ON SUMS OF VALENCIES IN PLANAR GRAPHS

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Planarity in graphs implies relatively small valencies and numbers of edges. In this note we find the maximum sum of valencies and the maximum number of incident edges for a set of n vertices in a planar graph with v vertices. Graphs considered are without multiple edges or loops.

THEOREM. Let G be a planar graph with vertices $A_1, \ldots, A_n, \ldots, A_v$ where $v > n \geq 3$. Denote by G_1 the graph obtained from G by deleting $A_{n+1}, \ldots, A_v$. Let the total number of edges of G be e and of G_1 be e_1, e_2 the number of edges of G joining vertices of the set $\{A_1, \ldots, A_n\}$ to those of $\{A_{n+1}, \ldots, A_v\}$, and s the sum of the valencies of $A_1, \ldots, A_n$ in G. Then

(i) $e_1 \leq 3n-6$ and $e_1 = 3n-6$ iff G_1 triangulates the plane.

(ii) $e_2 \leq n < 2v-4$ when $v = n+1$, and $e_2 \leq 2v-4$ when $v \geq n+2$ ($e_2 = n$ can hold when $v = n+1$, $e_2 = 2v-4$ can hold for each $v \geq n+2$).

(iii) $e_1 + e_2 \leq 3v-6$, and when $v \leq 3n-4$, $e_1 + e_2 = 3v-6$ iff no two of $A_{n+1}, \ldots, A_v$ are joined by an edge in G and G triangulates the plane.

(iv) $s \leq 3n+3v-12$, and when $v \leq 3n-4$, $s = 3n+3v-12$ iff G_1 triangulates the plane and $v-n$ of the regions of G_1 each contain exactly one vertex from among $A_{n+1}, \ldots, A_v$, this vertex being joined by an edge to all three vertices of G_1 adjacent to the region.

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(v) When $v \geq 3n-4$, $e_1 + e_2 \leq 3n+2v-10$ and $e_1 + e_2 = 3n+2v-10$ iff G_1 triangulates the plane, each of the $2n-4$ regions of G_1 contains one vertex from among $A_{n+1}, \ldots, A_v$ joined by an edge to all three vertices of G_1 adjacent to the region, and each of the remaining $v-3n+4$ vertices from among $A_{n+1}, \ldots, A_v$ is joined by an edge to two vertices of G_1.

(vi) When $v \geq 3n-4$, $s \leq 6n+2v-16$ and $s = 6n+2v-16$ iff G has the structure described in (v).

Proof. The proof of this theorem is based on the following well known results.

(1) Any planar graph with $w \geq 3$ vertices triangulates the whole plane iff the total number of edges is $3w-6$; in this case the number of regions into which the graph divides the plane is $2w-4$. Any planar graph with $w \geq 3$ vertices either triangulates the plane or is obtained from a planar graph with w vertices which triangulates the plane by deleting edges.

(2) If a planar graph has w vertices and e edges and divides the plane into r connected regions, then $w-e+r \geq 2$.

Proof of (i). (i) follows directly from (1) with $w = n$.

Proof of (ii). If $v = n+1$ then obviously $e_2 \leq n$ and equality may hold; also $n < 2v-4$, since $v > n \geq 3$ by hypothesis. If $e_2 \leq 3$ then $e_2 \leq n < 2v-4$ since $v > n \geq 3$. It only remains to assume that $v \geq n+2$ and $e_2 \geq 4$. Then let G' denote the graph obtained from G by deleting all edges except the e_2 edges joining vertices of $\{A_1, \ldots, A_n\}$ to vertices of $\{A_{n+1}, \ldots, A_v\}$; $G' = G$ possibly. Let r denote the number of connected regions into which G' divides the plane. By (2) applied to G' we have

$$v - e_2 + r \geq 2 .$$

Each of the connected regions into which G' divides the

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plane is adjacent to at least four edges of G', because $e_2 \geq 4$ and every circuit of G' (if any) contains an even number of edges and vertices since G' is bipartite. Also each edge of G' is adjacent to at most two regions. Hence

$$4r \leq 2e_2$$

because on the left each edge of G is counted at most twice.

Eliminating r from the two inequalities we have $e_2 \leq 2v-4$. $e_2 = 2v-4$ if, for example, two of $A_{n+1}, \ldots, A_v$ are joined by an edge to all of $A_1, \ldots, A_n$ and the rest of $A_{n+1}, \ldots, A_v$ to two of $A_1, \ldots, A_n$.

Proof of (iii). By (1) applied to G, $e_1+e_2 \leq e \leq 3v-6$, and $e_1+e_2 = 3v-6$ iff G triangulates the plane and $e_1+e_2 = e$, which is the case iff G triangulates the plane and no edge of G joins two of $A_{n+1}, \ldots, A_v$. If G has such a structure, then each of the connected regions into which G divides the plane contains at most one of $A_{n+1}, \ldots, A_v$; by (1) this implies $2n-4 \geq v-n$, i.e. $v \leq 3n-4$.

Proof of (iv). $s = 2e_1+e_2 = e_1+(e_1+e_2)$. $e_1 \leq 3n-6$ by (i) and $e_1+e_2 \leq e \leq 3v-6$ by (1); hence $s \leq 3n+3v-12$ with equality iff $e_1 = 3n-6$ and $e_1+e_2 = 3v-6$. By (i) and (iii) this is the case iff G_1 and G both triangulate the plane and no two of $A_{n+1}, \ldots, A_v$ are joined by an edge; consequently $s = 3n+3v-12$ iff G is as described in (iv) and then $v \leq 3n-4$.

Proof of (v). By (i) and (ii) $e_1+e_2 \leq 3n+2v-10$ with equality iff $e_1 = 3n-6$ and $e_2 = 2v-4$. By (i) $e_1 = 3n-6$ iff G_1 triangulates the plane; e_2 is clearly maximal, consistent with G_1 triangulating the plane, iff G has the structure described in (v); e_2 is then equal to $3(2n-4) + 2(v-3n+4) = 2v-4$ provided

$v \geq 3n-4$. Hence when $v \geq 3n-4$, $e_1+e_2 = 3n+2v-10$ iff G is as described in (v).

Proof of (vi). $s = e_1+(e_1+e_2)$. Hence, by (i) and (v), $s \leq 6n+2v-16$ and $s = 6n+2v-16$ iff G has the structure described in (v). This completes the proof of the theorem.

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