by Hayman and Lyons [7].

In Section 5, we fix $p$ with $1<p<\infty$ and define a mapping $T_m^{(p)}$ from $L^p(m)$ to $L^p(\partial D)$ that is analogous to $T_m$. In general, $T_m^{(p)}$ is unbounded, but it is an easy consequence of Carleson's theorem that $T_m^{(p)}$ is bounded if and only if $m$ is a Carleson measure, (Theorem 5.1). We have been unable to determine the Carleson measures $m$ (if any) for which $T_m^{(p)}L^p(m)=L^p(\partial D)$. However, non-tangential density of $m$ implies that the range of $T_m^{(p)}$ is dense in $L^p(\partial D)$, (Theorem 5.2).

In the final section, Section 6, we take $1<p<\infty$, $q=p(p-1)^{-1}$, and consider certain sums of Poisson kernels. Let $\{a_k:k\in\mathbb{N}\}$ be a countably infinite subset of $D$, and, for each $k$, let

$$Q_k=(1-|a_k|)^{1/q}p_{a_k}.$$

We define a, possibly unbounded, linear mapping $S$ from $l^p$ into $L^p(\partial D)$, such that, for $\lambda$ in the domain of $S$,

$$(S\lambda)(\zeta)=\sum_{k=1}^{\infty}\lambda_k Q_k(\zeta) \quad \text{a.e.}$$

Let $m$ be the measure on $D$ given by

$$m=\sum_{k=1}^{\infty}(1-|a_k|)\,\delta_{a_k},$$

with $\delta_a$ the unit mass concentrated at $a$. Then $S$ is a bounded linear mapping of $l^p$ into $L^p(\partial D)$ if and only if $m$ is a Carleson measure (Theorem 6.2). This theorem also gives other properties of $S$ when $m$ is a Carleson measure. In particular, $\ker S=\{0\}$. In Theorem 6.3, it is proved that $S$ is bounded and has closed range if and only if $\{a_k\}$ is an interpolating sequence for $H^\infty$. In this case, the range of $S$ is a closed subspace, whose elements correspond to unique elements of $l^p$, and this unique coefficient sequence is given in Corollary 6.4.

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## 2. The representation of $L^1(\partial D)$

It is often convenient to use the same symbol to denote an element of $L^1(\partial D)$ and its harmonic extension to $D$, but this convention would be confusing in the present context. Given an element $f$ of $L^1(\partial D)$, we therefore denote its harmonic extension to $D$ by $f^\dagger$, thus

$$f^\dagger(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})p_z(e^{it})\,dt \qquad (z\in D).$$

We use $[,]$ to denote the natural bilinear forms on $L^1(\partial D)\times L^\infty(\partial D)$ and on $L^1(m)\times L^\infty(m)$, that is

$$[f,g]=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})g(e^{it})\,dt \qquad (f\in L^1(\partial D),\ g\in L^\infty(\partial D)),$$

$$[\phi,\psi]=\int_D \phi(z)\psi(z)\,dm(z) \qquad (\phi\in L^1(m),\ \psi\in L^\infty(m)).$$

For $\phi$ in $L^1(m)$, we define $T_m\phi$ as in (1.1).

**Lemma 2.1.** *$T_m$ is a bounded linear mapping of $L^1(m)$ into $L^1(\partial D)$, and its adjoint $T_m^*$ is the harmonic extension mapping of $L^\infty(\partial D)$ into $L^\infty(m)$, that is*

$$T_m^*g=g^\dagger \qquad (g\in L^\infty(\partial D)).$$

**Remark.** We are following the usual convention here, in using the same symbol $g^\dagger$ to denote the element of $L^\infty(m)$ containing the bounded function $g^\dagger$.

**Proof** (of Lemma 2.1). Given $\phi$ in $L^1(m)$, the function $\psi$ on $D\times\partial D$, given by

$$\psi(z,\zeta)=|\phi(z)|p_z(\zeta),$$

is Borel measurable, since it is the product of two functions each Borel measurable on $D\times\partial D$. Thus, by Fubini's theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\int_D |\phi(z)|p_z(e^{it})\,dm(z)\right\}dt=\int_D\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} p_z(e^{it})\,dt\right\}|\phi(z)|\,dm(z)$$

$$=\int_D |\phi(z)|\,dm(z)=\|\phi\|_1.$$

Therefore, $T_m\phi \in L^1(\partial D)$ and $\|T_m\phi\|_1 \leqq \|\phi\|_1$. The mapping $T_m$ is a bounded linear mapping of $L^1(m)$ into $L^1(\partial D)$ with norm 1. With the dual space of $L^1$ identified with $L^\infty$ as usual, the adjoint $T_m^*$ becomes a bounded linear mapping of $L^\infty(\partial D)$ into $L^\infty(m)$. To show that this adjoint is the harmonic extension mapping of $L^\infty(\partial D)$ into $L^\infty(m)$, let $\phi \in L^1(m)$, $g \in L^\infty(\partial D)$. Then

$$[\phi, T_m^* g] = [T_m\phi, g] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\int_D \phi(z)p_z(e^{it})\,dm(z)\right\}g(e^{it})\,dt$$

$$= \int_D \phi(z)\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it})p_z(e^{it})\,dt\right\}dm(z)$$

$$= \int_D \phi(z)g^\dagger(z)\,dm(z) = [\phi, g^\dagger].$$

Since this holds for all $\phi$ in $L^1(m)$, we have $T_m^* g = g^\dagger$, with $g^\dagger$ identified in $L^\infty(m)$.

**Notation.** We denote by $A(m)$ the support of the measure $m$, that is the complement in $D$ of the largest open subset $G$ with $m(G) = 0$.

Given $b > 0$, $0 < \alpha < \pi/2$, let

$$D(b,\alpha) = \{z \in D : b > 1 - Re z > |Im z| \cot \alpha\}.$$

For $\zeta$ in $\partial D$ and $b, \alpha$ as above, we denote by $D(\zeta, b, \alpha)$ the triangular open set $\zeta D(b,\alpha)$.

The measure $m$, or a subset $A$ of $D$, is said to be *non-tangentially dense* for $\partial D$ if, for almost all $\zeta$ in $\partial D$, there exists $\alpha$ with $0 < \alpha < \pi/2$ such that, for all $b > 0$, we have $m(D(\zeta, b, \alpha)) > 0$, $A \cap D(\zeta, b, \alpha) \neq \varnothing$, respectively.

**Lemma 2.2.** *Let $f$ be a continuous real function on $D$, then*

$$\sup_{z \in A(m)} f(z) = \operatorname{ess\,sup}_{z \in D} f(z),$$

*where the essential supremum is relative to the measure $m$.*

**Proof.** Let $\alpha = \sup_{z \in A(m)} f(z)$ and $\beta = \operatorname{ess\,sup}_{z \in D} f(z)$. By definition of $A(m)$, $m(D \setminus A(m)) = 0$. Thus $f(z) \leqq \alpha$ a.e. $(m)$, and so $\beta \leqq \alpha$. Suppose that $\beta < \alpha$, and let $G = \{z \in D : f(z) > \beta\}$. Then $G \cap A(m) \neq \varnothing$, but $G$ is an open set with $m(G) = 0$, and so $G \cap A(m) = \varnothing$.

**Theorem 2.3.** *The following statements are equivalent to each other.*

(i) $T_m L^1(m) = L^1(\partial D)$

(ii) $T_m L^1(m) = L^1(\partial D)$, *and for each $f$ in $L^1(\partial D)$*,

$$\|f\|_1 = \inf\{\|\phi\|_1 : \phi \in L^1(m) \text{ and } T_m\phi = f\}.$$

(iii) *For all $g$ in $L^\infty(\partial D)$*,

$$\|g\|_\infty = \|g^\dagger\|_\infty,$$

*where the essential supremum $\|g^\dagger\|_\infty$ is relative to the measure $m$.*

(iv) *The measure $m$ is non-tangentially dense for $\partial D$.*

(v) *The support $A(m)$ is non-tangentially dense for $\partial D$.*

**Proof.** (i)⇒(iii). Let $T_m L^1(m) = L^1(\partial D)$. By the open mapping theorem, there exists a positive constant $\kappa$ such that the image of the open ball in $L^1(m)$ with centre 0 and radius $\kappa$ contains the closed unit ball in $L^1(\partial D)$. Thus, for all $f$ in $L^1(\partial D)$,

$$\inf\{\|\phi\|_1 : \phi \in L^1(m),\ T_m\phi = f\} \leqq \kappa\|f\|_1. \tag{2.1}$$

Let $g \in L^\infty(\partial D)$ and $\varepsilon > 0$. Then there exists $f$ in $L^1(\partial D)$ with $\|f\|_1 = 1$ and $|[f,g]| > \|g\|_\infty - \varepsilon$. By (2.1), there exists $\phi$ in $L^1(m)$ with $\|\phi\|_1 < \kappa + \varepsilon$ and $T_m\phi = f$. Therefore

$$\begin{aligned}\|g\|_\infty - \varepsilon < |[f,g]| &= |[T_m\phi, g]| \\ &= |[\phi, T_m^* g]| \leqq (\kappa+\varepsilon)\|T_m^* g\|_\infty.\end{aligned}$$

By Lemma 2.1, this gives

$$\|g\|_\infty \leqq \kappa\|g^\dagger\|_\infty,$$

and, by Lemma 2.2, we have

$$\|g\|_\infty \leqq \kappa \sup_{z \in A(m)} |g^\dagger(z)|.$$

By [**2**, Theorem 2], it follows that

$$\|g\|_\infty = \sup_{z \in A(m)} |g^\dagger(z)|. \tag{2.2}$$

An application of Lemma 2.2 completes the proof of (iii).

(iii)⇔(v). As we have seen, (iii) is equivalent to (2.2) holding for all $g$ in $L^\infty(\partial D)$. By [**2**, Theorem 2], this is equivalent to (v).

(v)⇔(iv). Since $D(\zeta, b, \alpha)$ is an open subset of $D$,

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$$m(D(\zeta, b, \alpha)) = 0 \Leftrightarrow A(m) \cap D(\zeta, b, \alpha) = \varnothing.$$

(iv)⇒(ii). Let (iv) hold, that is

$$\|T_m^* g\|_\infty = \|g\|_\infty \qquad (g \in L^\infty(\partial D)).$$

This implies that $T_m^*$ has closed range and zero kernel, and so, by Banach's closed range theorem [**4**, p. 488], $T_m L^1(m) = L^1(\partial D)$.

Let $X$ denote the quotient Banach space $L^1(m)/\ker T_m$, and define $S$ on $X$ by

$$Sx = T_m \phi \qquad (\phi \in x \in X).$$

Then $S$ is an invertible bounded linear mapping of $X$ onto $L^1(\partial D)$, and its adjoint $S^*$ is an invertible bounded linear mapping of $L^\infty(\partial D)$ onto $X^*$. If $\phi \in x \in X$ and $g \in L^\infty(\partial D)$, we have

$$(S^* g)(x) = [Sx, g] = [T_m \phi, g] = [\phi, T_m^* g]. \tag{2.3}$$

Let $g \in L^\infty(\partial D)$ and $\varepsilon > 0$. Then there exists $\phi$ in $L^1(m)$ with $\|\phi\|_1 = 1$ and $|[\phi, T_m^* g]| > \|T_m^* g\|_\infty - \varepsilon$. Let $x$ denote the coset of $\phi$ in $X$. Then, by (2.3),

$$|(S^* g)(x)| > \|T_m^* g\|_\infty - \varepsilon,$$

and $\|x\| \leqq \|\phi\|_1 = 1$. Therefore, the norm of the functional $S^* g$ satisfies

$$\|S^* g\| > \|T_m^* g\|_\infty - \varepsilon = \|g\|_\infty - \varepsilon.$$

Since $\varepsilon$ is arbitrary, we have proved that

$$\|S^* g\| \geqq \|g\|_\infty \qquad (g \in L^\infty(\partial D)),$$

and so

$$\|S^{-1}\| = \|(S^*)^{-1}\| \leqq 1.$$

Given $f$ in $L^1(\partial D)$, take $y = S^{-1} f$. Then $\|y\| \leqq \|f\|_1$, that is

$$\inf\{\|\phi\|_1 : \phi \in y\} \leqq \|f\|_1.$$

But $\phi \in y$ if and only if $T_m \phi = f$, and so

$$\inf\{\|\phi\|_1 : T_m \phi = f\} \leq \|f\|_1.$$

**Remark.** [**2**, Theorem 2] contains other equivalent conditions which can be added to the list in Theorem 2.3.

It is also natural to ask for what measures $m$ the range of $T_m$ is dense in $L^1(\partial D)$.

**Theorem 2.4.** *The following statements are equivalent to each other.*

(i) $T_m L^1(m)$ *is norm dense in* $L^1(\partial D)$.

(ii) *If* $g \in L^\infty(\partial D)$ *and* $g^\dagger(z)=0$ *almost everywhere on D relative to m, then* $g=0$.

(iii) *If* $g \in L^\infty(\partial D)$ *and* $g^\dagger(z)=0$ *for all z in the support of m, then* $g=0$.

**Proof.** (i)⇒(ii). Let $T_m L^1(m)$ be norm dense in $L^1(\partial D)$. Then $T_m^*$ has zero kernel. If $g \in L^\infty(\partial D)$ and $g^\dagger(z)=0$ almost everywhere $(m)$, then $T_m^* g$ is the zero element of $L^\infty(m)$, $g$ belongs to the kernel of $T_m^*$, $g=0$.

(ii)⇒(i). Let (ii) hold, that is $T_m^*$ has zero kernel, and, by the Hahn–Banach theorem, $T_m L^1(m)$ is norm dense in $L^1(\partial D)$.

(ii)⇔(iii). Apply Lemma 2.2 to $|g^\dagger|$.

**Remark.** Non-tangential density is plainly a sufficient condition for the statements in Theorem 2.4. It would be interesting to find a necessary and sufficient geometrical condition.

## 3. Measures $m$ for which $T_m$ has non-zero kernel

**Corollary 3.1.** *If* $T_m L^1(m) = L^1(\partial D)$, *then* $\ker T_m \neq \{0\}$.

**Proof.** Let $T_m L^1(m) = L^1(\partial D)$. Then $m(D)>0$ and there exists a Borel subset $E$ of $D$ with $0<m(E)<\infty$. Since $D$ is $\sigma$-compact, there exists a compact subset $K$ of $D$ with $0<m(K\cap E)<\infty$. Let $F=K\cap E$, $F^c=D\backslash F$, and define a measure $\mu$ on $D$ by $\mu(X)=m(X\cap F^c)$, for Borel subsets $X$ of $D$. Let $\eta=\inf\{1-|z|: z\in F\}$. For $b$ with $0<b<\eta$, $\zeta\in\partial D$, $0<\alpha<\pi/2$, we have $D(\zeta,b,\alpha)\subset F^c$, and so

$$\mu(D(\zeta,b,\alpha)) = m(D(\zeta,b,\alpha)).$$

By Theorem 2.3, it follows that $\mu$ is non-tangentially dense for $\partial D$, and so

$$T_\mu L^1(\mu) = L^1(\partial D). \tag{3.1}$$

Since $m(F)<\infty$, the characteristic function $\chi_F$ of $F$ belongs to $L^1(m)$, and so $T_m\chi_F \in L^1(\partial D)$. By (3.1), it follows that there exists $\phi$ in $L^1(\mu)$ with $T_\mu\phi = T_m\chi_F$. Since $\mu(F)=0$, we may assume that $\phi(z)=0$ for all $z$ in $F$. Thus.

$$\int_D |\phi(z)|\,dm(z) = \int_{F^c} |\phi(z)|\,dm(z) = \int_{F^c} |\phi(z)|\,d\mu(z) = \int_D |\phi(z)|\,d\mu(z),$$

and so $\phi\in L^1(m)$. Likewise $(T_m\phi)(\zeta)=(T_\mu\phi)(\zeta)$. Thus, $T_m\phi = T_\mu\phi = T_m\chi_F$, and so

$\phi-\chi_F\in\ker T_m$. Finally, since $\phi$ is identically zero on $F$, $|\phi-\chi_F|=|\phi|+|\chi_F|$, and therefore $\|\phi-\chi_F\|_1\geqq m(F)>0$.

**Notation.** We denote by $H^\infty$ the closed subspace of $L^\infty(\partial D)$ consisting of those $f$ with their Fourier coefficients $\hat{f}(n)$ zero for all negative $n$. As is well known, the set of bounded analytic functions on $D$ coincides with the set $\{f^\dagger: f\in H^\infty\}$.

**Lemma 3.2.** *If $\phi\in\ker T_m$, then*

$$\int_D \phi(z)(1-wz)^{-1}\,dm(z)=0 \qquad (w\in D).$$

**Proof.** Let $u_w(\zeta)=(1-w\zeta)^{-1}(w\in D,\zeta\in\partial D)$, and let $\phi\in L^1(m)$ with $T_m\phi=0$. For $w$ in $D$, we have $u_w\in H^\infty$, and so

$$0=[T_m\phi,u_w]=[\phi,T_m^*u_w]=[\phi,u_w^\dagger]$$

$$=\int_D \phi(z)(1-wz)^{-1}\,dm(z).$$

**Definition.** As in [3], we define the *firm boundary* of an open subset $G$ of $D$ to be the set of $\zeta$ in $\partial D$ such that, for every $\alpha$ with $0<\alpha<\pi/2$, there exists $b>0$ with $D(\zeta,b,\alpha)\subset G$.

**Theorem 3.3.** *Let $m$ be not non-tangentially dense for $\partial D$, but let $\phi$ belong to $L^1(m)$, such that $\|\phi\|_1>0$ and*

$$\int_D \phi(z)(1-wz)^{-1}\,dm(z)=0 \qquad (w\in D). \tag{3.2}$$

*Then there exists an open subset $G$ of $D$ such that*

(i) $m(G)=0$,

(ii) *the firm boundary of $G$ has positive Lebesgue measure,*

(iii) $\int_D \phi(z)(z-w)^{-1}\,dm(z)=0 \qquad (w\in G)$.

**Proof.** By Theorem 2.3, the support $A(m)$ is not non-tangentially dense for $\partial D$, and therefore, by [**2**, Theorem 2], there exists $g$ in $H^\infty$ such that

$$\sup_{z\in A(m)}|g^\dagger(z)|<1<\sup_{z\in D}|g^\dagger(z)|=\|g\|_\infty.$$

Let $G=\{z\in D:|g^\dagger(z)|>1\}$. Then $G$ is an open subset of $D$ with $m(G)=0$. Let $E=\{\zeta\in\partial D:|g(\zeta)|>1\}$. Since $\|g\|_\infty>1$, $E$ has positive Lebesgue measure, and for almost all $\zeta$ in $E$, $g^\dagger(z)\to g(\zeta)$ as $z\to\zeta$ non-tangentially. For such $\zeta$ and $\alpha$ with $0<\alpha<\pi/2$, there exists $b>0$ such that $|g^\dagger(z)|>1$ for all $z$ in $D(\zeta,b,\alpha)$, that is $D(\zeta,b,\alpha)\subset G$. This proves (ii).

Define $h$ on $G$ by

$$h(w)=\int_D \phi(z)(z-w)^{-1}\,dm(z) \qquad (w\in G).$$

For $w$ in $G$, we have $\operatorname{dist}(w,G^c)>0$, where $G^c=D\backslash G$. Thus $\psi_w$, defined by $\psi_w(z)=\phi(z)(z-w)^{-1}$, belongs to $L^1(m)$, and $h$ is well-defined on $G$ and analytic there.

Suppose that (iii) does not hold, and let $a$ be a point of $G$ with $h(a)\neq 0$. By geometric series expansion of $(1-wz)^{-1}$, (3.2) gives

$$\int_D \phi(z)z^k\,dm(z)=0 \qquad (k\geqq 0).$$

Thus, for $j,k=0,1,2,\ldots,$

$$\int_D \phi(z)z^j(1-wz)^{-k}\,dm(z)=0 \qquad (w\in D). \tag{3.3}$$

Let $|w|<1/4$. Then, for all $z$ in $D$, we have

$$(|a|+|z|)|w||1-wz|^{-1}<2/3,$$

and so, for $z$ in $D$,

$$(1-aw)^{-1}=(1-zw-(a-z)w)^{-1}=\sum_{k=0}^{\infty}(a-z)^k w^k(1-zw)^{-(k+1)},$$

with the series converging uniformly absolutely on $D$. Multiplication by $\psi_a(z)$ and integration gives

$$\begin{aligned}(1-aw)^{-1}h(a)&=\int_D \psi_a(z)(1-wz)^{-1}\,dm(z)-\sum_{k=1}^{\infty} w^k\int_D \phi(z)(a-z)^{k-1}(1-zw)^{-(k+1)}\,dm(z)\\&=\int_D \psi_a(z)(1-wz)^{-1}\,dm(z),\end{aligned}$$

by (3.3). This holds whenever $|w|<1/4$, and hence, by analyticity, for all $w$ in $D$.

Take $\psi(z)=\psi_a(z)/h(a)$. Then $\psi\in L^1(m)$, and

$$(1-aw)^{-1}=\int_D \psi(z)(1-wz)^{-1}\,dm(z) \qquad (w\in D).$$

It follows that

$$a^n=\int_D \psi(z)z^n\,dm(z)$$

for non-negative integers $n$, and hence that

$$p(a)=\int_D \psi(z)p(z)\,dm(z)$$

for all polynomials $p$. Given $f$ in $H^\infty$, there exists a sequence $\{p_n\}$ of polynomials such that $p_n(\zeta)\to f(\zeta)$ a.e. on $\partial D$ as $n\to\infty$, and $\|p_n\|_\infty \le M\,(n\in\mathbb{N})$, with $M<\infty$. Then $p_n^\dagger(z)\to f^\dagger(z)$ for all $z$ in $D$ and $|p_n^\dagger(z)|\le M\,(n\in\mathbb{N}, z\in D)$. By Lebesgue's dominated convergence theorem, it follows that

$$f^\dagger(a)=\int_D \psi(z)f^\dagger(z)\,dm(z),\ |f^\dagger(a)|\leqq\|\psi\|_1 \sup_{z\in A(m)}|f^\dagger(z)|.$$

Since $|g^\dagger(a)|>1>\sup_{z\in A(m)}|g^\dagger(z)|$, we arrive at a contradiction by taking $f=g^n$ with $n$ sufficiently large.

**Remarks.** If there exists an open subset $G$ of $D$ satisfying (i) and (ii) in the last theorem, it is easy to see that $m$ is not non-tangentially dense for $\partial D$.

In the special case when $m$ is the counting measure for a countable subset $A$ of $D$, Theorem 3.3 is known [3, Theorem 7]. It is noted there that this leads to various conditions on the set $A$ sufficient to imply that $A$ is non-tangentially dense for $\partial D$ whenever the kernel of $T_m$ is non-zero. In particular, this is the case if $A$ has no limit point in $D$. The following theorem shows that this conclusion still holds when the support of $m$ is a countable union of suitable compact sets.

**Theorem 3.4.** *Suppose that the support of m is the union of a sequence $\{E_n\}$ of compact subsets of D such that, for each n, the complement of $E_n$ is connected, the interior of $E_n$ is void, and $E_n$ has void intersection with the closure of the union of the remaining $E_k$.*

*Then* ker $T_m$ *is non-zero if and only if m is non-tangentially dense for $\partial D$.*

**Proof.** Let $\phi\in\ker T_m\backslash\{0\}$. By Lemma 3.2,

$$\int_D \phi(z)(1-wz)^{-1}\,dm(z)=0 \qquad (w\in D).$$

Suppose that $m$ is not non-tangentially dense for $\partial D$. Then, by Theorem 3.3, there exists a non-void open subset $G_0$ of $D$ such that $m(G_0)=0$ and

$$\int_D \phi(z)(z-w)^{-1}\,dm(z)=0 \qquad (w\in G_0). \tag{3.4}$$

Let $E$ denote the support of $m$. Since $E$ is of the form stated in the theorem, it is intuitive, and probably known, that $D\backslash E$ is connected. A proof can be based on the following statement of Alexander's lemma.

**Lemma 3.5.** [**8**, p. 101]. *Let $X_1, X_2$ be subsets of $D$ with $X_1$ compact, $X_2$ relatively closed in $D$ and $X_1 \cap X_2$ connected. If two points of $D \backslash (X_1 \cup X_2)$ are connected in $D \backslash X_1$ and in $D \backslash X_2$, then they are connected in $D \backslash (X_1 \cup X_2)$.*

Let $w_1, w_2 \in D \backslash E$, take $\rho$ with $\max(|w_1|, |w_2|) < \rho < 1$, let $\Delta = \{z: |z| \leqq \rho\}$, and let $P_n = \cup \{E_k: 1 \leqq k \leqq n\}$, $Q_n = \cup \{E_k: k > n\}$. Since $E$ is the support of $m$, it is relatively closed in $D$. Therefore $Q_n$ is relatively closed in $D$ and $Q_n \cap \Delta$ is compact. Therefore, there exists $N$ with $Q_N \cap \Delta$ void. Since $D \backslash E_k$ is connected for each $k$, repeated application of Lemma 3.5 shows that $D \backslash P_N$ is connected. Since $w_1, w_2 \in \operatorname{int} \Delta \subset D \backslash Q_N$, they are connected in $D \backslash Q_N$. Therefore, by Lemma 3.5, $w_1, w_2$ are connected in $D \backslash (P_N \cup Q_N)$, that is in $D \backslash E$.

It now follows by analyticity from (3.4) that

$$\int_D \phi(z)(z-w)^{-1}\, dm(z) = 0 \qquad (w \in D \backslash E). \tag{3.5}$$

We prove next that

$$\int_{E_n} \phi(z)(z-w)^{-1}\, dm(z) = 0 \qquad (w \in \mathbb{C} \backslash E_n, n \in \mathbb{N}). \tag{3.6}$$

Fix $n$ in $\mathbb{N}$, let $J_n = \cup \{E_j: j \neq n\}$, and define $f_n$ and $g_n$ by

$$f_n(w) = \int_{E_n} \phi(z)(z-w)^{-1}\, dm(z) \qquad (w \in \mathbb{C} \backslash E_n),$$

$$g_n(w) = \int_{J_n} \phi(z)(z-w)^{-1}\, dm(z) \qquad (w \in D \backslash J_n),$$

the integrals being well-defined since $\mathbb{C} \backslash E_n$ and $D \backslash J_n$ are open sets.

Since $E$ is the support of $m$, (3.5) gives

$$f_n(w) + g_n(w) = 0 \qquad (w \in D \backslash E).$$

Since $f_n$ and $g_n$ are analytic in $\mathbb{C} \backslash E_n$ and $D \backslash J_n$ respectively and $(\mathbb{C} \backslash E_n) \cap (D \backslash J_n) = D \backslash E$, we can define an entire function $f$ by taking $f(w) = f_n(w)$ on $\mathbb{C} \backslash E_n$ and $f(w) = -g_n(w)$ on $E_n$. When $|w| > 1$, we have $|z - w| > |w| - 1$ for all $z$ in $E_n$, and so

$$|f(w)| = |f_n(w)| \leqq (|w| - 1)^{-1} \|\phi\|_1.$$

Thus $f(w) \to 0$ as $|w| \to \infty$, and so $f$ is identically zero, and (3.6) is proved.

Expanding $(z-w)^{-1}$ in powers of $z/w$ for $|w| > 1 > |z|$, we now have

$$\sum_{k=0}^{\infty} w^{-(k+1)} \int_{E_n} \phi(z) z^k\, dm(z) = 0 \qquad (|w| > 1),$$

and therefore,

$$\int_{E_n} \phi(z)z^k\,dm(z)=0 \qquad (k=0,1,2,\ldots).$$

Since $E_n$ has connected complement and void interior, Mergelyan's theorem (see [**9**, p. 386]) gives

$$\int_{E_n} \phi(z)\psi(z)\,dm(z)=0$$

for all continuous complex functions $\psi$ on $E_n$.

It follows that the complex Borel measure $\mu$ on $E_n$, given by $d\mu(z)=\phi(z)\,dm(z)$, is the zero measure. Therefore

$$\int_{E_n} |\phi(z)|\,dm(z)=0 \qquad (n\in\mathbb{N}),$$

and so $\|\phi\|_1=0$.

**Corollary 3.6.** *Let $m$ be as in Theorem* 3.4. *Then the range of $T_m$ is closed in $L^1(\partial D)$ if and only if either the range of $T_m$ is $L^1(\partial D)$ or the range of $T_m^*$ is $L^\infty(m)$.*

**Proof.** This is an immediate consequence of Theorem 2.3, Theorem 3.4, and Banach's closed range theorem.

**Remark.** Let $BH(D)$ denote the space of bounded harmonic functions on $D$. Then $T_m^*L^\infty(\partial D)=L^\infty(m)$ if and only if, for every element $g$ of $L^\infty(m)$, there exists $h$ in $BH(D)$ with

$$h(z)=g(z) \qquad (\text{a.e. } m). \tag{3.7}$$

If the support of $m$ is a countable set $A=\{a_n: n\in\mathbb{N}\}$, $L^\infty(m)$ can be identified with the space $l^\infty$ of bounded sequences, and (3.7) holds if and only if $A$ is a harmonic interpolation set, that is every bounded sequence is of the form $\{h(a_n)\}$ with $h$ in $BH(D)$. By a theorem of Garnett [**5**], harmonic interpolation is equivalent to $H^\infty$ interpolation. In these circumstances, it is of interest to know whether there exist any measures $m$ of the kind considered in Theorem 3.4 that satisfy (3.7) but do not have countable support. Theorem 3.8 will show that no such measures $m$ exist, at least if we suppose that $m(E_k)<\infty$ for each $k$.

I am indebted to J. G. Clunie for a suggestion on which the proof of the following lemma is based.

**Lemma 3.7.** *Let $K$ be a compact subset of $\mathbb{C}$, $\mu$ a finite positive Borel measure on $K$, and suppose that, for every $\phi$ in $L^\infty(\mu)$, there exists a continuous function $f$ on $K$ with*

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$$f(z) = \phi(z) \text{ a.e. } (\mu). \tag{3.8}$$

*Then $\mu$ is supported by a finite subset of $K$.*

**Proof.** Let $a \in K$, and, for $n$ in $\mathbb{N}$, let $K_n = \{z \in K : |z-a| \leqq n^{-1}\}$, $\psi(n) = \mu(K_n)$. Either $\psi(n) = \mu(\{a\})$ for all sufficiently large $n$, or there exists a subsequence $n_k$ such that $\psi(n_{k+1}) < \psi(n_k)\,(k \in \mathbb{N})$. In the second case, define $\phi$ on $K$ by

$$\phi(z) = (-1)^k \qquad (z \in K_{n_k} \backslash K_{n_{k+1}}, k \in \mathbb{N}),$$

$$\phi(z) = 0 \text{ (all other } z \text{ in } K).$$

Plainly, $\phi$ is bounded, and so there exists a continuous function $f$ on $K$ satisfying (3.8). Since $\mu(K_{n_k} \backslash K_{n_{k+1}}) = \psi(n_k) - \psi(n_{k+1}) > 0$, there exists $z_k$ in $K_{n_k} \backslash K_{n_{k+1}}$ with $f(z_k) = (-1)^k$. But, since $\lim_{k \to \infty} z_k = a$, this contradicts the continuity of $f$.

We have proved that $\psi(n) = \mu(\{a\})$ for all sufficiently large $n$. Thus, for each $a$ in $K$, there exists $\rho_a > 0$, such that the relatively open set $G(a)$, defined by $G(a) = \{z \in K: |z-a| < \rho_a\}$, satisfies $\mu(G(a)) = \mu(\{a\})$. By compactness of $K$, there exists a finite subset $A = \{a_1, a_2, \ldots, a_n\}$ of $K$ with $K = \bigcup_{k=1}^n G(a_k)$. Since $K \backslash A \subset \bigcup_{k=1}^n (G(a_k) \backslash \{a_k\})$, we have $\mu(K \backslash A) = 0$. Thus the support of $\mu$ is a subset of the finite set $A$.

**Theorem 3.8.** *Let $m$ be as in Theorem 3.4, and let $m(E_k) < \infty$ for each $k$. If the range of $T_m^*$ is $L^\infty(m)$, then the support of $m$ is a countable set without limit points in $D$.*

**Proof.** Fix $n$ in $\mathbb{N}$, and let $\mu$ be the restriction of $m$ to the compact set $E_n$. Given $\phi_0$ in $L^\infty(E_n, \mu)$, take $\phi(z) = \phi_0(z)$ on $E_n$ and $\phi(z) = 0$ on $D \backslash E_n$. Then $\phi \in L^\infty(m)$, and so there exists $g$ in $BH(D)$ with $g(z) = \phi(z)$ a.e. $(m)$. On $E_n$, we have $g(z) = \phi_0(z)$ a.e. $(\mu)$, and $g$ is continuous. Since $E_n$ is the support of $\mu$, Lemma 3.7 shows that $E_n$ is a finite set.

## 4. Representation of continuous functions

**Question 1.** For what measures $m$ on $D$ is every continuous function on $\partial D$ of the form $T_m\phi$ with $\phi$ in $L^1(m)$?

We are a long way from a solution to this question, and most of this section is concerned with a second question.

**Question 2.** For what measures $m$ on $D$ is every positive continuous function on $\partial D$ of the form $T_m\phi$ with $\phi$ non-negative?

Plainly, a measure $m$ with the property in Question 2 also has the property in Question 1.

**Notation.** Let $C(\partial D)$, $C_R(\partial D)$, $C_+(\partial D)$ denote respectively the sets of complex, real, and non-negative continuous functions on $\partial D$. A continuous function $f$ on $\partial D$ is said to be *positive* if $f(\zeta)>0$ for every $\zeta$ in $\partial D$. We denote by $h^1$ the space of all differences of positive harmonic functions on $D$. Given a real Borel measure $\mu$ on $\partial D$, let

$$\tilde{\mu}(z)=\int_D p_z(\zeta)\,d\mu(\zeta).$$

It is well known that the mapping $\mu\to\tilde{\mu}$ is a bijection of the space of real Borel measures onto $h^1$.

If $m$ is non-tangentially dense for $\partial D$, then every $f$ in $C(\partial D)$ is of the form $T_m\phi$ as an element of $L^1(\partial D)$, that is

$$(T_m\phi)(\zeta)=f(\zeta) \text{ a.e.}$$

Even in the case when $m$ is the counting measure of a countable set, we do not know when this holds in the sense of pointwise convergence everywhere or of uniform convergence. It is, however, easy to prove the following theorem on uniform approximation to continuous functions by linear combinations of Poisson kernels.

**Theorem 4.1.** *Let $A$ be a subset of $D$ and let $V$ be the linear span of $\{p_a: a\in A\}$. Then $V$ is uniformly dense in $C(\partial D)$ if and only if the zero function is the only member of $h^1$ that vanishes on $A$.*

**Proof.** $V$ is uniformly dense in $C(\partial D)$ if and only if the real linear span $V_R$ of $\{p_a: a\in A\}$ is uniformly dense in $C_R(\partial D)$. By the Hahn–Banach theorem, this holds if and only if the zero measure is the only real Borel measure $\mu$ on $D$ with

$$\int_{\partial D} p_a(\zeta)\,d\mu(\zeta)=0 \qquad (a\in A),$$

that is with $\tilde{\mu}(a)=0\,(a\in A)$. The result now follows through the correspondence of members of $h^1$ with real Borel measures on $\partial D$.

**Remark.** It would be interesting to find a geometric characterization of the subsets $A$ of $D$ with the property in Theorem 4.1.

The remainder of this section is concerned with the representation of positive continuous functions on $\partial D$. Let $W(m)$ denote the set of all continuous functions $f$ on $\partial D$ that satisfy

$$f(\zeta)=\int_D \lambda(z)p_z(\zeta)\,dm(z) \qquad (\zeta\in\partial D), \tag{4.1}$$

with $\lambda$ a non-negative Borel measurable function on $D$. By Fubini's theorem, (4.1) implies

$$\|f\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it})\,dt = \int_D \lambda(z)\,dm(z) = \|\lambda\|_1. \tag{4.2}$$

Thus $\lambda \in L^1_+(m)$, the set of non-negative functions in $L^1(m)$, when $f \in W(m)$. Note also that, for all $\zeta$ in $\partial D$, (4.1) gives

$$f(\zeta) \geq \int_D \lambda(z)(1-|z|)(1+|z|)^{-1}\,dm(z),$$

and so every function in $W(m)$ is either positive or identically zero.

We say that $m$ is a *positive Poisson representing measure* (PPR measure) if $W(m)$ contains all positive continuous functions on $\partial D$.

**Lemma 4.2.** *The measure $m$ is a PPR measure if and only if $W(m)$ is uniformly dense in $C_+(\partial D)$.*

**Proof.** Let $W(m)$ be uniformly dense in $C_+(\partial D)$, and let $f_1$ be a positive continuous function on $\partial D$. Since $\inf_{\zeta \in \partial D} f_1(\zeta) > 0$, there exists $v_1$ in $W(m)$ with

$$0 < f_1(\zeta) - v_1(\zeta) < 1 \qquad (\zeta \in \partial D).$$

Take $f_2 = f_1 - v_1$. Then there exists $v_2$ in $W(m)$ with

$$0 < f_2(\zeta) - v_2(\zeta) < 1/2 \qquad (\zeta \in \partial D).$$

Continuing in this way, we obtain a sequence $\{v_n\}$ of elements of $W(m)$ and a sequence $\{f_n\}$ of continuous functions such that, for all $n$ in $\mathbb{N}$, $f_{n+1} = f_n - v_n = f_1 - (v_1 + v_2 + \cdots + v_n)$ and

$$0 < f_{n+1}(\zeta) < 1/n \qquad (\zeta \in \partial D).$$

This shows that $\sum_{k=1}^{\infty} v_k(\zeta)$ converges uniformly to $f_1(\zeta)$ on $\partial D$, and so

$$\|f_1\|_1 = \sum_{k=1}^{\infty} \|v_k\|_1. \tag{4.3}$$

For each $k$, there exists $\lambda_k$ in $L^1_+(m)$ with

$$v_k(\zeta) = \int_D \lambda_k(z) p_z(\zeta)\,dm(z) \qquad (\zeta \in \partial D),$$

and, by (4.2) and (4.3),

$$\sum_{k=1}^{\infty} \|\lambda_k\|_1 = \|f_1\|_1.$$

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Take $\lambda=\sum_{k=1}^{\infty}\lambda_k$. Then $\lambda\in L_+^1(m)$ and

$$f_1(\zeta)=\int_D \lambda(z)p_z(\zeta)\,dm(z) \qquad (\zeta\in\partial D),$$

that is $f_1\in W(m)$. This proves that $m$ is a PPR measure, and the converse is clear.

**Lemma 4.3.** *Let $E$ be a Borel subset of a compact subset of $D$ with $m(E)<\infty$, and let*

$$p_E(\zeta)=\int_E p_z(\zeta)\,dm(z) \qquad (\zeta\in\partial D).$$

*Then $p_E\in W(m)$.*

**Proof.** Let $K$ be a compact subset of $D$ containing $E$, let $R=\sup_{z\in K}|z|$ and $q_z(\zeta)=(1+\bar{z}\zeta)(1-\bar{z}\zeta)^{-1}$. Then $0\leqq R<1$ and $p_z(\zeta)=Re\,q_z(\zeta)$. For $\zeta_1,\zeta_2$ in $\partial D$,

$$q_z(\zeta_1)-q_z(\zeta_2)=2\bar{z}(\zeta_1-\zeta_2)(1-\bar{z}\zeta_1)^{-1}(1-\bar{z}\zeta_2)^{-1},$$

and so, for all $z$ in $K$,

$$|p_z(\zeta_1)-p_z(\zeta_2)|\leqq 2(1-R)^{-2}|\zeta_1-\zeta_2|.$$

Thus

$$|p_E(\zeta_1)-p_E(\zeta_2)|\leqq 2(1-R)^{-2}m(E)|\zeta_1-\zeta_2|,$$

and $p_E$ is continuous on $\partial D$. Since $p_E$ is also of the form (4.1) with $\lambda$ the characteristic function of $E$, the lemma is proved.

We recall from [3], that a subset $A$ of $D$ is a *positive Poisson basic set* (PPB set) if every positive continuous function $f$ on $\partial D$ is of the form

$$f(\zeta)=\sum_{n=1}^{\infty}\lambda_n p_{a_n}(\zeta) \qquad (\zeta\in\partial D),$$

with all $\lambda_n$ non-negative and $a_n$ in $A$.

**Theorem 4.4** *The following statements are equivalent to each other.*

(i) *$m$ is a PPR measure.*

(ii) *For every harmonic function $h$ in the space $h^1$,*

$$\sup_{z\in D} h(z)=\operatorname{ess\,sup}_{z\in D} h(z),$$

*the essential supremum being relative to the measure m.*

(iii) *The support of m is a PPB set.*

**Proof.** (i)⇒(ii). Let $m$ be a PPR measure and let $h\in h^1$. We may assume that $\operatorname{ess\,sup}_{z\in D} h(z)=M<\infty$, since otherwise there is nothing to prove. Also, since $h$ can be replaced by $h-M$, we may assume that $M=0$, and thus that

$$h(z)\leqq 0 \text{ a.e. } (m). \tag{4.4}$$

Let $\mu$ be the real Borel measure on $\partial D$ with $\tilde{\mu}=h$, and let $w\in D$. *Since $m$* is a PPR measure, there exists $\lambda$ in $L^1_+(m)$ with

$$p_w(\zeta)=\int_D \lambda(z)p_z(\zeta)\,dm(z).$$

By Fubini's theorem and (4.4), we therefore have

$$h(w)=\int_{\partial D} p_w(\zeta)\,d\mu(\zeta)=\int_D h(z)\lambda(z)\,dm(z)\leqq 0.$$

(ii)⇔(iii). By Lemma 2.2, (ii) is equivalent to

$$\sup_{z\in D} h(z)=\sup_{z\in A(m)} h(z),$$

for all $h$ in $h^1$. But, by [3, Theorem 10], this is equivalent to (iii).

(ii)⇒(i). Assume that (ii) holds, but that $m$ is not a PPR measure. Then, by Lemma 4.2, $W(m)$ is not uniformly dense in $C_+(\partial D)$, and so there exists $g_0$ in $C_+(\partial D)$ and a real Borel measure $\mu$ on $\partial D$ such that

$$\int_{\partial D} f(\zeta)\,d\mu(\zeta)\leqq 0 \qquad (f\in W(m)), \tag{4.5}$$

but

$$\int_{\partial D} g_0(\zeta)\,d\mu(\zeta)>0. \tag{4.6}$$

Let $h=\tilde{\mu}$, and let $G=\{z\in D: h(z)>0\}$. Let $K$ be a compact subset of $G$ and $E$ a Borel subset of $K$ with $m(E)<\infty$. Then, by (4.5) and Lemma 4.3,

$$\int_{\partial D} p_E(\zeta)\,\mathrm{d}\mu(\zeta)\leqq 0.$$

By Fubini's theorem, it follows that

$$\int_E h(z)\,dm(z) = \int_{\partial D}\left\{\int_E p_z(\zeta)\,dm(z)\right\}d\mu(\zeta) = \int_{\partial D} p_E(\zeta)\,d\mu(\zeta) \leqq 0.$$

Thus

$$m(E)\inf_{z\in K} h(z) \leqq \int_E h(z)\,dm(z) \leqq 0,$$

and so $m(E)=0$.

By $\sigma$-finiteness of $m$, $K$ is a countable union of Borel sets $E_n$ with $m(E_n)<\infty$, and so $m(K)=0$. Finally, since $G$ is a countable union of compact sets, $m(G)=0$, and therefore $h(z)\leqq 0$ a.e. $(m)$. By (ii), we therefore have $h(z)\leqq 0$ for all $z$ in $D$, $-\mu$ is a positive measure, contradicting (4.6).

## 5. Mapping into $L^p(\partial D)$

Let $1<p<\infty$. We define a linear mapping $T_m^{(p)}$ from $L^p(m)$ into $L^p(\partial D)$ with domain $\mathscr{D}=\mathscr{D}(T_m^{(p)})$ as follows. $\mathscr{D}$ is the set of all $\phi$ in $L^p(m)$ such that $f$, given by

$$f(\zeta) = \int_D |\phi(z)|\,p_z(\zeta)\,dm(z),$$

belongs to $L^p(\partial D)$. For $\phi$ in $\mathscr{D}$, $T_m^{(p)}\phi$ is the element of $L^p(\partial D)$ defined for almost all $\zeta$ in $\partial D$ by

$$(T_m^{(p)}\phi)(\zeta) = \int_D \phi(z)p_z(\zeta)\,dm(z).$$

**Theorem 5.1.** *$T_m^{(p)}$ is a bounded linear mapping of $L^p(m)$ into $L^p(\partial D)$ if and only if $m$ is a Carleson measure.*

**Proof.** Let $q=p(p-1)^{-1}$. For $\phi$ in $\mathscr{D}$ and $g$ in $L^q(\partial D)$, Fubini's theorem gives

$$[T_m^{(p)}\phi, g] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\int_D \phi(z)p_z(e^{it})\,dm(z)\right\}g(e^{it})\,dt$$

$$= \int_D \phi(z)g^\dagger(z)\,dm(z), \tag{5.1}$$

with $g^\dagger$ the harmonic extension of $g$. If $T_m^{(p)}$ is a bounded linear mapping of $L^p(m)$ into $L^p(\partial D)$, (5.1) gives

$$\left|\int_D \phi(z)g^\dagger(z)\,dm(z)\right| \leqq M\|\phi\|_p\|g\|_q \qquad (\phi \in L^p(m),\ g \in L^q(\partial D)),$$

with $M<\infty$. Therefore,

$$\|g^\dagger\|_q \leqq M\|g\|_q \qquad (g \in L^q(\partial D)),$$

and so $m$ is a Carleson measure (see Carleson's theorem, Garnett [**6**, p. 33], and Sarason [**10**, p. 5]).

Conversely, let $m$ be a Carleson measure. Then, again by Carleson's theorem, there exists a constant $C$ with

$$\|g^\dagger\|_q \leqq C\|g\|_q \qquad (g \in L^q(\partial D)).$$

Let $g \in L^q(\partial D)$ with $g \geqq 0$. For $\phi$ in $L^p(m)$, we have

$$\int_D |\phi(z)|g^\dagger(z)\,dm(z) \leqq \|\phi\|_p\|g^\dagger\|_q \leqq C\|\phi\|_p\|g\|_q,$$

that is

$$\int_D |\phi(z)|\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it})p_z(e^{it})\,dt\right\}dm(z) \leqq C\|\phi\|_p\|g\|_q.$$

By Fubini's theorem, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it})\left\{\int_D |\phi(z)|p_z(e^{it})\,dm(z)\right\}dt \leqq C\|\phi\|_p\|g\|_q. \tag{5.2}$$

Since this holds for all non-negative $g$ in $L^q(\partial D)$, it follows that the function $f$, given by

$$f(\zeta) = \int_D |\phi(z)|p_z(\zeta)\,dm(z),$$

belongs to $L^p(\partial D)$. Thus $\phi \in \mathscr{D}(T_m^{(p)})$ and, also from (5.2),

$$\|T_m^{(p)}\phi\|_p \leqq C\|\phi\|_p.$$

The following example shows that the range of $T_m^{(p)}$, with $m$ a Carleson measure and non-tangentially dense for $\partial D$, can fail to be the whole of $L^p(\partial D)$.

**Example.** Let $p=2$ and let $m$ be normalized area measure for $D$. With $n$ in $\mathbb{N}$, let $g_n(\zeta)=\zeta^n$ ($\zeta \in \partial D$). By (5.1), we have

$$(T_m^{(2)*}g_n)(z)=z^n,$$

and

$$\|g_n^\dagger\|_2^2=\frac{1}{\pi}\int_D |z^{2n}|\,dxdy=2\int_0^1 r^{2n+1}\,dr=(n+1)^{-1}.$$

Thus $\|T_m^{(2)*}g_n\|_2=(n+1)^{-1/2}$, while $\|g_n\|_2=1$. This is incompatible with $T_m^{(2)}L^2(m)=L^2(\partial D)$. For this, by Banach's closed range theorem [**4**, p. 488], would imply that $T_m^{(2)*}$ has closed range and zero kernel, and hence would have a bounded inverse on its range.

**Remark.** It should be noted that if $m$ is the counting measure of a countably infinite subset of $D$, then $m(D)=\infty$ and so $m$ is not a Carleson measure, $T_m^{(p)}$ is not bounded.

We do not know any example in which $T_m^{(p)}L^p(m)=L^p(\partial D)$.

**Theorem 5.2.** *Let $m$ be a Carleson measure and non-tangentially dense for $\partial D$. Then the range of $T_m^{(p)}$ is dense in $L^p(\partial D)$.*

**Proof.** Suppose that $T_m^{(p)}L^p(m)$ is not dense in $L^p(\partial D)$. Then, by the Hahn–Banach theorem, there exists $g$ in $L^q(\partial D)$ with $\|g\|_q \neq 0$ and $[T_m^{(p)}L^p(m),g]=\{0\}$. By (5.1), it follows that

$$g^\dagger(z)=0 \text{ a.e. } (m), \tag{5.3}$$

and, by taking real and imaginary parts, we may assume that $g$ is real valued. Let $G=\{z\in D: g^\dagger(z)\neq 0\}$. Then $G$ is an open set with $m(G)=0$.

Since $\|g\|_q\neq 0$, we may assume that $g$ takes positive values on a set of positive measure. Then there exist $c,d$ with $0<c<d<\infty$ such that $g^{-1}((c,d))$ has positive Lebesgue measure. By Fatou's theorem, there exists a subset $X$ of $g^{-1}((c,d))$ with positive Lebesgue measure such that, for all $\zeta$ in $X$, $g^\dagger(z)\to g(\zeta)$ as $z\to\zeta$ non-tangentially. Since $m$ is non-tangentially dense for $\partial D$, there exists $\zeta$ in $X$ such that there exists $\alpha$ with $0<\alpha<\pi/2$ and $m(D(\zeta,b,\alpha))>0$ for all $b>0$. When $b$ is sufficiently small, $g^\dagger(z)\in(c,d)$ for all $z$ in $D(\zeta,b,\alpha)$, and so $D(\zeta,b,\alpha)\subset G$. This contradicts $m(G)=0$.

## 6. $l^p$-sums of normalized Poisson kernels

Let $1<p<\infty$ and $q=p(p-1)^{-1}$. Let $\{a_k: k\in\mathbb{N}\}$ be a countable subset of $D$, and, for each $k$, let

$$Q_k=(1-|a_k|)^{1/q}p_{a_k}.$$

We shall be concerned with sums of the form

$$\sum_{k=1}^{\infty} \lambda_k Q_k$$

with $\lambda = \{\lambda_k\}$ in $l^p$.

Elementary calculations give the existence of positive constants $c_p, C_p$ such that, for all $a$ in $D$,

$$c_p(1-|a|)^{-1/q} \leqq \|p_a\|_p \leq C_p(1-|a|)^{-1/q}. \tag{6.1}$$

Thus the functions $Q_k$ are essentially the Poisson kernels $p_{a_k}$ normalized as elements of $L^p(\partial D)$.

Let $\mathscr{D}(S)$ denote the set of sequences $\lambda = \{\lambda_k\}$ in $l^p$ such that $\sum_{k=1}^{\infty} |\lambda_k| Q_k(\zeta)$ converges for almost all $\zeta$ in $\partial D$ to an element of $L^p(\partial D)$, and, for $\lambda$ in $\mathscr{D}(S)$ define $S\lambda$ by

$$(S\lambda)(\zeta) = \sum_{k=1}^{\infty} \lambda_k Q_k(\zeta) \text{ a.e.}$$

In this way, we obtain a linear mapping from $l^p$ into $L^p(\partial D)$ with domain $\mathscr{D}(S)$, which is plainly dense in $l^p$.

We note that $\lambda$ in $l^p$ belongs to $\mathscr{D}(S)$ if and only if the series $\sum_{k=1}^{\infty} |\lambda_k| Q_k$ converges in the norm of $L^p(\partial D)$, and that, in this case, $\sum_{k=1}^{\infty} \lambda_k Q_k$ converges to $S\lambda$ with respect to that norm.

For $z$ in $D$, let $\delta_z$ denote the unit mass concentrated at $z$. Let $m$ denote the $\sigma$-finite positive Borel measure on $D$ given by

$$m = \sum_{k=1}^{\infty} (1-|a_k|)\, \delta_{a_k}.$$

For $\phi$ in $L^p(m)$, let $V\phi$ be defined by

$$V\phi = \{(1-|a_k|)^{1/p} \phi(a_k)\}.$$

**Lemma 6.1.** (i) *$V$ is a linear isometry of $L^p(m)$ onto $l^p$.*

(ii) $V\mathscr{D}(T_m^{(p)}) = \mathscr{D}(S)$.

(iii) $SV = T_m^{(p)}$.

**Proof.** (i) For $\phi$ in $L^p(m)$,

$$\|V\phi\|_p^p = \sum_{k=1}^{\infty} (1-|a_k|)|\phi(a_k)|^p = \|\phi\|_p^p,$$

and so $V$ is a linear isometry of $L^p(m)$ into $l^p$. Given $\lambda = \{\lambda_k\}$ in $l^p$, define $\phi$ on $D$ by taking $\phi(a_k)=(1-|a_k|)^{-1/p}\lambda_k(k\in\mathbb{N})$ and $\phi(z)=0$ for all other $z$ in $D$. Then $\phi\in L^p(m)$ and $V\phi=\lambda$.

(ii) and (iii). With $\lambda=V\phi$, we have

$$\sum_{k=1}^{\infty}|\lambda_k|Q_k(\zeta)=\sum_{k=1}^{\infty}(1-|a_k|)|\phi(a_k)|p_{a_k}(\zeta)$$

$$=\int_D|\phi(z)|p_z(\zeta)\,dm(z),$$

so that $\mathscr{D}(S)=V\mathscr{D}(T_m^{(p)})$. With $\lambda=V\phi$ and $\phi$ in $\mathscr{D}(T_m^{(p)})$, similar equations hold with the moduli removed, and (iii) is proved.

**Theorem 6.2.** *$S$ is a bounded linear mapping of $l^p$ into $L^p(\partial D)$ if and only if $m$ is a Carleson measure. In this case*

(i) $\sum_{k=1}^{\infty}(1-|a_k|)<\infty$,

(ii) $S^*g=\{(1-|a_k|)^{1/q}g^\dagger(a_k)\}\qquad(g\in L^q(\partial D))$,

(iii) $\ker S=\{0\}$,

(iv) $Sl^p\neq L^p(\partial D)$.

**Proof.** By Lemma 6.1, $S$ is a bounded linear mapping of $l^p$ into $L^p(\partial D)$ if and only if $T_m^{(p)}$ is a bounded linear mapping of $L^p(m)$ into $L^p(\partial D)$, that is if and only if $m$ is a Carleson measure.

Suppose that $m$ is a Carleson measure. Then $m(D)<\infty$, that is (i) holds. With $\lambda$ in $l^p$ and $g$ in $L^q(\partial D)$, we have

$$[\lambda,S^*g]=[S\lambda,g]=\sum_{k=1}^{\infty}\lambda_k(1-|a_k|)^{1/q}[p_{a_k},g]$$

$$=\sum_{k=1}^{\infty}\lambda_k(1-|a_k|)^{1/q}g^\dagger(a_k)$$

$$=[\lambda,\{(1-|a_k|)^{1/q}g^\dagger(a_k)\}],$$

from which (ii) follows.

(iii) Let $\lambda\in\ker S$ and let $n\in\mathbb{N}$. By (i), there exists a Blaschke product $B_n$ with its zeros at the points $a_k$ with $k$ in $\mathbb{N}\backslash\{n\}$. We have $g_n$ in $L^\infty(\partial D)$ with $g_n^\dagger=B_n$, and so

$$0=[S\lambda,g_n]=[\lambda,S^*g_n]=\sum_{k=1}^{\infty}\lambda_k(1-|a_k|)^{1/q}B_n(a_k)$$

$$=\lambda_n(1-|a_n|)^{1/q}B_n(a_n).$$

Since $B_n(a_n)\neq 0$, we have $\lambda_n=0$. Note that $a_k\neq a_j$, when $j\neq k$ since $\{a_k: k\in\mathbb{N}\}$ was defined as a set.

(iv) If $Sl^p=L^p(\partial D)$, then $\ker S^*=\{0\}$. But, by (i), there exists a Blaschke product $B$ with its zeros at the points $a_k$ with $k$ in $\mathbb{N}$. Take $g$ in $L^\infty(\partial D)$ with $g^\dagger=B$. Then $g$ is a non-zero element of $L^q(\partial D)$ but, by (ii), $S^*g=0$.

**Theorem 6.3.** *$S$ is bounded and has closed range if and only if $\{a_k\}$ is an interpolating sequence for $H^\infty$.*

**Proof.** Suppose that $S$ is a bounded linear mapping of $l^p$ into $L^p(\partial D)$ and that $Sl^p$ is closed in $L^p(\partial D)$. By Theorem 6.2(iii), $\ker S=\{0\}$, and so, by Banach's closed range theorem [4, p. 488],

$$S^*L^q(\partial D)=l^q. \tag{6.2}$$

Let $d(a,b)=|a-b|/|1-\bar{a}b|$ for $a,b$ in $D$. We prove that

$$\inf\{d(a_j,a_k): j,k\in\mathbb{N}, j\neq k\}>0. \tag{6.3}$$

By (6.2) and the open mapping theorem, there exists a positive constant $M$ such that, given $\lambda$ in $l^q$ with $\|\lambda\|_q\leqq 1$, there exists $g$ in $L^q(\partial D)$ with

$$\{(1-|a_k|)^{1/q}g^\dagger(a_k)\}=\lambda,$$

and $\|g\|_q\leqq M$. Let $j\neq k$. Then there exists $g$ as above with $g^\dagger(a_j)=0$ and $(1-|a_k|)^{1/q}g^\dagger(a_k)=1$. Thus

$$(1-|a_k|)^{-1/q}=g^\dagger(a_k)-g^\dagger(a_j)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it})\{p_{a_k}(e^{it})-p_{a_j}(e^{it})\}\,dt$$

Let $\delta=d(a_j,a_k)$. By Harnack's inequality, we have

$$|p_{a_k}(\zeta)-p_{a_j}(\zeta)|\leqq 2\delta(1-\delta)^{-1}p_{a_k}(\zeta) \qquad (\zeta\in\partial D).$$

Thus, by (6.1),

$$(1-|a_k|)^{-1/q}\leqq\|g\|_q 2\delta(1-\delta)^{-1}\|p_{a_k}\|_p$$

$$\leqq MC_p 2\delta(1-\delta)^{-1}(1-|a_k|)^{-1/q}.$$

With $C=2MC_p$, we have $\delta(1-\delta)^{-1}\geqq C^{-1}$. Thus $\delta\geqq(1+C)^{-1}$, and (6.3) is proved.

Since $m$ is a Carleson measure and (6.3) holds, it follows that $\{a_n\}$ is an interpolating sequence for $H^\infty$, see Garnett [**5**, Lemma 2].

Suppose on the other hand that $\{a_n\}$ is an interpolating sequence for $H^\infty$. Then $m$ is a Carleson measure, and so $S$ is bounded. Also, see Shapiro and Shields [**11**, Theorem 2], every $l^q$ sequence is of the form

$$\{(1-|a_k|^2)^{1/q}f^\dagger(a_k)\}$$

with $f$ in $H^q$. Therefore $S^*L^q(\partial D)=l^q$, and, by Banach's closed range theorem, $S$ has closed range.

**Corollary 6.4.** *Let $\{a_k\}$ be an interpolating sequence for $H^\infty$ and let $X$ be the closed linear span of $\{p_{a_k}: k\in\mathbb{N}\}$ in $L^p(\partial D)$. Then each $f$ in $X$ is of the form*

$$f=\sum_{n=1}^{\infty}(B_n(a_n))^{-1}[f,B_n]p_{a_n}, \tag{6.4}$$

*where $B_n$ is the Blaschke product with its zeros at the points $a_k$ with $k\neq n$, and the series converges in the norm of $L^p(\partial D)$.*

**Proof.** By Theorem 6.3, $S$ is bounded and has closed range. Thus $Sl^p=X$, and, by Theorem 6.2 (iii) each $f$ in $X$ has a unique expression in the form

$$f=\sum_{k=1}^{\infty}\lambda_k(1-|a_k|)^{1/q}p_{a_k},$$

with $\lambda=\{\lambda_k\}$ in $l^p$ and the series convergent in the norm of $L^p(\partial D)$. Then

$$[f,B_n]=\sum_{k=1}^{\infty}\lambda_k(1-|a_k|)^{1/q}[p_{a_k},B_n]$$

$$=\sum_{k=1}^{\infty}\lambda_k(1-|a_k|)^{1/q}B_n(a_k)=\lambda_n(1-|a_n|)^{1/q}B_n(a_n).$$

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School of Mathematics
University of Leeds
Leeds LS2 9JT
England